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ON THE DISTRIBUTION OF THE MAXIMUM OF THE SEQUENCE OF SUMS OF INDEPENDENT RANDOM VARIABLES

BY T. GERGELY AND I. I. YEZHOW

Central Research Institute for Physics and University of Kiev

Let $\xi_1, \xi_2, \cdots$ be independent random variables. The distribution of $\max(0, \xi_1, \xi_1 + \xi_2, \cdots, \xi_1 + \cdots + \xi_n)$ is investigated by means of a method based on the construction of certain events with easily determined probabilities. These yield a new formula for the distribution of the maximum which is sometimes more useful than that given in literature.

0. Introduction. Let $\xi_1, \xi_2, \cdots$ be independent random variables. In this paper the distribution of $\max(0, \xi_1, \xi_1 + \xi_2, \cdots, \xi_1 + \cdots + \xi_n)$ is discussed. Distributions of this kind appear in several applications of probability theory, e.g. in queuing theory, in reliability theory, etc.

In [1], [2], [3], [4], [6], [7] different methods can be found for obtaining a formula for the maximum's distribution in the case where $\xi_1, \xi_2, \cdots$ are identically distributed.

The method investigated here leads to a new formula, which is sometimes more useful than that given, e.g. in [2].

In Section 1, a lemma for the maximum-distribution of the sequence consisting of differences of arbitrary integer-valued random variables is proved by a new method. By means of this lemma the main theorem is proved.

In Section 2, two special cases are investigated, namely, in 2.1 $\xi_1, \xi_2, \cdots$ are independent, bounded from below, integer-valued random variables and in 2.2 they are identically distributed. In the last Section the asymptotic behavior of the case described in Section 2.2 is discussed.

1. The main theorem. The proof of the main theorem for the distribution of the maximum of partial sums of independent integer-valued random variables is based on the following.

LEMMA 1. *Let $\{\sigma_n, n \geqq 0\}$ be an arbitrary sequence of integer-valued random variables. If $n > m$, then for all $l \geqq 0$*

$$\begin{aligned} &P\{\max(\sigma_k - \sigma_m, m \leqq k \leqq n) = l\} \\ &\quad = P\{\sigma_n - \sigma_m = l\} + \sum_{k=m}^{n-1} \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} [P\{\sigma_k - \sigma_m = l, \\ &\qquad \sigma_{k+1} - \sigma_k = -i - j, \max(\sigma_r - \sigma_{k+1}, k+1 \leqq r \leqq n) = i\} \\ &\qquad - P\{\sigma_k - \sigma_m = l + j, \sigma_{k+1} - \sigma_k = -i - j, \\ &\qquad \max(\sigma_r - \sigma_{k+1}, k+1 \leqq r \leqq n) = i\}] . \end{aligned} \tag{1}$$

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PROOF. Let us introduce the events

$$A = \{\max(\sigma_k - \sigma_m, m \leqq k \leqq n) = l\}, \qquad B = \{\sigma_n - \sigma_m = l\},$$
$$C_{kji} = \{\sigma_k - \sigma_m = l, \sigma_{k+1} - \sigma_m = l - i - j, \max(\sigma_r - \sigma_m, k < r \leqq n) = l - j\},$$
$$D_{kji} = \{\sigma_k - \sigma_m = l + j, \sigma_{k+1} - \sigma_m = l - i, \max(\sigma_r - \sigma_m, k < r \leqq n) = l\},$$
$$C = \bigcup_{k=m}^{n-1} \bigcup_{j=1}^{\infty} \bigcup_{i=0}^{\infty} C_{kji}, \qquad D = \bigcup_{k=m}^{n-1} \bigcup_{j=1}^{\infty} \bigcup_{i=0}^{\infty} D_{kji}.$$

It will be shown, that

$$A = (B \cup C) \cap A \quad \text{and} \quad D = (B \cup C) \cap A^c \tag{2}$$

where A^c is the complement of A.

Let event A take place. If $\sigma_n - \sigma_m = l$ then B takes place. But if $\sigma_n - \sigma_m < l$ then that maximal index for which $\sigma_k - \sigma_m = l$ will be designated by k $(m \leqq k < n)$. Then $\sigma_{k+1} - \sigma_m = l - t$ $(t > 0)$ and $\max(\sigma_r - \sigma_m, k < r \leqq n) = l - j$ $(j \geqq 1)$. Therefore the event $C_{kj,t-j}$ $(t \geqq j)$ takes place and with it C too. Thus the first equality in (2) is proved.

Let $B \cap A^c$ take place. Then $\sigma_n - \sigma_m = l$ and there exists at least one such index ν $(m < \nu < n)$, for which $\sigma_\nu - \sigma_m > l$.

Let k be denoted as the maximum of all such indexes ν and accept that $\sigma_k - \sigma_m = l + j$ $(j \geqq 1)$. If $\sigma_{k+1} - \sigma_m = l - i$ $(i \geqq 0)$ then D_{kji} takes place. Therefore $B \cap A^c \subset D$.

If $C \cap A^c$ takes place, then there exists $m < \nu < k < n$ such that $\sigma_\nu - \sigma_m > l$, $\sigma_k - \sigma_m = l$, $\sigma_{k+1} - \sigma_m = l - i - j$, and $\max(\sigma_r - \sigma_m, k < r \leqq n) = l - j$ $(i \geqq 0, j \geqq 1)$.

If Θ is the maximum of such indexes ν and $\sigma_\Theta - \sigma_m = l + j'$ $(\Theta \leqq k, j' \geqq 1)$ then for some $i' \geqq 0$ $\sigma_{\Theta+1} - \sigma_m = l - i'$ and $D_{\Theta j' i'}$ takes place. Therefore $C \cap A^c \subset D$.

Since D and A are disjoint it is still necessary to prove the inclusion

$$D \subset B \cup C \tag{3}$$

in order to prove the second equality in (2).

Let D take place. If $\sigma_n - \sigma_m = l$ then B takes place. However if $\sigma_n - \sigma_m < l$, then there exist such values k, i, j, that $\sigma_k - \sigma_m = l + j$, $\sigma_{k+1} - \sigma_m = l - i$, $\max(\sigma_r - \sigma_m, k < r \leqq n) = l$, $\sigma_n - \sigma_m < l$.

Let us designate ρ as the maximal index, for which $\sigma_\rho - \sigma_m = l$ $(k < \rho < n)$ and take $\sigma_{\rho+1} - \sigma_m = l - j'$ $(j' \geqq 1)$.

It is evident that for some $j^* \geqq 1$ $\max(\sigma_r - \sigma_{m'}, \rho < r \leqq n) = l - j^*$ $(j^* \leqq j')$ and $C_{\rho j^*, j' - j^*}$ takes place. Herewith (3) is proved.

Events B and C are disjoint. According to (2)

$$P(A) = P(A \cap B) + P(A \cap C)$$
$$P(D) = P(A^c \cap B) + P(A^c \cap C)$$

from where

$$P(A) = P(B) + P(C) - P(D)$$

or

$$P(A) = P(B) + \sum_{k=m}^{n-1} \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} [P(C_{kji}) - P(D_{kji})] ,$$

which is identical with (1). The lemma is proved.

COROLLARY 1. *If* $\{\sigma_n, n \geqq 0\}$ *is an arbitrary sequence of integer-valued random variables satisfying the condition*

$$P\{\sigma_{k+1} - \sigma_k \geqq -1\} = 1$$

for all k, *then*

$$\begin{aligned} &P\{\max(\sigma_k - \sigma_m, m \leqq k \leqq n) = 0\} \\ &\quad = P\{\sigma_n - \sigma_m = 0\} + \sum_{k=m}^{n-1} [P\{\sigma_k - \sigma_m = 0, \sigma_{k+1} - \sigma_k = -1, \\ &\qquad \max(\sigma_r - \sigma_{k+1}, k+1 \leqq r \leqq n) = 0\} \\ &\quad - P\{\sigma_k - \sigma_m = 1, \sigma_{k+1} - \sigma_k = -1, \\ &\qquad \max(\sigma_r - \sigma_{k+1}, k+1 \leqq r \leqq n) = 0\}] , \end{aligned} \tag{4}$$

and

$$\begin{aligned} &P\{\max(\sigma_r - \sigma_m, m \leqq r \leqq n) \geqq l\} \\ &\quad = P\{\sigma_n - \sigma_m \geqq l\} \\ &\qquad + \sum_{k=m}^{n-1} P\{\sigma_k - \sigma_m = l, \max(\sigma_r - \sigma_m, k < r \leqq n) = l - 1\} . \end{aligned} \tag{5}$$

Let $\xi_1, \xi_2, \cdots$ be independent integer-valued random variables.

$$\rho_k^{[m,n]} = P\{\xi_{m+1} + \xi_{m+2} + \cdots + \xi_n = k\} , \qquad \rho^{[n]} = \rho^{[n-1,n]} \qquad (n > m \geqq 0, k = 0, \pm 1, \cdots) . \tag{6}$$

THEOREM. *If*

$$\sigma_0 = 0 , \qquad \sigma_n = \xi_1 + \cdots + \xi_n \qquad (n = 1, 2, \cdots) ,$$

and

$$P\{\max(\sigma_k - \sigma_m, m \leqq k \leqq n) = l\} = \mu_l^{[m,n]} \tag{7}$$

then

$$\mu_l^{[m,n]} = \rho_l^{[m,n]} + \sum_{k=m}^{n-1} \sum_{i=0}^{\infty} a_{li}^{[m,k]} \mu_i^{[k+1,n]} \qquad (l \geqq 0, n > m \geqq 0) \tag{8}$$

where

$$a_{li}^{[m,k]} = \sum_{j=1}^{\infty} (\rho_l^{[m,k]} - \rho_{l+j}^{[m,k]}) \rho_{-(i+j)}^{[k+1]} . \tag{9}$$

PROOF. Since $\xi_1, \xi_2, \cdots$ are independent integer-valued random variables, then

$$\begin{aligned} &P\{\sigma_k - \sigma_m = q, \sigma_{k+1} - \sigma_k = s, \max(\sigma_r - \sigma_{k+1}, k+1 \leqq r \leqq n) = i\} \\ &\quad = p\{\sigma_k - \sigma_m = q\} P\{\sigma_{k+1} - \sigma_k = s\} \\ &\qquad \times P\{\max(\sigma_r - \sigma_{k+1}, k+1 \leqq r \leqq n) = i\} , \end{aligned}$$

where $q, s, i = 0, \pm 1, \pm 2, \cdots$.

Using this and the notations (6) and (7), from (1) we obtain (8). The theorem is thus proved. The equation (8) enables us to find $\mu_l{}^{[m,n]}$. Let

$$\boldsymbol{\mu}^{[m,n]} = \{\mu_l{}^{[m,n]}; l \geqq 0\}, \qquad \boldsymbol{\rho}^{[m,n]} = \{\rho_l{}^{[m,n]}; l \geqq 0\}$$

be vector-columns and

$$A^{[m,k]} = ||a_{li}^{[m,k]}; l, i \geqq 0|| \qquad \text{be a matrix.}$$

Then in accordance with (8)

$$\boldsymbol{\mu}^{[m,n]} = \boldsymbol{\rho}^{[m,n]} + \sum_{k=m+1}^{n} A^{[m,k-1]}\boldsymbol{\mu}^{[k,n]} \quad (0 \leqq m < n) \tag{10}$$

and

$$\boldsymbol{\mu}^{[n,n]} = \boldsymbol{\rho}^{[n,n]} = \{\delta_{0l}; l \geqq 0\} = \boldsymbol{\delta}\,. \tag{11}$$

Employing these equations we have:

$$\boldsymbol{\mu}^{[n-1,n]} = \boldsymbol{\rho}^{[n-1,n]} + A^{[n-1,n]}\boldsymbol{\delta}\,,$$

$$\boldsymbol{\mu}^{[n-2,n]} = \boldsymbol{\rho}^{[n-2,n]} + A^{[n-2,n-2]}\boldsymbol{\rho}^{[n-1,n]} + (A^{[n-2,n-2]}A^{[n-1,n-1]} + A^{[n-2,n-1]})\boldsymbol{\delta}\,,$$

and in general

$$\boldsymbol{\mu}^{[m,n]} = \boldsymbol{\rho}^{[m,n]} + \sum_{i=1}^{n-m} \sum_{m<k_1<\cdots<k_i\leqq n} A^{[m,k_1-1]}A^{[k_1,k_2-1]} \cdots A^{[k_{i-1},k_i-1]}\boldsymbol{\rho}^{[k_i,n]}$$

or

$$\boldsymbol{\mu}^{[k_0,n]} = \boldsymbol{\rho}^{[k_0,n]} + \sum_{i=1}^{n-k_0} \sum_{k_0<k_1<\cdots<k_i\leqq n} \prod_{j=0}^{i-1} A^{[k_j,k_{j+1}-1]}\boldsymbol{\rho}^{[k_i,n]}\,. \tag{12}$$

2. Special cases.

2.1. Let $\xi_1, \xi_2, \cdots$ be independent integer-valued random variables bounded below, i.e. for some fixed integer $c \geqq 0$ and for all $k = 1, 2, \cdots$

$$P\{\xi_k \geqq -c\} = 1\,. \tag{13}$$

In this case from (9) it follows that for $i \geqq c$ $a_{li}^{[m,k]} = 0$, and for $i < c$

$$a_{li}^{[m,k]} = \sum_{j=1}^{c-i} (\rho^{[m,k]} - \rho_{l+j}^{[m,k]})\rho_{-(i+j)}^{[k+1]} \qquad (0 \leqq i \leqq c-1)\,. \tag{14}$$

Therefore from (8) we obtain

$$\mu_l{}^{[m,n]} = \rho_l{}^{[m,n]} + \sum_{k=m}^{n-1} \sum_{i=0}^{c-1} a_{li}^{[m,k]}\mu_i{}^{[k+1,n]} \qquad (l = 0, 1, \cdots)\,. \tag{15}$$

It follows from this, that $\mu_l{}^{[m,n]}$ for $l \geqq c$ can be expressed very simply by means of $\mu_i{}^{[k,n]}$ $(i < c, k > m)$. Therefore only the probabilities $\mu_l{}^{[m,n]}$ $(0 \leqq l < c,$ $n > m \geqq 0)$ need to be defined. Let

$$\boldsymbol{\mu}_c{}^{[m,n]} = \{\mu_l{}^{[m,n]}, 0 \leqq l \leqq c-1\}, \qquad \boldsymbol{\rho}_c{}^{[\mu,n]} = \{\boldsymbol{\rho}_l{}^{[m,n]}, 0 \leqq l \leqq c-1\}$$

be vector columns and $A_c{}^{[m-k]} = ||a_{li}^{[m,k]}; l, i = 0, \cdots, c-1||$ be a matrix. On the analogy of (12) we have:

$$\boldsymbol{\mu}_c{}^{[m_0,n]} = \rho_c{}^{[m_0,n]} + \sum_{i=1}^{n-m_0} \sum_{m_0<m_1<\cdots<m_i\leqq n} \prod_{j=0}^{i-1} A_c{}^{[m_j,m_{j+1}-1]}\rho_c{}^{[m_i,n]} \tag{16}$$

where

$$\boldsymbol{\mu}_c{}^{[n,n]} = \rho_c{}^{[n,n]} = \boldsymbol{\delta}_c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

NOTE. If $c = 1$ (that is, all the ξ_n do not assume to take less then -1), then

$$\mu_l^{[m,n]} = \rho_l^{[m,n]} + \textstyle\sum_{k=m}^{n-1} (\rho_l^{[m,k]} - \rho_{l+1}^{[m,k]})\rho_{-1}^{[k+1]}\mu_0^{[k+1,n]} \,,$$

$$(17) \qquad \mu_0^{[m_0,n]} = \rho_0^{[m_0,n]} + \textstyle\sum_{i=1}^{n-m_0} \sum_{m_0<\cdots<m_i\leq n} \prod_{j=0}^{i-1} (\rho_0^{[m_j,m_{j+1}-1]} - \rho_1^{[m_j,m_{j+1}-1]})\rho_{-1}^{[m_j+1]}\rho_0^{[m_i,n]} \,.$$

It is noteworthy that (17) easily follows from (4).

2.2. Suppose now that $\xi_1, \xi_2, \cdots$ are independent identically distributed integer-valued random-variables, i.e. for all $n = 1, 2, \cdots$

$$P\{\xi_n = k\} = \rho_k \,, \qquad P\{\xi_1 + \cdots + \xi_n = k\} = \rho_k^{[n]} \quad (k = 0, \pm 1, \cdots) \,.$$

Then

$$P\{\max(\sigma_k - \sigma_m, m \leqq k \leqq n) = l\} = \mu_l^{[n-m]} \quad (l = 0, 1, \cdots; n > m) \,.$$

If

$$a_{li}^{[k]} = \textstyle\sum_{j=1}^{\infty} (\rho_l^{[k]} - \rho_{l+j}^{[k]})\rho_{-(i+j)} \,,$$

then, according to (8)

$$(18) \qquad \mu_l^{[n]} = \rho_l^{[n]} + \textstyle\sum_{k=0}^{n-1} \sum_{i=0}^{\infty} a_{li}^{[k]}\mu_i^{[n-k-1]} \qquad (l \geqq 0, n \geqq 1) \,;$$

whence passing over to generating functions we get

$$(19) \qquad \mu_l(z) = \rho_l(z) + z \textstyle\sum_{i=0}^{\infty} a_{li}(z)\mu_i(z)$$

where

$$\textstyle\sum_{n=0}^{\infty} \mu_l^{[n]}z^n = \mu_l(z) \,, \qquad \sum_{n=0}^{\infty} \rho_l^{[n]}z^n = \rho_l(z) \,, \qquad \sum_{n=0}^{\infty} a_{li}^{[n]}z^n = a_{li}(z)$$

$$(i, l = 0, 1, \cdots; |z| \leqq 1) \,;$$

(18) is nothing other than an infinite system of linear algebraical equations connecting with functions $\mu_l(z)$.

Let

$$\boldsymbol{\mu}(z) = \{\mu_l(z); l \geqq 0\} \,, \qquad \boldsymbol{\rho}(z) = \{\rho_l(z); l \geqq 0\}$$

be vector columns and $zA(z) = ||za_{li}(z); l, i \geqq 0||$ be a linear operator in the Banach space of bounded sequences $\mathbf{x} = \{x_l; l \geqq 0\}$;

$$||\mathbf{x}|| = \sup_l |x_l| \,, \qquad zA(z)\mathbf{x} = \mathbf{y} \,,$$

$\mathbf{y} = \{z \sum_{i=0}^{\infty} a_{li}(z)x_i, l \geqq 0\}$, $||zA(z)|| = |z| \sup_l \sum_{i=0}^{\infty} |a_{li}(z)|$.

In this notation (19) takes the form:

$$(20) \qquad \boldsymbol{\mu}(z) = \boldsymbol{\rho}(z) + zA(z)\boldsymbol{\mu}(z) \,.$$

Let us suppose:

$$(21) \qquad E = \left\{z : \sup_l \textstyle\sum_{j=1}^{\infty} |\rho_l(z) - \rho_{l+j}(z)| P\{\xi_1 \leqq -j\} < \dfrac{1}{|z|}\right\} .$$

Then $||zA(z)|| < 1$ if $z \in E$. Therefore (20) has one unique bounded solution if all $z \in E$:

$$(22) \qquad \boldsymbol{\mu}(z) = |I - zA(z)|^{-1}\boldsymbol{\rho}(z) \qquad (z \in E) \,.$$

Here I is the identity operator and $|I - zA(z)|^{-1}$ the inverse operator of $I - zA(z)$. If the set E includes at least one accumulation point then by means of the principle of extension of analytic continuation, the right part (22) enables the vector $\boldsymbol{\mu}(z)$ to be uniquely determined over the whole domain of analyticity. In the domain of analyticity of the vector $\boldsymbol{\mu}(z)$ the open sphere $|z| < 1$ is always included. It is a sufficient condition that E contain some neighborhood of 0, e.g. the following:

$$\sum_{j=1}^{\infty} P\{\xi_1 \leqq -j\} = \sum_{n=1}^{\infty} n\rho_{-n} < \infty . \tag{23}$$

In fact in this case

$$\left\{z : |z| < \frac{1}{1 + \sum_{n=1}^{\infty} n\rho_{-n}}\right\} \subset E .$$

Note that the condition (23) is equivalent to the fact that $M\xi_1$ is either final or equal to $+\infty$. And thus, if

$$A(z) = \|\textstyle\sum_{j=1}^{\infty} |\rho_l(z) - \rho_{l+j}(z)|\rho_{-(i+j)}; l, i \geqq 0\| ,$$

and

$$|I - zA(z)|^{-1} = \textstyle\sum_{n=0}^{\infty} z^n A^n(z) = \|b_{li}(z); l, i \geqq 0\| ,$$

then, according to (21)

$$\mu_l(z) = \textstyle\sum_{i=0}^{\infty} b_{li}(z)\rho_i(z) \qquad (l \geqq 0) . \tag{24}$$

The convenience of the right part of (24) is that it can be simplified by means of assumption (13). Thus, let (13) be satisfied, that is

$$\textstyle\sum_{k=-c}^{\infty} \rho_k = 1 .$$

According to (18) in the definition only $\mu_0(z), \cdots, \mu_{c-1}(z)$ are needed. Introducing the matrix

$$A_c(z) = \|\textstyle\sum_{j=1}^{c-i} |\rho_l(z) - \rho_{l+j}(z)|\rho_{-(i+j)}; l, i = 0, \cdots, c-1\| ,$$

and the vector-columns $\boldsymbol{\mu}_c(z) = \{\mu_l(z); 0 \leqq l \leqq c-1\}$, $\boldsymbol{\rho}_c(z) = \{\rho_l(z); 0 \leqq l \leqq c-1\}$ we obtain

$$\boldsymbol{\mu}_c(z) = \boldsymbol{\rho}_c(z) + zA_c(z)\boldsymbol{\mu}_c(z) . \tag{25}$$

Solving the already finite system of linear algebraical equations (25) we get

$$\mu_l(z) = \textstyle\sum_{i=0}^{c-1} b_{li}^c(z)\rho_i(z) \qquad 0 \leqq l \leqq c-1 \tag{26}$$

where the matrices

$$\|b_{li}^c(z); l, i = 0, \cdots, c-1\| \qquad \text{and} \qquad I - zA_c(z)$$

are inverse with respect to one another.

REMARK. In the special case $c = 1$ we have for any $l > 0$

$$\mu_l^{[n]} = \rho_l^{[n]} + \rho_{-1} \textstyle\sum_{k=0}^{n-1} (\rho_l^{[k]} - \rho_{l+1}^{[k]}) \dfrac{1}{2\pi i} \oint_{|z|=r} \dfrac{\rho_0(z) z^{k+2-n}}{1 - z\rho_{-1}[\rho_0(z) - \rho_1(z)]} \tag{27}$$

where $\gamma > 0$ is sufficiently small, for example it is obviously sufficient to require that

$$\gamma < \frac{1}{1+\rho_{-1}} .$$

3. The asymptotic behavior of $\mu_l^{[n]}$ as $n \to \infty$. Let $\xi_1, \xi_2, \cdots,$ be the same sequence as in Section 2.2. Let

$$\zeta_n = \max_{0 \leqq k \leqq n} \sigma_k = \max(0, \xi_1, \xi_1 + \xi_2, \cdots, \xi_1 + \cdots + \xi_n) .$$

Since

$$\max(0, \xi_1, \xi_1 + \xi_2, \cdots, \xi_1 + \cdots + \xi_n) = \max(0, \xi_1 + \max(0, \xi_2, \xi_2 + \xi_3, \cdots, \xi_2 + \cdots + \xi_n)) ,$$

and $\max(0, \xi_2, \xi_2 + \xi_3, \cdots, \xi_2 + \cdots + \xi_n)$ has the same distribution as ζ_{n-1} and is independent of ξ_1, ζ_n has the same distribution with all n as $\zeta_n{}^*$, where $\{\zeta_n{}^*\}$ is homogeneous Markov chain which satisfies the following condition:

$$\zeta_0{}^* = 0 , \qquad \zeta_{n+1}^* = \max(0, \zeta_n{}^* + \xi_{n+1}) \qquad (n = 0, 1, \cdots) ; \tag{28}$$

ξ_{n+1} is independent of $\zeta_n{}^*$ and has the same distribution as in Section 2.2. Thereby the problem is reduced to the determination of the stationary distribution of the homogeneous Markov chain $\{\zeta_n{}^*\}$. Let

$$\lim_{n\to\infty} P\{\zeta_n{}^* = l\} = \lim_{n\to\infty} \mu_l{}^{[n]} = \mu_l \qquad (n \geqq 0) .$$

According to (28) we have;

$$\mu_0 = \sum_{k=0}^{\infty} \mu_k \sum_{i \leqq -k} \rho_i , \qquad \mu_l = \sum_{k=0}^{\infty} \mu_k \rho_{l-k} \qquad (l > 0) .$$

Let us assume that $\rho_{-(c+1)} = \rho_{-(c+2)} = \cdots = 0$. Then

$$\mu_l = \sum_{k=0}^{l+c} \mu_k \rho_{l-k} \qquad (l = 1, 2, \cdots) . \tag{29}$$

Let us introduce the generating function:

$$\sum_{l=0}^{\infty} \mu_l z^l = \mu(z) \qquad (|z| \leqq 1) .$$

According to (29),

$$\mu(z) = \frac{\mu_0 - \sum_{j=0}^{c} \mu_j \sum_{i=j}^{c} \rho_{-i} z^{j-i}}{1 - \rho(z)} \tag{30}$$

where $\rho(z) = Mz^{\xi_1}$. The numerator on the right side of (30) contains $c + 1$ unknown $\mu_0, \mu_1, \cdots, \mu_c$. Let us show how to obtain these values. $1 - \rho(z)$ corresponding to Rouche's theorem [1] has roots in the domain of $|z| < 1$. Each root is regarded as the same number as is its multiplicity. In fact

$$1 - \rho(z) = \frac{1}{z^c} (z^c - \sum_{i=-c}^{\infty} \rho_i{}^{z^{i+c}})$$

and on the boundary

$$[(|z| = 1) \cap (|z - 1| > \varepsilon)] \cup [(z < 1) \cap (|z - 1| = \varepsilon)]$$

where ε is a suitable small positive number we have

$$|z^c| > |\sum_{i=-c}^{\infty} \rho_i z^{i+c}| \,.$$

Let us denote the roots of the equation $1 - \rho(z) = 0$ belonging to the circle $|z| \leqq 1$, by $1, \alpha_1, \cdots, \alpha_{c-1}$. Since $\mu(z)$ is an analytic function in that circle, it therefore follows from (30) that

$$\text{(31)} \qquad \begin{aligned} &\mu_0 - \sum_{j=0}^{c} \mu_j \sum_{i=j}^{c} \rho_{-i} = 0 \\ &\mu_0 - \sum_{j=0}^{c} \mu_j \sum_{i=j}^{c} \rho_{-i} \alpha_m{}^{j-i} = 0 \qquad (1 \leqq m \leqq c-1)\,. \end{aligned}$$

Further $\mu(1) = 1$. Applying the l'Hospital's rule we get:

$$\text{(32)} \qquad 1 = \frac{\sum_{j=0}^{c} \mu_j \sum_{i=j}^{c} (j-i)\rho_{-i}}{M\xi_1}\,.$$

By doing so the determination of $\mu_0, \cdots, \mu_c$ is reduced to the solution of the system of linear algebraic equations (30)—(32). In particular, from (32) it follows that $M\xi_1 < 0$. In our propositions concerning the distribution ξ_1 this condition is necessary and sufficient in order that

$$\sup_{k \geqq 0} \sigma_k = \zeta$$

be a proper random quantity (see for instance [8]). Note that if there are multiple roots among the roots α_i, then the system of equations (31) contains more unknown quantities than equations. We are able to get the missing equations by equating to zero the corresponding derivatives of the numerator in (30).

REMARK. If $c = 1$, then

$$\mu_0 = \mu_0(\rho_0 + \rho_{-1}) + \mu_1 \rho_{-1}\,, \qquad M\xi_1 = -\rho_{-1}\mu_0$$

or

$$\text{(33)} \qquad \mu_0 = -\frac{1}{\rho_{-1}} M\xi_1\,, \qquad \mu(z) = M\xi_1 \frac{1-z}{z - z\rho(z)}\,.$$

By employing elementary transformations, $\mu(z)$ can be reduced to the form

$$\text{(34)} \qquad \mu(z) = \frac{\mu_0}{1 - A(z)}$$

where

$$A(z) = \sum_{i=1}^{\infty} a_i z^i\,, \qquad a_i = \frac{1}{\rho_{-1}} \sum_{l=i}^{\infty} \rho_l \qquad (i \geqq 1)\,.$$

Take $a_i{}^{1*} = a_i$, and if $k > 1$

$$\begin{aligned} a_i{}^{k*} &= 0\,, && i < k \\ &= \sum_{j=k-1}^{i-1} a_j{}^{(k-1)*} a_{i-j}\,, && k \leqq i\,. \end{aligned}$$

Then $\sum_{i=k}^{\infty} a_i{}^{k*} z^i = A^k(z)$ and, according to (34)

$$\text{(35)} \qquad \mu_i = \mu_0 \sum_{k=0}^{i} a_i{}^{k*} \qquad (i \geqq 0)\,.$$

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T. GERGELY
CENTRAL RESEARCH INSTITUTE FOR PHYSICS
H-1525 BUDAPEST
P.O.B. 49, HUNGARY

I. I. YEZHOW
DEPARTMENT OF THEORY OF PROBABILITY
UNIVERSITY OF KIEV
KIEV, USSR