The Annals of Probability
1975, Vol. 3, No. 2, 281-288

THE LAW OF LARGE NUMBERS FOR SUBSEQUENCES OF A STATIONARY PROCESS

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Convergence in mean of $N^{-1} \sum_{k=1}^{N} X_{t_k}$ is studied for stationary processes classified according to parameter space and type of spectral measure.

0. Introduction. In this paper we consider the problem of estimating the mean of a stationary process by using a subsequence of the possible observations. To begin assume x_n is a weakly stationary sequence with mean m and covariance

$$R(n) = E((x_k - m)(x_{k+n} - m)) = \int_0^{1-} \exp(2\pi i\lambda n)\, dF(\lambda)\ .$$

The measure given by F is called the spectral measure of the process. If $dF(0)=0$ the process is called ergodic in the wide sense and in this case $N^{-1} \sum_{n=1}^{N} x_n$ converges to m in mean square. Unlike the case for uncorrelated random variables $N^{-1} \sum_{K=1}^{N} x_{n_k}$ does not converge for arbitrary subsequences n_K. In this paper we find conditions on subsequences for which there is convergence for certain classes of processes.

Hanson and Pledger (1969) studied a closely related problem in the discrete parameter case and gave a fairly complete solution. Section 1 starts with a different approach to the problem, includes a derivation of some known results (Hanson–Pledger (1969) and Blum–Eisenberg (1974)) and continues with a consideration of the same problem in the continuous parameter case. The same approach is used in Section 2 for the class of processes weakly mixing in the wide sense, i.e., those with continuous spectral functions. Some new sampling subsequences are shown to be admissible here. A sequence t_K is called admissible for a stationary process x_t if $N^{-1} \sum_{K=1}^{N} x_{t_K}$ converges to m in mean square.

In Blum–Hanson (1960) it is shown that any subsequence is admissible for a process which is strongly mixing in the wide sense, i.e., a process whose covariance $R(n) \to 0$ as $n \to \infty$. Conversely, they show that if every subsequence is admissible then the process must be strongly mixing. In Section 3 an extension of these results is given for continuous parameter processes.

In all this work we neglect the question of whether m could be better estimated by some other linear combination of observations since this would require more detailed knowledge of the spectral measure of the process.

Received August 6, 1973; revised July 16, 1974.

[1] Research supported by N.S.F. Grant GP 27536.

[2] Research supported by N.S.F. Grant GP 37771.

AMS 1970 *subject classifications.* 28A65, 60610, 62M99.

Key words and phrases. Estimation of the mean, spectral measure, ergodic theorem, weak convergence to Haar measure.

1. Ergodic processes. The first theorem occurs in greater generality in Blum–Eisenberg (1974) and includes results of Hanson–Pledger (1969) (i.e., the equivalence of (1) and (5)).

THEOREM 1A. *The following are equivalent*:

(1) $N^{-1}\sum_{k=1}^{N} x_{n_k} \to_{L^2} m$ *for every stationary ergodic process* x_n.

(2) $\int_0^{1-} [|\sum_{k=1}^{N} \exp(2\pi i\lambda n_k)|^2/N^2]\, dF(\lambda) \to 0$ *for all spectral measures with* $dF(0) = 0$.

(3) $N^{-1}\sum_{k=1}^{N} \exp(2\pi i\lambda n_k) \to 0$ *for all nonintegral* λ.

(4) *For irrational* λ, $N^{-1}\sum_{k=1}^{N} \exp(2\pi i\lambda n_k m) \to 0$ *for* $m = 1, 2, 3, \cdots$, *and for integral* p

$$\frac{1}{N}\sum_{k=1}^{N} \exp(2\pi i n_k m/p) \to 0 \qquad \textit{for } m = 1, 2, \cdots, p-1 .$$

(5) λn_k *is uniformly distributed modulo* 1 *for irrational* λ *and* $n_k \bmod p$ *is uniformly distributed on* 0 *to* $p - 1$ *for integral* p.

PROOF. (1) $\leftrightarrow$ (2). The map $x_n - m$ to $\exp(2\pi i\lambda n)$ in $L^2(dF)$ is an isometry.

(2) $\to$ (3). Let dF be a point mass at λ mod 1.

(3) $\to$ (2). Dominated convergence utilizing $dF(0) = 0$.

(3) $\to$ (4). For the first part, substitute λm for λ and in the second part substitute m/p for λ, $m = 1, \cdots, p - 1$.

(4) $\to$ (3). Let m vary from 1 to $p - 1$.

(4) $\leftrightarrow$ (5). The Lévy continuity theorem for probability measures on the unit circle T states that $\mu_n \to \mu$ weakly if and only if the Fourier coefficients $\hat{\mu}_N(m) = \int_T \exp(2\pi imt) d_{\mu_N}(\exp(2\pi it))$ converge pointwise to $\hat{\mu}(m)$ for all integers m (Billingsley (1968) page 50). For the first part of the equivalence of (4) and (5) let μ_N be the uniform measure on the points $\exp(2\pi i\lambda n_k)$, $k = 1, \cdots, N$. Then (4) says $\hat{\mu}_N(m) \to 0$ for $m \neq 0$. That is, $\hat{\mu}_N(m) \to \hat{\mu}(m)$, where μ is normalized Lebesgue measure on the circle. This holds if and only if $\mu_N \to \mu$ weakly which is true if and only if λn_k is uniformly distributed mod 1.

For the second part of the equivalence of (4) and (5) let μ_N be the uniform measure on $\exp(2\pi i n_k/p)$, $k = 1, \cdots, N$ and note that (4) says μ_N converges weakly to the uniform measure on $\exp(2\pi ij/p)$, $j = 0, \cdots, p - 1$. □

Condition (3) is interesting in that it shows that for processes with discrete spectral measure and almost periodic sample paths there will be convergence almost surely in (1). It is striking that this condition is sufficient for (1) to hold for processes with continuous spectral measure.

Uniform distribution mod 1 has been widely studied. Condition (5) of the theorem is a condition of simultaneous uniform distribution for different sequences and will rarely be satisfied.

Examples of admissible sequences are 1, 2, 3, $\cdots$ as well as 1, $\underline{10}$, $\underline{11}$, $\underline{101}$, 102, 103, $\cdots$ consisting of longer and longer strings of consecutive integers arbitrarily spaced, and 1, 2, 4, 5, 7, 9, $\cdots$ consisting of an odd number followed by two consecutive evens, three consecutive odds, etc. All can be checked using (3).

Examples of sequences which do not work are $n_k = 2k$ and $n_k = 3^k$ as can be verified by (5).

An analogue of this theorem for continuous parameter processes is stated here without proof. In the continuous case we assume x_t has covariance $R(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, dF(\lambda)$.

THEOREM 1B. *The following are equivalent*:

(1) $N^{-1} \sum_{k=1}^{N} X_{t_k} \to m$ *for every stationary ergodic process.*

(2) $N^{-2} \int_{-\infty}^{\infty} |\sum_{k=1}^{N} \exp(i\lambda t_k)|^2\, dF(\lambda) \to 0$ *for every spectral measure with* $dF(0) = 0$.

(3) $N^{-1} \sum_{k=1}^{N} \exp(i\lambda t_k) \to 0$ *for all* $\lambda \neq 0$.

(4) $N^{-1} \sum_{k=1}^{N} \exp(i\lambda t_k m) \to 0$ *for all* $\lambda \neq 0$, $m = 1, 2, \cdots$

(5) λt_k *is uniformly distributed modulo* 1 *for all* $\lambda \neq 0$.

Obviously no sequence of integers satisfies (5) when $\lambda = 1$. However, a theorem of Fejer gives the existence of many admissible sequences t_k. Namely, if $f(x)$ is a differentiable function such that $f'(x) \downarrow 0$ but $\lim_{x\to\infty} x|f'(x)| = \infty$ then the sequence $t_k = f(k)$ is uniformly distributed mod 1 (Kuipers and Niederreiter (1974) page 14).

But if f satisfies the hypotheses of Fejer's theorem so does $g(x) = \lambda f(x)$, $\lambda > 0$. Hence $g(k) = \lambda f(k)$ is uniformly distributed mod 1 for any $\lambda > 0$ and it is also seen (5) holds for $\lambda < 0$ as well. Hence $f(k)$ satisfies (5). An example of such an admissible sequence is $t_k = \sqrt{k}$.

At this point we should mention that in a recent paper Niederreiter (1973) has shown that integer sequences n_k satisfying Theorem 1A, part (5) can be gotten from sequences t_k satisfying Theorem 1B, part (5) by taking $n_k = [t_k]$, the greatest integer less than t_k.

We now consider some results for random sampling schemes for a continuous time process.

Let $y_n = x_{n\alpha}$ for some constant α. Then

$$\frac{1}{N} \sum_{n=1}^{N} y_n - m = \int_{-\infty}^{\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(i\lambda n\alpha)\, dZ(\lambda) \to \sum_{k=-\infty}^{\infty} dZ\left(\frac{2\pi k}{\alpha}\right)$$

and $E((N^{-1} \sum_{n=1}^{N} y_n - m)^2) \to \sum_{k=-\infty}^{\infty} dF(2\pi k/\alpha)$. But if α is chosen at random with continuous probability distribution dP then writing $y_n(\omega, \omega') = x_{n\alpha(\omega)}(\omega')$,

$$E\left(\left(\frac{1}{N} \sum_{n=1}^{N} y_n - m\right)^2\right) \to \int \sum_{k=-\infty}^{\infty} dF\left(\frac{2\pi k}{\alpha}\right) dP(\alpha) = 0$$

since $\sum dF(2\pi k/\alpha) = 0$ except for at most countably many α. In this case there is a set A of ω's with $P(A) = 1$ for which there is convergence in $L^2(\omega')$. But this set A is not universal in that it depends on the spectral measure F of the x_t process.

A more complicated case of random sampling is where $y_n(\omega, \omega') = x((T_1 + \cdots + T_n)(\omega), \omega')$, where we write $x(t, \omega)$ for $x_t(\omega)$ and where T_i are independent identically distributed random variables with characteristic function $\varphi(\lambda)$ satisfying $|\varphi(\lambda)| < 1$ for $\lambda \neq 0$. That is, the T_i do not have a lattice distribution. Without loss of generality assume $n < m$. Then

$$\begin{aligned} E(y_n y_m) &= E_\omega E_{\omega'}(x(T_1 + \cdots + T_n)(\omega), \omega')x((T_1 + \cdots + T_m(\omega), \omega')) \\ &= E_\omega R((T_{n+1} + \cdots + T_m)(\omega)) \\ &= E_\omega(\textstyle\int e^{i\lambda}(T_{n+1} + \cdots + T_m)\, dF(\lambda)) = \int \varphi(\lambda)^{m-n}\, dF(\lambda)\,. \end{aligned}$$

Hence $E((N^{-1} \sum_{n=1}^N y_n)^2) = \int N^{-2} \sum_{n=1}^N \sum_{m=1}^N \varphi^{|m-n|}(\lambda)\, dF(\lambda)$. But for any a with $|a| < 1$,

$$\begin{aligned} \left|\frac{1}{N^2} \sum_{m=1}^N \sum_{n=1}^N a^{|m-n|}\right| &= \left|\frac{1}{N} + \frac{2}{N} \sum_{k=1}^N \frac{N-k}{N} a^k\right| \\ &\leqq \frac{1}{N} + \frac{2}{(1-|a|)N} \to 0\,. \end{aligned}$$

Substituting $\varphi(\lambda)$ for a we have $E(N^{-1} \sum y_n)^2 \to 0$. Again this does not say that $N^{-1}\sum_{n=1}^N \exp(i\lambda(T_1 + \cdots + T_n)(\omega)) \to 0$ for any fixed $\lambda \neq 0$. However, Robbins (1953) has shown that for each $\lambda \neq 0$ there is convergence to 0 a.s. and Blum and Cogburn in some unpublished work have shown there is simultaneous convergence for all $\lambda \neq 0$ with probability 1. That is $t_m = T_1 + \cdots + T_m$ is admissible with probability 1.

If it is also assumed that dF has compact support then there is no problem in estimating the mean of a continuous time process by discrete observations. Namely, if x_t has spectral measure dF with support $[-T, T]$ then $N^{-1} \sum_{n=1}^N x_{n\alpha} \to m$ for every $0 < \alpha < 2\pi/T$. This follows because if $\alpha < 2\pi/T$ then $|\alpha\lambda| < 2\pi$ for $|\lambda| \leqq T$. Hence $N^{-1} \sum_{n=1}^N \exp(i\lambda n\alpha) \to 0$ for $\lambda \neq 0$ in $[-T, T]$. If $\alpha = 2\pi/T$ and $\lambda = T$ there is no such convergence. In fact, if dF consists of point masses at $\pm T$ then $R(t) = \cos tT$ and $R(2\pi n/T) = 1$. Hence $x_{2\pi n/T} = x_0$.

Another important class of processes are the continuous time periodic processes $x_t = \sum_{n\neq 0} Z_n \exp(2\pi int) + m$.

This time the theorem takes the form:

THEOREM 1C. *The following are equivalent*:

(1) $N^{-1} \sum_{k=1}^N x_{t_k} \to m$ *for every stationary ergodic process with period* 1.

(2) $\sum_{n\neq 0} N^{-2} |\sum_{k=1}^N \exp(2\pi int_k)|^2 c_n \to 0$ *for all* $c_n > 0$, $\sum c_n < \infty$.

(3) $N^{-1} \sum \exp(2\pi int_k) \to 0$ *all* $n \neq 0$.

(4) t_k *modulo one are uniformly distributed on* $[0, 1)$.

Examples of such sequences are t_k of the form $k\alpha$, where α is irrational.

More generally we have that the orbit $T^k s$ of almost all points s in [0, 1] under an ergodic measure preserving transformation T is uniformly distributed. Namely, let $f_n(u) = \exp(2\pi inu)$. Then by the pointwise ergodic theorem

$$\frac{1}{N}\sum_{k=1}^N f_n(T^k s) = \frac{1}{N}\sum_{k=1}^N \exp(2\pi inT^k s) \to 0 \quad \text{a.s.}$$

Thus for almost all s this is true simultaneously for $n = 1, 2, 3, \cdots$. By Theorem 1C, $t_k = T^k s$ for these s satisfy (1). Note that to say a sequence is uniformly distributed in a space with probability one is stronger than saying it visits any particular set A with the correct frequency with probability one.

A very important use of the estimate of m using observations X_{n_k} for an admissible sequence n_k is in the prediction of the average value of other observations X_n. If this is the purpose of the estimate it is not necessary to assume the process is ergodic, but merely stationary. By the spectral representaion $X_n = \int_0^{1-} \exp(2\pi i\lambda n)\, dZ(\lambda)$ and we see that $X_n - dZ(0)$ is an ergodic stationary process with mean 0. In this case $N^{-1}\sum_{k=1}^N X_{n_k}$ will converge in mean square to $dZ(0)$ rather than m, but this does no harm since $N^{-1}\sum_{n=1}^N X_n$ also converges in mean square to $dZ(0)$.

2. Weakly mixing processes. As there is essentially no difference in the mathematics we consider only discrete time processes. The obvious analogue of Theorem 1 is

THEOREM 2. *$N^{-1}\sum_{k=1}^N x_{n_k} \to m$ in L^2 for all weakly mixing processes if and only if $\int_0^{1-} N^{-2}\,|\sum_{k=1}^N \exp(2\pi i\lambda n_k)|^2\, dF(\lambda) \to 0$ for all continuous spectral functions $F(\lambda)$.*

It is thus sufficient that $N^{-1}\sum_{k=1}^N \exp(2\pi i\lambda n_k) \to 0$ except for countably many λ. This permits lattice sequences $n_k = 2k$, for example, to be used.

On the other hand the sequence $n_k = 3^k$ is still inadmissible. By the Schwarz inequality, it is necessary that $\int_0^1 N^{-1}\sum_{k=1}^N \exp(2\lambda i\lambda n_k)\, dF(\lambda) \to 0$ for all continuous spectral measures $dF(\lambda)$. In particular let $F(\lambda)$ be the Cantor function. λ of the form $\sum_1^\infty \omega_n 3^{-n}$, $\omega_n = 0$ or 2 form the Cantor set and the measure given by the Cantor function corresponds to making the ω_n independent with $P(\omega_n = 0) = \frac{1}{2}$. Thus

$$\int_0^1 \frac{1}{N}\sum_{k=1}^N \exp(2\pi i\lambda 3^k)\, dF(\lambda) = \frac{1}{N}\sum_{k=1}^N E\exp(2\pi i 3^k\lambda)$$

where E stands for expectation over the probability space of the ω_n's. But $\exp(2\pi i 3^k\lambda)$ all have the same distribution over that probability space so that

$$\frac{1}{N}\sum_{k=1}^N E\exp(2\pi i 3^k\lambda) = E\exp(2\pi i\lambda) = \prod_{k=1}^\infty \left(\frac{\exp(2\pi i 2\cdot 3^{-k}) + 1}{2}\right)$$

which is seen to be nonzero.

Hanson and Pledger show it is sufficient that for all sets S of density 0, $\mu_N(S) \to 0$, where μ_N assigns measure N^{-1} to each of $n_1, \cdots, n_N$. It is not easy to derive this result even using Theorem 2. We merely note here a corollary to their result that any sequence with positive lower density is admissible for weakly mixing processes.

Examples of admissible sequences are certain additional random sequences. Let α be irrational and let n_k be the kth integer j such that $\exp(2\pi ij\alpha)$ is in a fixed set A in the unit circle D, i.e., the kth entry time of $\exp(2\pi ij\alpha)$ into A. The Lebesgue measure of A is assumed to be positive but the Lebesgue measure of ∂A is assumed to equal zero. More complicated but similar sequences were studied by Brunel and Keane (1969). We give a simple self-contained proof that such entry time sequences are admissible for weakly mixing processes.

LEMMA. *If α and β are rationally independent, i.e., there are no nonintegers m and n for which $\alpha m + \beta n$ is an integer, then if n_k is the kth entry of $\exp(2\pi im\alpha)$ into A as above, then $N^{-1}\sum_{k=1}^{N}\exp(2\pi i\beta n_k) \to 0$.*

PROOF. Let μ_N be the uniform measure on the points $(\exp(2\pi i\alpha j), (\exp(2\pi i\beta j))$, $j = 1, \cdots, N$, in D^2. Then

$$\int_D \int_D \exp(2\pi i(mr + ns))\, d\mu_N(e^{2\pi ir}, e^{2\pi is}) = \frac{1}{N}\sum_{j=1}^{N}\exp(2\pi ij(m\alpha + n\beta)) \to 0$$

for m and n nonzero since $m\alpha + n\beta$ are irrational. By the Lévy continuity theorem for probability measures on D^2 this implies μ_N converges weakly to Haar measure on D^2. The function $C_A(\exp(2\pi ir))\exp(2\pi is)$ is bounded and continuous a.e. on D^2, where C_A is the characteristic function of A. Hence

$$\int\int C_A(\exp(2\pi ir))\exp(2\pi is)\, d\mu_N(e^{2\pi ir}, e^{2\pi is}) \to \int_0^1\int_0^1 C_A(\exp(2\pi ir))\exp(2\pi is)\, dr\, ds = 0\,.$$

But the left side is just $N^{-1}\sum_{n_k \leq N}\exp(2\pi in_k\beta)$, where n_k is the kth entry into A. The lemma says $N^{-1}\sum_{k=1}^{N}\exp(2\pi in_k\beta) \to 0$. To prove this let ν_N be the measure giving measure $1/N$ to $\exp(2\pi i\alpha j)$ for $j = 1, \cdots, N$. Then as above ν_N converges weakly to Haar measure on D and $\int_T C_A(\exp(2\pi i\alpha r))\, d\nu_N(e^{2\pi ir}) = N^{-1}$ (number of entries for $j \leqq N$) $\to$ Haar measure of $A > 0$. Hence $N^{-1}\sum_{k=1}^{N}\exp(2\pi in_k\beta)$ must also approach 0. □

COROLLARY. *The sequence of entry times of $\exp(2\pi i\alpha n)$ into A as above is admissible for weakly mixing processes.*

PROOF. $N^{-1}\sum_{k=1}^{N}\exp(2\pi i\beta n_k) \to 0$ except for β such that $\alpha m + \beta n$ is an integer for m and n not both zero. There are at most countably many such β. Hence

$$\int_0^{1-}\frac{1}{N^2}\left|\sum_{k=1}^{N}\exp(2\pi i\lambda n_k)\right|^2 dF(\lambda) \to 0$$

for continuous measures dF.

3. Strongly mixing processes. In Blum–Hanson (1961) it is shown that a process is strongly mixing if and only if all subsequences are admissible for it. In this section we extend this result to continuous time processes.

THEOREM 3. *Let μ_n be a sequence of probability measures on the real line such that $\mu_n(I) \to 0$ uniformly over all intervals I of fixed length. Then if X_t is a strongly mixing process $\int X_t\, d\mu_n(t) \to m$ in L^2.*

PROOF. $E((\int X_t\, d\mu_n(t) - m)^2) = \int\int R(s-t)\, d\mu_n(s)\, d\mu_n(t)$. Since X_t is strongly mixing, for all $\varepsilon > 0$ there is an interval I such that for $u \notin I$, $|R(u)| < \varepsilon/2$.

$$|\int\int R(s-t)\, d\mu_n(s)\, d\mu_n(t)| \leqq \frac{\varepsilon}{2} + |\int\int_{s-t\in I} R(s-t)\, d\mu_n(s)\, d\mu_n(t)|$$

$$\leqq \sup_t \mu_n(I+t) + \frac{\varepsilon}{2}.$$

Since $\sup_t \mu_n(I+t) \to 0$, for n large enough

$$|\int\int R(s-t)\, d\mu_n(s)\, d\mu_n(t)| < \varepsilon. \qquad \square$$

Examples of such measures μ_n satisfying the hypotheses are μ_n of the form $\mu_n(A) = \lambda(A \cap E_n)/\lambda(E_n)$, where λ is Lebesgue measure and E_n are any sets with $\lambda(E_n) \to \infty$. That is, $\sup_{|I|\leqq L} \mu_n(I) \leqq L/\lambda(E_n) \to 0$. This corresponds to sampling over the sets E_n.

For discrete time processes the Blum–Hanson result says that if the law of large numbers holds for a process for all subsequences then the process is strongly mixing. We now show that this converse is true for $\mu_n(A) = \lambda(A \cap E_n)/\lambda(E_n)$, where λ is Lebesgue measure.

THEOREM 3A. *If $\lambda(E_n)^{-1} \int_{E_n} X_t\, dt \to m$ for all E_n with $\lambda(E_n) \to \infty$, where λ is Lebesgue measure, then X_t is strongly mixing.*

PROOF. If $\lambda(E_n)^{-1} \int_{E_n} X_t\, dt \to m$ then

$$\frac{1}{\lambda(E_n)} \int_{E_n} (X_t - m)(X_0 - m)\, dt = \frac{1}{\lambda(E_n)} \int_{E_n} R(t)\, dt \to 0.$$

If X_t is not strongly mixing, there is an $\varepsilon > 0$ with $\limsup |R(t)| > 2\varepsilon$. Thus there is a sequence t_n with $R(t_n) > \varepsilon$. But $R(t)$ is uniformly continuous. Hence there is a δ with $R(t) > \varepsilon$ on the set $\bigcup_{k=1}^{\infty} (t_n - \delta, t_n + \delta)$. Let $E_n = \bigcup_{k=1}^{n} (t_n - \delta, t_n + \delta)$. Then $\lambda(E_n) \to \infty$. Yet $|\lambda(E_n)^{-1} \int_{A_n} R(t)\, dt| > \varepsilon$. This is a contradiction. $\square$

We also have

THEOREM 3B. *If $N^{-1} \sum_{k=1}^{N} X_{n_k} \to m$ for all sequences of integers n_k then X_t is weakly mixing.*

PROOF. As before $N^{-1} \sum_{k=1}^{N} R(n_k) \to 0$ for every subsequence n_k. By letting n_k first be the subsequence where $R < 0$ and then the subsequence where $R > 0$ it follows that $N^{-1} \sum_{n=1}^{N} |R(n)| \to 0$. By Wiener's theorem $R(n) = \int_0^{2\pi} e^{i\lambda n}\, dF(\lambda)$, where F is continuous. But $dF(\lambda) = \sum_{k=-\infty}^{\infty} dG(\lambda + 2\pi k)$, where $R(t) = \int e^{i\lambda t}\, dG(\lambda)$. Hence G must be continuous. $\square$

More can be said if X_t is assumed to be bandlimited. If X_t has spectral measure dF with support $[-\pi, \pi]$ and if $N^{-1} \sum_{k=1}^{N} X_{n_k} \to m$ for all subsequences n_k then X_t is strongly mixing. This follows from the Blum–Hanson theorem and the fact in Salem (1963), that if $\int_{-\pi}^{\pi} e^{i\lambda n}\, dF(\lambda) \to 0$ as $n \to \infty$ through the integers then $\int_{-\pi}^{\pi} e^{i\lambda t}\, dF(\lambda) \to 0$ as $t \to \infty$.

We have not been able to construct a process which is not strong mixing but for which all subsequences of the integers are admissible.

REFERENCES

[1] BILLINGSLEY, PATRICK (1968). *Convergence of Probability Measures.* Wiley, New York.

[2] BLUM, JULIUS and EISENBERG, BENNETT (1974). Generalized summing sequences and the mean ergodic theorem. *Proc. Amer. Math. Soc.* **42** 423–429.

[3] BLUM, J. R. and HANSON, D. (1960). On the mean ergodic theorem for subsequences. *Bull. Amer. Math. Soc.* **66** 308–311.

[4] BRUNEL, A. and KEANE, M. (1969). Ergodic theorems for operator sequences. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **12** 231–240.

[5] HANSON, D. L. and PLEDGER, GORDON (1969). On the mean ergodic theorem for weighted averages. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **13** 141–149.

[6] KUIPERS, L. and NIEDERREITER, H. (1974). *Uniform Distribution of Sequences.* Wiley-Interscience, New York.

[7] NIEDERREITER, H. (1973). On a paper of Blum, Eisenberg and Hahn concerning ergodic theory and the distribution of sequences in the Bohr group. To appear.

[8] ROBBINS, H. (1953). On the equidistribution of sums of independent random variables. *Proc. Amer. Math. Soc.* 786–799.

[9] SALEM, RAPHAEL (1963). *Algebraic Numbers and Fourier Analysis.* Heath Mathematical Monographs, D. C. Heath and Company, Boston.

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