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TWO DIMENSIONAL SEMI-STABLE MARKOV PROCESSES

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A Markov Process is called semi-stable if it is invariant under certain dilation of time and space. The particular case of semi-stable continuous branching Markov Processes is studied.

1. Introduction. One-dimensional continuous branching processes are identical with the class of limits of discrete branching processes as shown in J. Lamperti [7]. Further, in [3], he also showed that limits of such processes are necessarily semi-stable ($\{x_t\}$ is semi-stable in the sense of [3] means that the random variables $\{x_t\}$ and $\{x_{at}\}$ have the same distribution except for a dilation of scale of state space). In that sense, limits of discrete branching processes with one type are essentially characterized. It is very plausible that the results of [7] can be extended to higher dimensions which would essentially characterize multitype Galton Watson branching processes. If that is the case, then the limit is necessarily semi-stable by [3]. A systematic study of semi-stable Markov processes on the nonnegative real line and their characterization was obtained in [10]. The main result was extended to higher dimensions by the author [1].

In this paper, semi-stable continuous branching Markov processes in R^2 are characterized by using Watanabe's result [13]. The generalization to R^n is routine. It turns out that there is no semi-stable continuous branching process of order less than 1, while all of them are direct products of independent processes. In the case of order 1, the components themselves are diffusion processes with generator $cx(d^2f/dx^2)$. The only one-dimensional semi-stable continuous branching Markov process of order greater than 1 has generator of the form

$$Bf(x) = cx \int_0^\infty \left[f(x+y) - f(x) - \xi(y)\frac{df}{dx}\right] y^{-2-1/\alpha}\, dy + dx\frac{df}{dx}$$

where $\xi \in C_0^2(R^+)$, $\xi(x) = x$ for $|x| \leqq \beta$, β some positive number and

$$d = -c\alpha\beta^{-1/\alpha} + c\int_\beta^\infty \xi(y) y^{-2-1/\alpha}\, dy\,.$$

This is a time-change of a stable process as pointed out in [14].

It is also interesting to note that one-dimensional semi-stable continuous branching processes of order 1 have $\exp[\lambda/(1+\beta+t\lambda)]$ as their Laplace transform for some β while those of order $\alpha > 1$ have $\exp\{\lambda/[1+\beta\lambda^{1/\alpha}]^\alpha\}$ as their Laplace transform. These were obtained in [7].

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Let D be the positive quadrant of R^2 with Δ as the point of infinity. Then $D \cup \{\Delta\}$ is compact.

DEFINITION. A Markov process $\{x_t\}$ on $D \cup \{\Delta\}$ having Δ as a trap is called a continuous branching process having p_t as a transition function if

(a) For $x, y \in D$, $t \geqq 0$, $\{p_t\}$ satisfies for each $E \subset D$, $p_t(x+y, E) = \int_D p_t(x, E-u)p_t(y, du)$; and

(b) There exists $t > 0$ and $x \in D$, $x \neq (0,0)$ such that

$$p_t(x, (0,0)) < 1 .$$

A continuous branching process (C.B.P) is called semi-stable of order $\alpha > 0$ if

$$p_t(x, E) = p_{at}(a^\alpha x, a^\alpha E) \qquad \text{for all } a > 0 . \tag{0}$$

Further, a C.B.P. on $D \cup \{\Delta\}$ is called regular if $E_{(1,0)}(e^{-\lambda \cdot x_t})$ and $E_{(0,1)}(e^{-\lambda \cdot x_t})$ are right differentiable in t at $t = 0$. Throughout the following discussion, $\{x_t\}$ will be a semi-stable regular C.B.P. of order α. By a theorem due to S. Watanabe [11], any regular C.B.P. is characterized by its generator, $\mathscr{A}$, on $C_0^2(D)$ where $C_0^n(D)$ is the set of all functions whose nth partials are continuous and vanish at Δ and $C_0^\infty = \bigcap C_0^n$. Let $\xi_1, \xi_2 \in C_0^\infty(D)$ be any two functions so that $\xi_1(x) = x_1$, $\xi_2(x) = x_2$ on some neighborhood U of the origin. Then the process is completely characterized by

$$\begin{aligned} \mathscr{A}f(x) = {} & \beta_1^2 x_1 \frac{\partial^2 f}{\partial x_1^2} + \beta_2^2 x_2 \frac{\partial^2 f}{\partial x_2^2} + (ax_1 + bx_2)\frac{\partial f}{\partial x_1} \\ & + (cx_1 + dx_2)\frac{\partial f}{\partial x_2} - (\gamma x_1 + \delta x_2) f(x) \\ & + x_1 \int \left[f(x+y) - f(x) - \xi_1(y)\frac{\partial f}{\partial x_1}\right] \zeta_1(dy) \\ & + x_2 \int \left[f(x+y) - f(x) - \xi_2(y)\frac{\partial f}{\partial x_2}\right] \zeta_2(dy) , \end{aligned} \tag{1}$$

where $x = (x_1, x_2)$, $f \in C_0^2(D)$, $\gamma, \delta, b, c \geqq 0$, and ζ_1, ζ_2 are nonnegative measures on D satisfying

$$\int_U (y_1^2 + y_2)\zeta_1(dy) + \int_U (y_1 + y_2^2)\zeta_2(dy) + (\zeta_1 + \zeta_2)(D - U) < \infty . \tag{2}$$

From the integral, it is obvious that we can assume that the measures have zero mass at the origin. In addition, if $\{x_t\}$ is semi-stable, we can conclude more through the following theorem:

MAIN THEOREM 1. *If $\alpha = 1$, then the generator $\mathscr{A}$ of $\{x_t\}$ on $C_0^2(D)$ has the form*

$$\mathscr{A}f(x) = \beta_1^2 x_1 \frac{\partial^2 f}{\partial x_1^2} + \beta_2^2 x_2 \frac{\partial^2 f}{\partial x_2^2} .$$

If $\alpha \neq 1$, *then* $\alpha > 1$ *and* $\mathscr{A}$ *has the form*

$$\begin{aligned}\mathscr{A}f(x) = c_1 x_1 \int_0^\infty &\left[f(x_1 + y_1, x_2) - f(x_1, x_2) - \xi_1(y_1, 0) \frac{\partial f}{\partial x_1} \right] y_1^{-(1/\alpha)-2}\, dy_1 \\ &+ c_2 x_2 \int_0^\infty \left[f(x_1, x_2 + y_2) - f(x_1, x_2) - \xi_2(0, y_2) \frac{\partial f}{\partial x_2} \right] y_2^{-(1/\alpha)-2}\, dy_2 \\ &+ a x_1 \frac{\partial f}{\partial x_1} + d x_2 \frac{\partial f}{\partial x_2} ,\end{aligned}$$

where

$$a = -C_1 \alpha \beta^{-1/\alpha} + C_1 \int_\beta^\infty \xi_1(y, 0) y^{-(1/\alpha)-2}\, dy \qquad \textit{and}$$

$$d = -C_2 \alpha \beta^{-1/\alpha} + C_2 \int_\beta^\infty \xi_2(0, y) y^{-(1/\alpha)-2}\, dy \qquad \textit{with}$$

$\{x : |x| \leqq \beta\} \subset U$, $x = (x_1, x_2)$. *Further, in either case, it is the direct product of two semi-stable* C.B.P. *on* R^+ *of the same order* α.

2. Proof of the theorem. Before proving the theorem, we need some observations and a lemma. Let T_t and p_t be the semi-group and transition function of $\{x_t\}$. For $f \in C_0^2(D)$, define $f_a \in C_0^2(D)$ by $f_a(x) = f(ax)$, $a > 0$. By calculation, we have

$$\mathscr{A}f(x) = a \mathscr{A} f_{a^{-\alpha}}(a^\alpha x) \qquad \text{for all } a > 0 . \tag{3}$$

Similarly, we have

$$\left.\frac{\partial f_{a^{-\alpha}}}{\partial x_i}\right|_x = a^{-\alpha} \left.\frac{\partial f}{\partial x_i}\right|_{a^{-\alpha}x} \qquad \text{and} \tag{4}$$

$$\left.\frac{\partial^2 f_{a^{-\alpha}}}{\partial x_i^2}\right|_x = a^{-2\alpha} \left.\frac{\partial^2 f}{\partial x_i^2}\right|_{a^{-\alpha}x} . \tag{5}$$

LEMMA 1. *For all* $s > 0$, $E \subset D$, *E measurable*, $i = 1, 2$,

$$\zeta_i(E) = s^{1+\alpha} \zeta_i(s^\alpha E) . \tag{6}$$

PROOF. It is enough to prove that

$$\zeta_i([r_1, r_2] \times [r_3, r_4]) = s^{1+\alpha} \zeta_i([s^\alpha r_1, s^\alpha r_2] \times [s^\alpha r_3, s^\alpha r_4]) \tag{7}$$

where $r_1 \vee r_3 > 0$. Let $x = (x_1, x_2)$ so that $x_1, x_2 > 0$, $A = [r_1, r_2] \times [r_3, r_4]$, $x \notin A + x$ where $A + x = [r_1 + x_1, r_2 + x_1] \times [r_3 + x_2, r_4 + x_2]$, $s > 0$ fixed. Consider a sequence of functions $f^n \in C_0^2(D)$ so that $f^n = 0$ on some neighborhood of x disjoint from $A + x$, $f^n = 1$ on $A + x$, f^n decreases to χ_{A+x}. Then it is obvious that

$$\mathscr{A}f^n(x) = x_1 \int_D f^n(x + y) \zeta_1(dy) + x_2 \int_D f^n(x + y) \zeta_2(dy)$$

which converges monotonically to $x_1 \zeta_1(A) + x_2 \zeta_2(A)$. On the other hand,

$$s \mathscr{A} f^n_{s^{-\alpha}}(s^\alpha x) = s^{1+\alpha}\left[x_1 \int_D f^n(x + s^{-\alpha} y) \zeta_1(dy) + x_2 \int_D f^n(x + s^{-\alpha} y) \zeta_2(dy)\right]$$

converges to

$$\begin{aligned}s^{1+\alpha}&\left[x_1 \int_D \chi_{A+x}(x + s^{-\alpha} y) \zeta_1(dy) + x_2 \int_D \chi_{A+x}(x + s^{-\alpha} y) \zeta_2(dy)\right] \\ &= s^{1+\alpha}[x_1 \zeta_1(s^\alpha A) + x_2 \zeta_2(s^\alpha A)] .\end{aligned}$$

Letting $x_2 \to 0$, we have

$$s^{1+\alpha}\zeta_1(s^\alpha A) = \zeta_1(A) .$$

Thus, by the unique extension of the measure, we have $\zeta_1(E) = s^{1+\alpha}\zeta_1(s^\alpha E)$ for all measurable sets $E \subset D$. Hence we also have $\zeta_2(E) = s^{1+\alpha}\zeta_2(s^\alpha E)$, which completes the proof of the lemma.

PROOF OF MAIN THEOREM 1. First we want to prove that there are only two interesting cases. Let $f \in C_0^2(D)$ be a function such that

$$f(\mathbf{r} + y) = y_1^2 + y_2^2 \qquad \text{for all } y \in V ,$$

where V is a neighborhood of the origin, $\mathbf{r} = (r, r)$, $r > 0$ fixed. Then

$$\mathscr{A}f(\mathbf{r}) = 2\beta_1^2 r + 2\beta_2^2 r + r \textstyle\int_D f(\mathbf{r} + y)(\zeta_1 + \zeta_2)(dy) , \qquad \text{and}$$
$$s\mathscr{A}f_{s^{-\alpha}}(s^\alpha \mathbf{r}) = 2\beta_1^2 s^{1-\alpha} r + 2\beta_2^2 s^{1-\alpha} r + s^{1+\alpha} r \textstyle\int f_{s^{-\alpha}}(s^\alpha \mathbf{r} + y)(\zeta_1 + \zeta_2)(dy)$$
$$= 2s^{1-\alpha} r(\beta_1^2 + \beta_2^2) + r \textstyle\int f(\mathbf{r} + u)(\zeta_1 + \zeta_2)(du) .$$

Since these are equal, we must have either

$$\beta_1^2 + \beta_2^2 = 0 \qquad \text{or} \qquad \alpha = 1 .$$

We handle the case $\alpha = 1$ first and proceed to prove that $\zeta_1 + \zeta_2 = 0$, using the condition (2). For any $0 < a < b$, let

$$W[a, b] = [0, b] \times [0, b] - [0, a) \times [0, a) ;$$
$$W(a, b] = [0, b] \times [0, b] - [0, a] \times [0, a] ;$$
$$W[a, b] = [0, b) \times [0, b) - [0, a) \times [0, a) ; \qquad \text{and}$$
$$W(a, b) = [0, b) \times [0, b) - [0, a] \times [0, a] .$$

It is easy to verify that for $a > 0$, $s > 0$, $i = 1, 2$,

$$\zeta_i(W[a, b]) = s^{1+\alpha}\zeta_i(W[s^\alpha a, s^\alpha b]) , \qquad \text{so that}$$
$$\zeta_i(W[a, b]) = \zeta_i(W(a, b)) .$$

Letting $b > a > 0$, and $s_1^\alpha = b/a$, $s_2^\alpha = a/b$, it is obvious that

$$D - \{0\} = \textstyle\bigcup_{i=1}^\infty W[s_1^{\alpha i}a, s_1^{\alpha i}b) \cup \bigcup_{i=0}^\infty W[s_2^{\alpha i}a, s_2^{\alpha i}b] .$$

But $\zeta_1(W[a, b]) = s_j^{i+\alpha i}\zeta_1(W[s_j^{\alpha i}a, s_j^{\alpha i}b])$.
Then if $\zeta_1(D - \{0\}) > 0$, we would have $\zeta(W[s_j^{\alpha i}a, s_j^{\alpha i}b]) > 0$ for all i, j because each one is simply a scalar multiple of the other. Now if we choose $b > a > 0$ so that $W[a, b] \subset U$, we will have for all $i = 1, 2, \cdots$

$$W[s_2^{\alpha i}a, s_2^{\alpha i}b] \subset U .$$

The given condition on ζ_1 and ζ_2 requires that

$$\textstyle\int_U (y_1^2 + y_2)\zeta_1(dy) < \infty .$$

But

$$\int_U (y_1^2 + y_2)\zeta_1(dy) \geqq \sum_{i=0}^{\infty} \int_{W[s_2^{\alpha i}a, s_2^{\alpha i}b]} (s_2^{\alpha i}a)^2 \zeta_1(dy)$$
$$= \sum_{i=0}^{\infty} s_2^{2\alpha i} a^2 (s_2^{i})^{-\alpha-1} \zeta_1(W[a, b])$$
$$= \sum_{i=0}^{\infty} a^2 \zeta_1(W[a, b]) \qquad (\alpha = 1)$$
$$= \infty$$

which is a contradiction. So we must have $\zeta_1(D - \{0\}) = 0$.

Similarly, we must have $\zeta_2(D - \{0\}) = 0$. This would imply that the generator is of the form

$$\mathscr{A}f(x) = \beta_1^2 x_1 \frac{\partial^2 f}{\partial x_1^2} + \beta_2^2 x_2 \frac{\partial^2 f}{\partial x_2^2} + (ax_1 + bx_2)\frac{\partial f}{\partial x_1} + (cx_1 + dx_2)\frac{\partial f}{\partial x_2}$$
$$- (\gamma x_1 + \delta x_2) f(x) .$$

Next we are going to prove that all terms involving first and zero derivatives vanish. To see this, consider $f \in C_0^2(D)$ so that $f(x) = 1$ for all x in a neighborhood of $\mathbf{1} = (1, 1)$. Then we have

$$\mathscr{A}f(\mathbf{1}) = -(\gamma + \delta) \qquad \text{and}$$
$$s\mathscr{A}f_{s^{-\alpha}}(s^{\alpha}\mathbf{1}) = s^2(-\gamma - \delta) .$$

Equating them, we must have $\gamma = \delta = 0$.

The proof of $a = b = 0$ goes similarly by choosing an f so that $\partial^2 f/\partial x_1^2 = 0$ and $\partial f/\partial x_2 = 0$ at a point, but $\partial f/\partial x_1 \neq 0$. This sort of proof also shows that $c = d = 0$. Therefore, we arrive at the conclusion that

$$\mathscr{A}f(x) = \beta_1^2 x_1 \frac{\partial^2 f}{\partial x_1^2} + \beta_2^2 x_2 \frac{\partial^2 f}{\partial x_2^2} , \qquad \text{when } \alpha = 1 .$$

Next, we are going to handle the case where $\alpha \neq 1$. This would mean that $\beta_1^2 + \beta_2^2 = 0$. The generator then can be expressed as

$$(ax_1 + bx_2)\frac{\partial f}{\partial x_1} + (cx_1 + dx_2)\frac{\partial f}{\partial x_2} - (\gamma x_1 + \delta x_2) f(x)$$
$$+ x_1 \int_D \left[f(x + y) - f(x) - \xi_1(y)\frac{\partial f(x)}{\partial x_1} \right] \zeta_1(dy)$$
$$+ x_2 \int_D \left[f(x + y) - f(x) - \xi_2(y)\frac{\partial f(x)}{\partial x_2} \right] \zeta_2(dy) .$$

If we consider an $f \in C_0^2(D)$ such that the first partial derivatives vanish at a point but not the function itself, and use our observation, we can conclude that

$$\mathscr{A}f(x) = -(\gamma x_1 + x_2\delta) f(x) + (x_1 + x_2) \int_D [f(x + y) - f(x)](\zeta_1 + \zeta_2)(dy)$$

and

$$s\mathscr{A}f_{s^{-\alpha}}(s^{\alpha}x) = s[-(\gamma s^{\alpha}x_1 + \delta s^{\alpha}x_2) f(x)$$
$$+ s^{\alpha}(x_1 + x_2) \int_D [f(x + y) - f(x)](\zeta_1 + \zeta_2)(ds^{\alpha}y)] .$$

Because of the fact that they are equal and the integrals match up by Lemma

1, we must have

$$(\gamma x_1 + \delta x_2)f(x) = s^{1+\alpha}(\gamma x_1 + \delta x_2)f(x)$$

which holds only if $\gamma = \delta = 0$.

Next, we consider a function which has $\partial f/\partial x_1 = 0$ but $\partial f/\partial x_2 \neq 0$. Using similar arguments, we have

$$(cx_1 + dx_2)\frac{\partial f}{\partial x_2} + x_2 \int_D \left[f(x + y) - f(x) - \xi_2(y)\frac{\partial f}{\partial x_2}\right]\zeta_2(dy)$$
$$= s^{1+\alpha}x_2 \int_D \left[f(x + y) - f(x) - \xi_2(s^\alpha y)s^{-\alpha}\frac{\partial f}{\partial x_2}\right]\zeta_2(ds^\alpha y) + s(cx_1 + dx_2)\frac{\partial f}{\partial x_2} .$$

That gives

$$(1 - s)(cx_1 + dx_2)\frac{\partial f}{\partial x_2} = x_2 \int_D \left[\xi_2(y)\frac{\partial f}{\partial x^2} - \xi_2(s^\alpha y)s^{-\alpha}\frac{\partial f}{\partial x_2}\right]\zeta_2(dy) .$$

Thus

$$(1 - s)(cx_1 + dx_2) = x_2 \int_D [\xi_2(y) - \xi_2(s^\alpha y)s^{-\alpha}]\zeta_2(dy) .$$

By the very fact that the right-hand side does not depend on x_1, we conclude that $c = 0$ and

$$d = \frac{1}{1 - s}\int_D [\xi_2(y) - \xi_2(s^\alpha y)s^{-\alpha}]\zeta_2(dy) \qquad \text{for } s \neq 1 . \tag{8}$$

Similar arguments will lead to the conclusion that $b = 0$ and for $s \neq 1$

$$a = \frac{1}{1 - s}\int_D [\xi_1(y) - \xi_1(s^\alpha y)s^{-\alpha}]\zeta_1(dy) . \tag{9}$$

The next characterization of ζ_1 and ζ_2 depends very much on the fact that $\int_D (y_1^2 + y_2)\zeta_1(dy) < \infty$, and

$$\zeta_1(E) = s^{1+\alpha}\zeta_1(s^\alpha A) \qquad \text{for all } A \subset D .$$

The first step is to prove that ζ_1 concentrates on the x-axis of D. To prove this, let $k > 0$, $I \subset R^+$, and define

$$V_k(I) = \{(x_1, x_2) \in D : x_2 \in I, x_2 \geqq kx_1\} .$$

It is obvious that

$$\zeta_i(V_k(I)) = s^{1+\alpha}\zeta_i(V_k(s^\alpha I)) \qquad \text{for } s > 0 .$$

In particular, for $s^\alpha = a/b$,

$$\begin{aligned}\{(x_1, x_2) \in D : (x_1, x_2) \neq (0, 0), x_2 \geqq kx_1\} &= \textstyle\bigcup_{i=-\infty}^{\infty} V_k((s^{\alpha i}a, s^{\alpha i}b]) , \qquad b > a > 0 ,\\ &= H_k .\end{aligned}$$

Hence, $\zeta_i(H_k) > 0$ iff $\zeta_i(V_k((s^{\alpha i}a, s^{\alpha i}b])) > 0$ where $s^\alpha = a/b$, $i = 1, 2, \cdots, 0, -1, -2, \cdots$. For any $k > 0$, choose $b > a > 0$ so that $V_k(a, b] \subset U$. Thus

$$\begin{aligned}\int (y_1^2 + y_2)\zeta_1(dy) &\geqq \textstyle\sum_{i=0}^\infty (\int s^{\alpha i}a\zeta_1(dy) \text{ over } V_k((s^{\alpha i}a, s^{\alpha i}b])) \qquad \text{where } s^\alpha = a/b\\ &= \textstyle\sum_{i=0}^\infty s^{\alpha i}a(s^i)^{-1-\alpha}\zeta_1(V_k((a, b]))\\ &= \textstyle\sum_{i=0}^\infty (s^{-\alpha})^i a\zeta_1(V_k((a, b])) .\end{aligned}$$

The last quantity is infinite if $\zeta_1(V_k((a, b])) > 0$. But the first quantity is finite, so we have

$$\zeta_1(V_k(a, b]) = 0 \qquad \text{or} \qquad \zeta_1(H_k) = 0 .$$

Take a sequence, k_n, which converges monotonically to zero. Then we have

$$\zeta_1(D - R^+ \times \{0\}) \leqq \sum_{n=0}^{\infty} \zeta_1(H_{k_n}) = 0 .$$

Therefore, ζ_1 has to concentrate on the set $R^+ \times \{0\}$.

Similarly, ζ_2 has to concentrate on $\{0\} \times R^+$. If ζ_1 and ζ_2 are zero measures, then the problem is uninteresting, for it corresponds to a trivial process. We thus examine the case where ζ_1 is not a zero measure.

For the time being, let us consider ζ_1 as a measure on R^+ with zero mass at the origin. Let

$$g(x) = \zeta_1([x, \infty)) \qquad \text{for all } x > 0 ,$$

then $g(x)$ is a non-increasing function on R^+ and, in fact, is monotone, hence differentiable almost everywhere. Let $u > 0$ be a point at which $g'(u)$ exists. Now,

$$g(x) = \zeta_1([x, \infty)) = s^{1+\alpha}\zeta_1([s^\alpha x, \infty)) = s^{1+\alpha}g(s^\alpha x) .$$

Hence,

$$g'(u) = s^{1+2\alpha}g'(s^\alpha u) \qquad \text{for all } s > 0 .$$

Therefore, by choosing s^α properly, we see that $g'(x)$ exists for all $x > 0$. By the fundamental theorem of calculus, we have

$$g(y) - g(x) = \int_x^y g'(t)\,dt \qquad \text{for all } x, y > 0 .$$

Letting $y \to \infty$, we have $-g(x) = \int_x^\infty g'(t)\,dt$. But $g'(x) = x^{-(1/\alpha)-2}g'(1)$ and

$$\zeta_1([x, \infty)) = \int_x^\infty -g'(t)\,dt = -g'(1)\int_x^\infty x^{-(1/\alpha)-2}\,dx .$$

By the condition (2), we have

$$I = \int_U (y_1^2 + y_2)\zeta_1(dy) < \infty$$

where U lies within the unit circle, and

$$\begin{aligned} I &\geqq \int_U y_1^2\zeta_1(dy) \\ &\geqq \int_0^\beta y_1^2(-g'(1)y_1^{-(1/\alpha)-2})\,dy_1 \qquad \text{for some } \beta > 0 \\ &= -g'(1)\int_0^\beta y_1^{-(1/\alpha)}\,dy_1 . \end{aligned}$$

If we assume that $g'(1) \neq 0$, then the last integral is finite iff $\alpha > 1$. The argument holds also for ζ_2 which concentrates on the positive y-axis. Therefore, we have proved that if $\alpha \neq 1$, then ζ_1 and ζ_2 must concentrate on the x- and y-axes, respectively, with

$$\begin{aligned} \zeta_2(0, dx) &= C_2 x^{-2-(1/\alpha)}\,dx \qquad \text{and} \\ \zeta_1(dx, 0) &= C_1 x^{-(1/\alpha)-2}\,dx \end{aligned}$$

for some C_1 and C_2 nonnegative and $\alpha > 1$.

Substituting ζ_1 and ζ_2 into (8) and (9), we have

$$d = \frac{C_2}{1-s} \int_0^\infty [\xi_2(0, y) - \xi_2(0, s^\alpha y)s^{-\alpha}]y^{-(1/\alpha)-2}\,dy \quad \text{and}$$

$$a = \frac{C_1}{1-s} \int_0^\infty [\xi_1(y, 0) - \xi_1(s^\alpha y, 0)s^{-\alpha}]y^{-(1/\alpha)-2}\,dy .$$

Recalling the fact that $\zeta_2(x) = x_2$ for $x \in U$, a neighborhood of the origin, and letting β be a number so that $\xi_2(0, y) = y$ for all $y \leqq \beta$, we get, assuming $s > 1$:

$$\begin{aligned}
d &= \frac{C_2}{1-s}\{\textstyle\int_0^{\beta/s^\alpha} [\xi_2(0, y) - \xi_2(0, s^\alpha y)s^{-\alpha}]y^{-(1/\alpha)-2}\,dy \\
&\quad + \textstyle\int_{\beta/s^\alpha}^\infty [\xi_2(0, y) - \xi_2(0, s^\alpha y)s^{-\alpha}]y^{-(1/\alpha)-2}\,dy\} \\
&= \frac{C_2}{1-s}\{\textstyle\int_{\beta/s^\alpha}^{\beta} \xi_2(0, y)y^{-(1/\alpha)-2}\,dy + \int_\beta^\infty \xi_2(0, y)y^{-(1/\alpha)-2}\,dy \\
&\quad - \textstyle\int_{\beta/s^\alpha}^\infty s^{-\alpha}\xi_2(0, s^\alpha y)y^{-(1/\alpha)-2}\,dy\} \\
&= \frac{C_2}{1-s}(-\alpha)[\beta^{-1/\alpha} - (\beta/s^\alpha)^{-1/\alpha}] + \frac{C_2}{1-s}\textstyle\int_\beta^\infty \xi_2(0, y)y^{-(1/\alpha)-2}\,dy \\
&\quad - \frac{C_2}{1-s}\textstyle\int_\beta^\infty \xi_2(0, y)y^{-(1/\alpha)-2}s\,dy \\
&= -\alpha C_2\beta^{-1/\alpha} + C_2\textstyle\int_\beta^\infty \xi_2(0, y)y^{-(1/\alpha)-2}\,dy .
\end{aligned}$$

Similarly,

$$c = -\alpha C_1\beta^{-1/\alpha} + C_1 \int_\beta^\infty \xi_1(y, 0)y^{-(1/\alpha)-2}\,dy \quad \text{where}$$

$\xi_1(y, 0) = y$ for all $y \leqq \beta$. Therefore, the generator has the desired form and is proved.

We would like to conclude that in both cases the process is, in fact, a direct product of two processes on R^+ of the same order. Let us note that $\mathscr{A}$ can be regarded as the sum of two operators; each contains $C_0^2(R^+)$ as domain and each one is the generator of a process in R^+. Let $\mathscr{A} = \mathscr{A}_1 + \mathscr{A}_2$ so that $\mathscr{A}_i$ only involves x_i. Consider two processes $\{y_t\}$ and $\{z_t\}$, both of which are semi-stable continuous branching process on R^+ of order α with generator $\mathscr{A}_1$ and $\mathscr{A}_2$, respectively. Then (y_t, z_t) will be a semi-stable C.B.P. on D of order α with generator $\mathscr{A}^*$. If f_1 and f_2 are twice differentiable functions on R^+ which vanish at infinity, then

$$(f_1 \cdot f_2)(x_1, x_2) = f_1(x_1) \cdot f_2(x_2) \in C_0^2(D)$$

and

$$(\mathscr{A}^* f_1 \cdot f_2)(x) = f_1(x_1)\mathscr{A}_2 f_2(x_2) + f_2(x_2)\mathscr{A}_1 f_1(x_1) .$$

However,

$$\begin{aligned}
(\mathscr{A} f_1 \cdot f_2)(x) &= \mathscr{A}_1 f_1(x_1) f_2(x_2) + \mathscr{A}_2 f_1(x_1) f_2(x_2) \\
&= f_2(x_2)\mathscr{A}_1 f_1(x) + f_1(x_1)\mathscr{A}_2 f_2(x_2) .
\end{aligned}$$

Hence, $\mathscr{A}^* = \mathscr{A}$ and, by the uniqueness part of Watanabe's theorem, $\{x_t\}$ has to be equal to (y_t, z_t). That is, $\{x_t\}$ is the direct product of two semi-stable C.B.P. on R^+. This completes the proof of the theorem.

REFERENCES

[1] KIU, S. W. (0000). Semi-stable Markov processes in R^n. To appear.
[2] LAMPERTI, J. W. (1962a). A new class of probability theorems. *J. Math. Mech.* **11** 749–772.
[3] LAMPERTI, J. W. (1962b). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 62–78.
[4] LAMPERTI, J. W. (1964). On extreme order statistics. *Ann. Math. Statist.* **35** 1726–1737.
[5] LAMPERTI, J. W. (1966). *Probability*. Benjamin, New York.
[6] LAMPERTI, J. W. (1967a). Limit distributions for branching processes. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press **2** 225–241.
[7] LAMPERTI, J. W. (1967b). The limit of a sequence of branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **7** 271–288.
[8] LAMPERTI, J. W. (1967c). Continuous state branching processes. *Bull. Amer. Math. Soc.* **73** 382–386.
[9] LAMPERTI, J. W. (1967d). On random time substitutions and the Feller property in *Markov Processes and Potential Theory*, ed. J. Chover. Wiley, New York. 87–101.
[10] LAMPERTI, J. W. (1972). Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **22** 205–225.
[11] LOÈVE, M. (1960). *Probability Theory*, (2nd ed.). Van Nostrand, Princeton.
[12] STONE, C. (1963). Zeros of semi-stable processes. *Illinois J. Math.* **7** 631–637.
[13] WATANABE, S. (1969). On two dimensional Markov processes with branching property. *Trans. Amer. Math. Soc.* **136** 447–466.
[14] WENDEL, J. G. (1964). Zero-free intervals of semi-stable Markov processes. *Math. Scand.* **14** 21–35.

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