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AN EXAMPLE IN WHICH STATIONARY STRATEGIES ARE NOT ADEQUATE

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An example is given of a gambling problem such that, on the one hand, for every initial state, there is a strategy which brings one to the goal with arbitrarily high probability and, on the other hand, for some initial state, every stationary strategy reaches the goal with probability zero.

Introduction. In [1, Section 3.9], the question was raised whether there is, for every leavable gambling house Γ and bounded utility u, a stationary family of nearly optimal strategies. Somewhat more precisely, let $U(f)$ be the most that is achievable from an initial state of f, and let $S(f)$ be the most that is achievable with stationary families of strategies. Say that stationary families are *adequate* if $S \equiv U$, and the question is whether stationary families are indeed adequate. For many gambling problems (Γ, u), the question has been answered in the affirmative ([1], [2], [3]). In those cases, stationary families were shown to be not only adequate but even uniformly adequate. Say that the stationary families are *uniformly adequate* for (Γ, u) if, for every $\varepsilon > 0$, there is a stationary family $\bar{\sigma}$ of strategies such that, for every initial state f, $\bar{\sigma}$ achieves at least $U(f) - \varepsilon$. Ornstein [2] has given an example of a gambling problem in which stationary families are not uniformly adequate. It is the purpose of this note to give an example in which stationary families are not even adequate. In Ornstein's example every gamble is discrete. In that example, as for all leavable problems with a bounded u in which all gambles are discrete, stationary families are adequate (Proposition 1, below). It is not surprising, therefore, that, in our example, purely finitely additive gambles play a crucial role. Introduce those finitely additive gambles that are needed for the example, thus.

DEFINITION OF $\delta(f-)$. For any bounded, real-valued function Q defined on a linearly ordered set, F, there is a notion of lim inf on the left of a point of f, namely

$$\liminf_{f'<f} Q = \sup_{f'<f} \inf_{f' \leq g < f} Q(g)\,, \tag{1}$$

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which is meaningful unless f should be the least element of F. A definition of lim sup is obtained from (1) by interchanging "sup" and "inf" throughout.

Plainly, if the set of f' less than f has a largest element, say f^*, then the lim inf of Q on the left at f is simply $Q(f^*)$.

LEMMA 1. *For any linearly ordered set F and any $f \in F$ for which there is an f' with $f' < f$ there is a finitely additive probability measure, defined on all subsets of F—to be designated by $\delta(f-)$—such that, for any bounded, real-valued function Q defined on F,*

$$\liminf_{f'<f} Q \leqq \delta(f-)Q \leqq \limsup_{f'<f} Q \,. \tag{2}$$

PROOF. If the set of f' less than f possesses a largest element, say f^*, then $\delta(f-)$ is simply the dirac-delta measure $\delta(f^*)$ that assigns probability one to the singleton $\{f^*\}$. For the general case, let $\mathscr{F}$ be the field of subsets of F consisting of all finite disjoint unions of intervals. Designate by $\delta(f-)$, that unique probability measure on $\mathscr{F}$ that assigns probability one to every interval $[f', f)$ with $f' < f$. As is well known and easily seen, every probability measure on the subfield $\mathscr{F}$ can be extended to be a gamble defined on all subsets of F. Designate any such extension of $\delta(f-)$ by "$\delta(f-)$", too. That $\delta(f-)$ satisfies (2) is trivial to verify. □

THE EXAMPLE. Let F be the set of ordinals less than or equal to the first uncountable ordinal, Ω, and, for each $f \in F$, $f \geqq 1$, and each real number ε, $0 < \varepsilon < 1$, let

$$\gamma_\varepsilon(f) = (1-\varepsilon)\delta(f-) + \varepsilon\delta(\Omega-) \,. \tag{3}$$

Let Γ be that house based on F which is defined thus. At 0 only the one-point, dirac-delta measure that assigns probability one to the singleton $\{0\}$ is available. That is, $\Gamma(0) = \{\delta(0)\}$. For $f > 0$, $\Gamma(f)$ is the set of all $\gamma_\varepsilon(f)$, $0 < \varepsilon < 1$. For clarity, note that $\Gamma(\Omega)$ is simply $\{\delta(\Omega-)\}$. Let $u(f)$ be 1 or 0 according as f is 0 or greater than 0. That is, the gambler desires to arrive at 0; other fortunes are worthless to him.

Proof that stationary families are inadequate for (Γ, u). As is easily verified by induction, it is possible to go from any f to 0 with arbitrarily high probability, that is, $U(f) = 1$ for all f.

The proof will be complete once it is shown that, for every stationary family of strategies, the probability of reaching 0 from Ω is zero. To this end, associate with each $f > 0$ a number $\varepsilon(f)$, $0 < \varepsilon(f) < 1$, and let $\gamma(f) = \gamma_{\varepsilon(f)}(f)$. That is,

$$\gamma(f) = (1-\varepsilon(f))\delta(f-) + \varepsilon(f)\delta(\Omega-) \,. \tag{4}$$

Complete the definition of γ by letting $\gamma(0) = \delta(0)$. Associated with γ is the stationary family of strategies that prescribes the gamble $\gamma(f)$ whenever the state is f.

The program is to show that, starting at Ω and using this stationary family, the probability of reaching 0 is 0. This will of course complete the proof. The program can be restated, thus. Let Γ^* be that subhouse of Γ which has available at f the gamble $\gamma(f)$ only, and let $U^*(f)$ be the probability, starting at f, of reaching 0 in the house Γ^*. It suffices to show that $U^*(\Omega) = 0$. More will be shown. Namely,

LEMMA 2. *There is a countable ordinal f' such that $U^*(f) = 0$ for all $f \geqq f'$.*

PROOF. For each initial state f, let $Q(f)$ be the probability in Γ^* of taking a monotone path to the goal 0. That is, $Q(0) = 1$; for $f > 0$, $M(f)$ is the set of histories $h = (f_1, f_2, \cdots)$ such that, for some positive integer n,

$$f > f_1 > \cdots > f_n = 0\,; \tag{5}$$

and $Q(f)$ is the probability of the event $M(f)$. As will now be shown,

$$Q(f') \geqq Q(f) \qquad \text{where} \quad f > f'\,, \tag{6}$$

and

$$Q(f) = (1 - \varepsilon(f))Q(f-) \qquad \text{for} \quad f \geqq 1\,, \tag{7}$$

where

$$Q(f-) = \inf_{f' < f} Q(f')\,. \tag{8}$$

An induction on f will establish both (6) and (7), thus. As is evident, (6) and (7) hold for $f = 1$. Suppose they obtain for all f less than some ordinal $g \leqq \Omega$, and compute thus.

$$\begin{aligned} Q(g) &= \textstyle\int_{f<g} Q(f)\, d\gamma(f\,|\,g) \\ &= (1 - \varepsilon(g))\delta(g-)Q \\ &= (1 - \varepsilon(g))Q(g-)\,, \end{aligned} \tag{9}$$

where the first equality holds in virtue of the definition of Q, the second holds for all bounded Q in virtue of the definition of $\gamma(g)$, and the last holds since, in view of the inductive assumption,

$$Q(g-) = \lim_{f' < g} Q(f')\,, \tag{10}$$

so, for $f = g$, equality holds in (2). This completes the proof of both (6) and (7). Since there can be at most a countable number of f at which $Q(f) > 0$ and (7) holds, there is a countable ordinal f' such that

$$Q(f) = 0 \qquad \text{for all} \quad f \geqq f'\,. \tag{11}$$

To complete the proof of Lemma 2, it suffices to show that $Q \geqq U^*$, which incidentally shows that Q is U^*, since the reverse inequality is trivial. Since Q plainly majorizes u, it suffices, in view of [1, Theorem 2.12.1], to verify that

$$\gamma(f)Q \leqq Q(f) \qquad \text{for all} \quad f\,. \tag{12}$$

For this, compute thus.

$$
\begin{aligned}
\gamma(f)Q &= (1 - \varepsilon(f))\delta(f-)Q + \varepsilon(f)\delta(\Omega-)Q \\
&= (1 - \varepsilon(f))\delta(f-)Q \\
&= (1 - \varepsilon(f))Q(f-) \\
&= Q(f)\,,
\end{aligned} \tag{13}
$$

where the respective equalities are justified by (4); (2) and (11); (6) and (8); (7). This completes the proof of Lemma 2.

In view of Lemma 2, no stationary family has any positive probability of reaching 0 from Ω, that is, $S(\Omega) = 0$. Since $U(\Omega) = 1$, stationary families are inadequate.

Of course, if for every f, the dirac-delta measure $\delta(f)$ is added to the gambles available at f, obtaining thereby a slight enlargement of Γ, namely Γ^L, the so-called leavable closure of Γ, U is still identically equal to 1 and no stationary family of strategies is of any use at Ω. Thus, stationary families are inadequate for Γ^L.

In [1] a distinction was made between stationary families of strategies and stationary strategies. As is trivial to verify for the Γ and Γ^L of the example above, nothing can be achieved with stationary strategies that cannot be achieved with stationary families. Hence, even stationary strategies are inadequate.

Conceivably, if F is denumerable, then stationary families are uniformly adequate for leavable houses with a bounded u, but it is unknown whether this hypothesis even implies that stationary families are adequate.

Stationary families are adequate for discrete houses. Let U_m be the utmost that is achievable if gambling cannot continue beyond time m, where m is a finite, positive integer, and let $U_\omega = \lim U_m$.

PROPOSITION 1. *Let Γ be a leavable house in which every gamble is discrete and let u be bounded. Then $U = S = U_\omega$, and, consequently, stationary families are adequate.*

PROOF. As shown in [3], S majorizes U_ω. Hence, $U \geqq S \geqq U_\omega$. Moreover, as is implied by [1, Theorem 2.15.5. g], under the hypothesis of Proposition 1, U_ω is U. This completes the proof.

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