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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR STATIONARY RANDOM FIELDS

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In this paper, the concept of φ-mixing is extended to random fields, and a central limit theorem analogous to Theorem 20.1 of Billingsley (Convergence of Probability Measures, Wiley (1968)) is obtained for stationary, φ-mixing random fields.

1. Introduction. In this paper we extend the concept of φ-mixing to random fields and obtain a functional central limit theorem for such random fields. This theorem may be regarded either as a generalization of Theorem 20.1 of Billingsley (1968) to "multivariate time," or as a generalization of Corollary 1 of Wichura (1969) to dependent random variables. A central limit theorem, in the classical form, for stationary random fields has been obtained by M. Rosenblatt (1970) under hypotheses somewhat weaker than those used here.

Let Z^q denote the set of all q-tuples of integers ($q \geqq 1$, a positive integer). The points in Z^q will be denoted by $\mathbf{m}$, $\mathbf{n}$, etc., or sometime, when necessary, more explicitly by $(m_1, m_2, \cdots, m_q)$, $(n_1, n_2, \cdots, n_q)$ etc. Z^q is partially ordered by stipulating $\mathbf{m} \leqq \mathbf{n}$ iff $m_i \leqq n_i$ for each i, $1 \leqq i \leqq q$. We write $\mathbf{0}$ and $\mathbf{1}$ respectively for points $(0, 0, \cdots, 0)$ and $(1, 1, \cdots, 1)$ in Z^q.

Let $\{\xi_\mathbf{n} : \mathbf{n} \in Z^q\}$ be a random field, i.e., a collection of random variables indexed by time-set Z^q. The random field is said to be stationary if for each finite subset S of Z^q, and each $\mathbf{m} \in Z^q$, the joint distribution of $\{\xi_{\mathbf{n}+\mathbf{m}} : \mathbf{n} \in S\}$ is the same as that of $\{\xi_\mathbf{n} : \mathbf{n} \in S\}$. Here $\mathbf{n} + \mathbf{m}$ is the usual coordinatewise sum.

For each j ($1 \leqq j \leqq q$) and $r \geqq 0$, let $\mathscr{A}^+(j; r)$ be the σ-field generated by $\{\xi_{n_1,n_2,\cdots,n_q} : n_j \geqq r$, other n_i's unrestricted$\}$ and let $\mathscr{A}^-(j; r)$ be the σ-field generated by $\{\xi_{n_1,n_2,\cdots,n_q} : n_j \leqq r$, other n_i's unrestricted$\}$. For $r \geqq 1$, we write $\varphi(j; r) = \sup\{|P(B \mid A) - P(B)| : A \in \mathscr{A}^-(j; 0) \text{ and } B \in \mathscr{A}^+(j; r), P(A) > 0\}$ and $\varphi(r) = \max_{1 \leqq j \leqq q} \varphi(j; r)$. Also set $\varphi(0) = 1$. Clearly $\{\varphi(r)\}$ is a decreasing sequence of real numbers. If $\varphi(r) \to 0$ we say that the random field $\{\xi_\mathbf{n}\}$ is φ-mixing. This is a natural extension to multivariate time parameter of the well-known concept of φ-mixing for sequences of random variables.

The random field $\{\xi_\mathbf{n}\}$ may be defined only for $\mathbf{n} \geqq \mathbf{1}$. In this case we define $\varphi(j; r) = \sup |P(B \mid A) - P(B)|$, where the supremum is taken over all sets A, B such that for some m, A is in the σ-field generated by $\{\xi_{n_1,n_2,\cdots,n_q} : 1 \leqq n_j \leqq m$, other n_i's $\geqq 1\}$ and B in the σ-field generated by $\{\xi_{n_1,n_2,\cdots,n_q} : n_j \geqq m + r$, other n_i's $\geqq 1\}$. Also $\varphi(r) = \max_{1 \leqq j \leqq q} \varphi(j; r)$. Given such a random field with "one-sided" time set, we can construct a new random field with time set all of Z^q

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and with the same finite-dimensional distributions and the same φ-values. This fact may be proved along the same lines as in the case of univariate time. Thus, without loss of generality, we will assume that the random field $\{\xi_{\mathbf{n}}\}$ is defined over all of Z^q.

We suppose throughout that $\{\xi_{\mathbf{n}}\}$ is a stationary, φ-mixing random field with $E\xi_0 = 0$ and $E\xi_0^2 < \infty$. For $\mathbf{n} \geqq \mathbf{1}$, define the partial sum

$$S_{\mathbf{n}} = \sum_{\mathbf{1} \leqq \mathbf{j} \leqq \mathbf{n}} \xi_{\mathbf{j}} .$$

If one of the coordinates of $\mathbf{n}$ is equal to 0 and others are $\geqq 0$, it is convenient to set $S_{\mathbf{n}} = 0$.

Let T be the closed unit interval $[0, 1]$ and T^q the q-fold Cartesian product of T. Let C_q be the set of all continuous functions on T^q with the uniform metric and, as in Bickel and Wichura (1971), let us denote by D_q the Skorohod function space on T^q. All the properties of D_q that we need can be found in Bickel and Wichura (1971). A subset B of T^q is called a block if it is of the form $\prod_{j=1}^q (s_j, t_j]$, $(s_j, t_j]$'s being half-closed subintervals of $[0, 1]$. If $X = \{X(\mathbf{t}) : \mathbf{t} \in T^q\}$ is a stochastic process, then the increment $X(B)$ of X around a block $B = \prod_{j=1}^q (s_j, t_i]$ is given by

$$X(B) = \sum_{\varepsilon_1=0,1} \sum_{\varepsilon_2=0,1} \cdots \sum_{\varepsilon_q=0,1} (-1)^{q-\Sigma\varepsilon_j} \times X(s_1 + \varepsilon_1(t_1 - s_1), s_2 + \varepsilon_2(t_2 - s_2), \cdots, s_q + \varepsilon_q(t_q - s_q)) .$$

On T^q as well as Z^q we use the maximum norm, i.e., if $t = (t_1, t_2, \cdots, t_q) \in T^q$ or $n = (n_1, n_2, \cdots, n_q) \in Z^q$, then $||t|| = \max_{1\leqq j\leqq q} |t_j|$ and $||n|| = \max_{1\leqq j\leqq q} |n_j|$.

The Wiener process $W = \{W(\mathbf{t}) : \mathbf{t} \in T^q\}$ on T^q is characterized by

(a) $P\{W \in C_q\} = 1$,

(b) if $B_1, B_2, \cdots, B_k$ are pairwise disjoint blocks in T^q,

then the increments $W(B_1), W(B_2), \cdots, W(B_k)$ are independent normal random variables with means zero and variances $\lambda(B_1), \lambda(B_2), \cdots, \lambda(B_k)$, λ being the q-dimensional Lebesgue measure T^q.

2. The Central Limit and related theorems. Let us write $r(\mathbf{j}) = E(\xi_0 \xi_{\mathbf{j}})$. If $\mathbf{n} = (n_1, n_2, \cdots, n_q)$, let $|\mathbf{n}|$ stand for the product $n_1 n_2 \cdots n_q$. Define $|\mathbf{t}|$ similarly for $\mathbf{t} \in T^q$. In this paper the limit $\mathbf{n} = (n_1, n_2, \cdots, n_q) \to \infty$ will mean $\min_{1\leqq i\leqq q} n_i \to \infty$.

LEMMA 1. *Let*

$$\sum_{r=1}^{\infty} r^{q-1} \varphi^{\frac{1}{2}}(r) < \infty . \tag{1}$$

Then (2), (3) *and* (4) *below hold.*

$$\sum_{\mathbf{j} \in Z^q} |r(\mathbf{j})| < \infty . \tag{2}$$

$$|\mathbf{n}|^{-1} E(S_{\mathbf{n}}^2) \to \sum_{\mathbf{j} \in Z^q} r(\mathbf{j}) = \sigma^2 , \quad \text{say} ; \qquad as\ \mathbf{n} \to \infty . \tag{3}$$

$$|\mathbf{n}|^{-1} E(S_{\mathbf{n}}^2) \leqq A(q, \varphi) E(\xi_0^2) , \qquad \forall \mathbf{n} \geqq \mathbf{1};\ where \tag{4}$$

$$A(q, \varphi) = 1 + 2q \sum_{r=1}^{\infty} (2r + 1)^{q-1} \varphi^{\frac{1}{2}}(r) .$$

PROOF. Note that there are, at most, $2q(2r+1)^{q-1}$ points $\mathbf{j} \in Z^q$ such that $||\mathbf{j}|| = r$ where r is a positive integer. Using this fact, the proof can be easily completed along the lines of Lemma 3 on page 172 of Billingsley (1968). □

Throughout the rest of this paper, σ^2 will be as defined in (3), and we will assume $\sigma > 0$.

For $\mathbf{t} = (t_1, t_2, \cdots, t_q) \in T^q$ and $\mathbf{n} = (n_1, n_2, \cdots, n_q) \geqq 1$, let $X_{\mathbf{n}}(\mathbf{t}) = (\sigma^2|\mathbf{n}|)^{-\frac{1}{2}} \times S_{[n_1t_1],[n_2t_2],\cdots,[n_qt_q]}$, where $[\cdot]$ is the usual greatest integer function. The stochastic process X_n has sample paths in D_q. The main theorem of this note is:

THEOREM 1. *Let $\{\xi_{\mathbf{n}}\}$ be a stationary, φ-mixing random field with $E(\xi_0) = 0$, $E(\xi_0^2) < \infty$. Suppose (1) holds and σ^2 defined in (3) is > 0. Then the net $\{X_{\mathbf{n}} : \mathbf{n} \geqq \mathbf{1}\}$ of stochastic processes converges weakly, in D_q, to the q-parameter Wiener process.*

The techniques used in the proof of this theorem are, with some exceptions, essentially those used by Billingsley (1968) to prove Theorem 20.1 there. The proof will be carried out in the following series of lemmas.

For each i, $1 \leqq i \leqq q$, let

$$0 = a_1^{(i)} < b_1^{(i)} < a_2^{(i)} < b_2^{(i)} < \cdots < a_{n_i}^{(i)} < b_{n_i}^{(i)} = 1$$

be real numbers. Call a collection of blocks in T^q "strongly separated" if it is of the form

$$\{\prod_{i=1}^q (a_{k_i}^{(i)}, b_{k_i}^{(i)}] : 1 \leqq k_i \leqq n_i, 1 \leqq i \leqq q\},$$

or if it is a subfamily of such a family of blocks.

LEMMA 2. *Let Y be a stochastic process on T^q with sample paths in D_q such that,*

(i) $E(Y(\mathbf{t})) = 0$, $E(Y^2(\mathbf{t})) = |\mathbf{t}|$, $t \in T^q$;

(ii) *Y has continuous sample paths, and*

(iii) *increments of Y around any collection of strongly separated blocks are independent random variables.*

Then Y is the Wiener process on T^q.

PROOF. It suffices to prove that $Y(\mathbf{t})$ is normally distributed for each $\mathbf{t}$; and this can be easily accomplished by induction on q in conjunction with Theorem 19.1 of Billingsley (1968). □

For $x \in D_q$ and $0 < \delta < 1$, define the modulus $w(x; \delta)$ by

$$w(x; \delta) = \sup\{|x(\mathbf{t}) - x(\mathbf{s})| : ||\mathbf{t} - \mathbf{s}|| \leqq \delta\}.$$

LEMMA 3. *Let $\{Y_{\mathbf{n}}\}$ be a net of stochastic processes in D_q such that,*

(i) $EY_{\mathbf{n}}(\mathbf{t}) \to 0$, $EY_{\mathbf{n}}^2(\mathbf{t}) \to |\mathbf{t}|$ *as $n \to \infty$, for each t;*

(ii) *the set $\{Y_{\mathbf{n}}^2(\mathbf{t})\}$ is uniformly integrable for each* $\mathbf{t}$,

(iii) *if $B_1, B_2, \cdots, B_k$ are a collection of strongly separated blocks, then the increments $Y_{\mathbf{n}}(B_1), Y_{\mathbf{n}}(B_2), \cdots, Y_{\mathbf{n}}(B_k)$ are asymptotically independent in the sense that* *if $H_1, H_2, \cdots, H_k$ are arbitrary linear Borel sets, then the difference*

$$P\{Y_{\mathbf{n}}(B_1) \in H_1, Y_{\mathbf{n}}(B_2) \in H_2, \cdots, Y_{\mathbf{n}}(B_k) \in H_k\} \\ - P\{Y_{\mathbf{n}}(B_1) \in H_1\} P\{Y_{\mathbf{n}}(B_2) \in H_2\} \cdots P\{Y_{\mathbf{n}}(B_k) \in H_k\}$$

goes to zero as $\mathbf{n} \to \infty$ and,

(iv) *for each $\varepsilon > 0$, $\eta > 0$, we can find a $\delta > 0$ such that $P\{w(Y_{\mathbf{n}}, \delta) > \varepsilon\} < \eta$ for all sufficiently large $\mathbf{n}$. Then $\{Y_{\mathbf{n}}\}$ converges weakly, in D_q, to the Wiener process.*

PROOF. Using Lemma 2 in this paper, the proof is similar to that of Theorem 19.2 of Billingsley (1968). □

For the net $\{X_{\mathbf{n}}\}$ of stochastic processes in Theorem 1, the conditions (i) and (iii) of Lemma 3 are trivially seen to be satisfied. Using Lemma 4 below and the estimate (4) in Lemma 1, one can repeat the arguments on page 176 of Billingsley (1968) to show that the condition (ii) of Lemma 3 is also satisfied by $\{X_{\mathbf{n}}\}$. It remains, therefore, to prove that $\{X_{\mathbf{n}}\}$ also satisfies the condition (iv) of Lemma 3.

LEMMA 4. *In addition to the conditions of Theorem 1, suppose that $|\xi_0| < C < \infty$. Then, we can find a constant $B = B(q, \varphi)$ depending only on q and the φ-sequence such that, for all $\mathbf{n} \geqq \mathbf{1}$,*

$$ES_{\mathbf{n}}^4 \leqq BC^4|\mathbf{n}|^2 . \tag{5}$$

PROOF. Fix a positive integer k_0 so large that $16\varphi(k_0) < 10^{-8}$ and $10^8 k_0 < 2^{k_0}$. Applying the univariate time, Lemma 4 on page 172 of [2], we can find a constant B^* such that

$$ES^4_{n_1, n_2, \cdots, n_q} \leqq B^* C^4 n_1^2 n_2^2 \cdots n_q^2 \tag{6}$$

for all $n_1 \geqq 1$, and $n_2, n_3, \cdots, n_q \leqq 2^{k_0}$. Assume, without loss of generality, that $B^* \geqq 12A^2(q, \varphi)$ where $A(q, \varphi)$ is defined in Lemma 1.

We now show by induction that (6) holds for all $n_1 \geqq 1$, all n_2 of the form 2^r, $(1 \leqq r < \infty)$, and $n_3, n_4, \cdots, n_q \leqq 2^q$. For this it suffices to show that if (6) holds for some $n_2 \geqq 2^{k_0}$, then it holds for $2n_2$ as well. To alleviate the notation let us take $n_3 = n_4 = \cdots = n_q = 1$. Now write,

$$\begin{aligned} T_1 &= S_{n_1, n_2, 1, \cdots, 1} \\ T_2 &= S_{n_1, 2n_2+k_0, 1, \cdots, 1} - S_{n_1, n_2+k_0, 1, \cdots, 1} \\ R_1 &= S_{n_1, n_2+k_0, 1, \cdots, 1} - S_{n_1, n_2, 1, \cdots, 1} \\ R_2 &= S_{n_1, 2n_2, 1, \cdots, 1} - S_{n_1, 2n_2+k_0, 1, \cdots, 1} . \end{aligned}$$

Thus,

$$S_{n_1, 2n_2, 1, \cdots, 1} = T_1 + T_2 + R_1 + R_2 ,$$

and so by Minkowski's inequality

$$E^{\frac{1}{4}} S^4_{n_1, 2n_2, 1, \cdots, 1} \leqq E^{\frac{1}{4}}(T_1 + T_2)^4 + E^{\frac{1}{4}} R_1^4 + E^{\frac{1}{4}} R_2^4 .$$

Now

$$E^{\frac{1}{4}} R_1^4 = E^{\frac{1}{4}} R_2^4 \leqq [B^* C^4 n_1^2 k_0^2]^{\frac{1}{4}} \leqq 10^{-4} [B^* C^4 n_1^2 n_2^2]^{\frac{1}{4}} .$$

Also,

$$E(T_1 + T_2)^4 = 2ET_1^4 + 4E(T_1^3T_2) + 6E(T_1^2T_2^2) + 4(T_1 T_3^3) .$$

We have, by the inequality (20.23) of Billingsley (1968),

$$E(T_1^3T_3) \leqq 2\varphi^{\frac{3}{4}}(k_0)E(T_1^4) \leqq 10^{-6}B^*C^4n_1^2n_2^2 ;$$
$$E(T_1 T_2^3) \leqq 2\varphi^{\frac{1}{4}}(k_0)E(T_1^4) \leqq 10^{-2}B^*C^4n_1^2n_2^2 ;$$
$$E(T_1^2T_2^2) \leqq E(T_1^2)E(T_2^2) + 2\varphi^{\frac{1}{2}}(k_0)E(T_1^4)$$
$$\leqq A_2(q, \varphi)C^4n_1^2n_2^2 + 10^{-4}B^*C^4n_1^2n_2^2$$

where in the last step we have used Lemma 1. Thus

$$E(T_1^2T_2^2) \leqq [\tfrac{1}{12} + 10^{-4}]B^*C^4n_1^2n_2^2 .$$

Combining all the preceding estimates,

$$E^{\frac{1}{4}}S^4_{n_1,2n_2,1,\cdots,1}$$
$$\leqq [B^*C^4n_1^2n_2^2]^{\frac{1}{4}}\{[2 + \tfrac{1}{2} + 4 \times 10^{-6} + 6 \times 10^{-4} + 4 \times 10^{-2}]^{\frac{1}{4}} + 10^{-2}\}$$
$$\leqq [B^*C^4n_1^2(2n_2)^2]^{\frac{1}{4}} .$$

Thus (6) holds for all $n_1 \geqq 1$, all n_2 of the form $n_2 = 2^r$, $1 \leqq r < \infty$, and $n_3, n_4, \cdots, n_q \leqq 2^{k_0}$. From this one can show that

$$ES^4_{n_1,n_2,\cdots,n_q} \leqq 256B^*C^4n_1^2\, n_2^2 \cdots n_q^2 \tag{7}$$

for all $n_1 \geqq 1$, all $n_2 \geqq 1$, and $n_3, n_4, \cdots, n_q \leqq 2^{k_0}$. To do this, write n_2 as sum of powers of 2; and apply the Minkowski inequality and (7) to the corresponding decomposition of $S_{n_1,n_2,\cdots,n_q}$. The details are straightforward and, therefore, omitted. Continuing in this fashion terminates the proof of the lemma. □

LEMMA 5. *In addition to the conditions of Theorem* 1, *suppose* $|\xi_0| \leqq C < \infty$. *Then the condition* (iv) *of Lemma* 3 *is satisfied by* $\{X_n\}$.

PROOF. The equation (1) and Theorem 1 of Bickel and Wichura (1971) yield a natural extension of Theorem 12.3 of Billingsley (1968) to multivariate time. Using this extension and Lemma 4 above proves this lemma. □

Now to complete the proof of Theorem 1, it remains to remove the boundedness assumption on $\{\xi_0\}$ in Lemma 5. We use a truncation argument to achieve this.

For $C > 0$, let

$$\xi_{\mathbf{n}}^{(C)} = \xi_{\mathbf{n}} \quad \text{if} \quad |\xi_{\mathbf{n}}| \leqq C$$
$$= 0 \quad \text{otherwise.}$$

Write

$$U_n = \textstyle\sum_{1\leqq j\leqq n} \{\xi_j^{(C)} - E\xi_j^{(C)}\} , \quad \text{and}$$
$$V_n = S_n - U_n .$$

Now Lemma 5 applies to the random field $\{\xi_n^{(C)} - E\xi_n^{(C)}\}$. Thus to complete the proof of Theorem 1 it is enough to show the following: given $\varepsilon > 0$, $\eta > 0$, we can find a truncation level $C = C(\varepsilon, \eta)$, such that

$$P\{\max_{1\leq \mathbf{j}\leq \mathbf{n}} |V_{\mathbf{j}}| > \varepsilon\sigma|\mathbf{n}|^{\frac{1}{2}}\} < \eta , \qquad \text{for all } \mathbf{n} . \tag{8}$$

To establish (8) we need a variant of the standard Ottaviani–Skorohod inequality. See e.g. Theorem 2, page 120 of Gikhman and Skorohod (1969). To state this inequality let $\zeta_1, \zeta_2, \cdots, \zeta_n$ be random variables taking values in a normed linear space with norm $||\cdot||$. In our application, this normed linear space will be the Euclidean space $R^d = \{(x_1, \cdots, x_d): x_i\text{'s real}\}$ with the *maximum norm*, i.e., $||(x_1, x_2, \cdots, x_d)|| = \max_{1\leq i\leq d} |x_i|$. Now write $\varphi(1)$ for $\sup\{|P(B \mid A) - P(B)|\}$ where the supremum is taken over all sets A, B such that, for some m, A is in the σ-field generated by $\zeta_1, \zeta_2, \cdots, \zeta_m$ and B is in the σ-field generated by $\zeta_{m+1}, \zeta_{m+2}, \cdots, \zeta_n$.

LEMMA 6. *Suppose $\varphi(1) < \frac{1}{4}$, and let $a > 0$, $0 < \eta < \frac{1}{4}$ be such that*

$$P\{||\textstyle\sum_{i=k+1}^{n} \zeta_i|| \leq a\} > 1 - \eta , \qquad \textit{for all } k = 0, 1, \cdots, n-1 . \tag{9}$$

Then,

$$P\{\max_{1\leq k\leq n} ||\textstyle\sum_{i=1}^{k} \zeta_i|| > 2a\} < 2\eta . \tag{10}$$

PROOF. Straightforward, using φ-mixing instead of independence. □

Now to establish (8), first note that by Lemma 1,

$$\begin{aligned} EV_{\mathbf{n}}^2 &\leq A(q, \varphi)E\{\xi_{\mathbf{n}} - \xi_{\mathbf{n}}^{(C)}\}^2|\mathbf{n}| \\ &\leq A(q, \varphi)|\mathbf{n}| \textstyle\int_{\{|\xi_0|>C\}} \xi_0^2\, dP . \end{aligned} \tag{11}$$

Now make a temporary assumption that $\varphi(1) < \frac{1}{4}$, and fix $\varepsilon > 0$, $0 < \eta < \frac{1}{4}$. Choose C so large that,

$$\varepsilon^{-2}\sigma^{-2}2^{2q}A(q, \varphi) \textstyle\int_{\{|\xi_0|>C\}} \xi_0^2\, dP < \eta 2^{-q} . \tag{12}$$

We will now show that (8) holds for this choice of C (under the additional assumption $\varphi(1) < \frac{1}{4}$). Now by (12),

$$P\{|V_{\mathbf{j}}| > \varepsilon\sigma 2^{-q}|\mathbf{n}|^{\frac{1}{2}}\} \leq \eta 2^{-q} , \qquad \text{if } \mathbf{1} \leq \mathbf{j} \leq \mathbf{n} . \tag{13}$$

Let $m_2, m_3, \cdots, m_q$ be fixed but arbitrary integers such that $1 \leq m_i \leq n_i$, for $i = 2, 3, \cdots, q$. Here $n = (n_1, n_2, \cdots, n_q)$ so that n_i is the ith coordinate of n. Let $m_2, m_3, \cdots, m_q$ be fixed but arbitrary integers between 1 and n. Applying Lemma 6 to the n_1 random variables

$$\{V_{j,m_2,\cdots,m_q} - V_{j-1,m_2,\cdots,m_q} : 1 \leq j \leq n_1\}$$

we obtain from (13),

$$P\{\max_{1\leq j_1\leq n_1} |V_{j_1,m_2,m_3,\cdots,m_q}| > \varepsilon\sigma 2^{-q+1}|\mathbf{n}|^{\frac{1}{2}}\} < \eta 2^{-q+1} . \tag{14}$$

Note that (14) is valid for all choices of $m_2, m_3, \cdots, m_q$ such that $1 \leq m_i \leq n_i$. Now fix integers $m_3, m_4, \cdots, m_q$ arbitrarily such that $1 \leq m_i \leq n_i$ and apply Lemma 6 to the following n_1-dimensional random vectors (with the maximum norm used on R^{n_1}):

$$\zeta_1 = \{V_{1,1,m_3,\cdots,m_q}, V_{2,1,m_3,\cdots,m_q}, \cdots, V_{n_1,1,m_3,\cdots,m_q}\},$$
$$\zeta_2 = \{V_{1,2,m_3,\cdots,m_q} - V_{1,1,m_3,\cdots,m_q}, V_{2,2,m_3,\cdots,m_q} - V_{2,1,m_3,\cdots,m_q}, \cdots, V_{n_1,2,m_3,\cdots,m_q} - V_{n_1,1,m_3,\cdots,m_q}\},$$
$$\vdots$$
$$\zeta_{n_2} = \{V_{1,n_2,m_3,\cdots,m_q} - V_{1,n_2-1,m_3,\cdots,m_q}, \cdots, V_{n_1,n_2,m_3,\cdots,m_q} - V_{n_1,n_2-1,m_3,\cdots,m_q}\}.$$

Thus we get,

$$P\{\max_{1\leqq j_1\leqq n_1, 1\leqq j_2\leqq n_2} |V_{j_1,j_2,m_3,\cdots,m_q}| > \varepsilon\sigma 2^{-q}|\mathbf{n}|^{\frac{1}{2}}\} < \eta 2^{-q+2}. \tag{15}$$

Continuing in this fashion (in the next step e.g. we apply Lemma 6 to $n_1 n_2$-dimensional random vectors), we obtain (8) under the assumption that $\varphi(1) < \frac{1}{4}$. To remove this assumption, find r_0 such that $\varphi(r_0) < \frac{1}{4}$. Then (8) is true for each of the following $r_0{}^q$ random fields:

$$\{\xi_{j_1 r_0+p_1, j_2 r_0+p_2, \cdots, j_q r_0+p_q} : (j_1, j_2, \cdots, j_q) \in Z^q\}$$

where $0 \leqq p_1, p_2, \cdots, p_q \leqq r_0 - 1$ are fixed for each random field. Now V's for the original random field are sums of V's for these new random (sub) fields. Using this fact it is easy to see that (8) holds for the original random field. This concludes the proof of Theorem 1. □

THEOREM 2. *Let $(\Omega, \mathscr{F}, P)$ be the probability space which supports the random field in Theorem* 1. *Then Theorem* 1 *remains true if P is replaced by any probability measure P_0 which is absolutely continuous with respect to P.*

PROOF. This can be proved by a straightforward adaptation, to multivariate time, of the proof of Theorem 20.2 of [2]. □

A similar adaptation of the proof of Theorem 20.3 of Billingsley (1968) allows us to obtain

THEOREM 3. *Assume the framework of Theorem* 1. *For each j, $1 \leqq j \leqq q$, let $\{\nu_n{}^{(j)} : 1 \leqq n < \infty\}$ be a sequence of positive-integer valued random variables such that, for some positive constants $a_n{}^{(j)} \uparrow \infty$ $(n \to \infty)$, $\nu_n{}^{(j)}/a_n{}^{(j)}$ converges to a strictly positive random variable in probability, as $n \to \infty$. Write $\nu_n = (\nu_n{}^{(1)}, \nu_n{}^{(2)}, \cdots, \nu_n{}^{(q)})$. Then the sequence of stochastic processes $\{Y_n\}$ defined by $Y_n(\mathbf{t}) = X_{\nu_n}(\mathbf{t})$ converges weakly to the Wiener process in D_q.* □

In conclusion it may be interesting to note that, even in the case $q = 1$, it is not known whether the condition (1) is really necessary or whether it could be replaced simply by the φ-mixing property.

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