The Annals of Probability
1975, Vol. 3, No. 4, 643-663

ERGODIC THEOREMS FOR WEAKLY INTERACTING INFINITE SYSTEMS AND THE VOTER MODEL

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A theorem exhibiting the duality between certain infinite systems of interacting stochastic processes and a type of branching process is proved. This duality is then used to study the ergodic properties of the infinite system. In the case of the vector model a complete understanding of the ergodic behavior is obtained.

1. Introduction. In this paper we use the duality between certain systems of infinitely many interacting stochastic processes and a type of branching process to study the long term behavior of the infinite systems. The models which we study here are similar in many respects to the contact processes of Harris (1973). In fact, Sections 3 and 4 of our paper may be viewed as part of a continuing program initiated by Vasershtein (1969) and Dobrushin (1971) and carried forward by Harris (1973) to find conditions which guarantee that the interactions in an infinite system of interacting stochastic processes are weak enough so that the system is ergodic (i.e., has a unique stationary measure to which the distribution at time t converges weakly as t goes to infinity). This program is rather complex and involves coupling several systems together in order to reduce the problem to one concerning a process which we call a proximity process. Rather than attempt to describe the original problem and the couplings involved in the reduction to the proximity process we refer the reader to the original papers of Vasershtein (1969), Dobrushin (1971), and Harris (1973), and to a recent paper by Griffeath (1974). We concentrate our attention on the proximity processes, which are described below.

Let I be a countable set and let $S = \{0, 1\}^I$. A proximity process is a particular type of Markov process η_t, with state space S. Thus $\eta_t(i)$, the value of the ith coordinate at time t, is a stochastic process which takes the values zero and one. The time parameter may be either discrete or continuous; however, since it is easiest to define the transition function for the discrete time processes we do that here. The general continuous time proximity process is described in Section 2. The transition function for the discrete time proximity process is a product measure on S each of whose factors depends (in a way to be described) on the configuration at the time of the transition. Thus the transition function is

Revised June 19, 1974; revised November 18, 1974.

[1] Research supported in part by AOR contract DAHC04-72-C-0005, Project 10150-M.

[2] Research supported in part by N.S.F. Grant GP-33431X, and in part by Alfred P. Sloan Foundation research fellowship.

AMS 1970 *subject classifications*. Primary 60K35; Secondary 60J10.

Key words and phrases. Infinite particle system, ergodic theorem, branching process with interference, Markov chain, harmonic function.

given by

$$Q(\eta, \cdot) = \prod_{i \in I} \nu_{\alpha(i,\eta),i}(\cdot)\,, \tag{1.1}$$

where $\alpha(i, \eta)$ is defined below and $\nu_{\rho,i}$ is the probability on $\{0, 1\}_i$ which puts mass ρ on $\{1\}$. In order to define $\alpha(i, \eta)$ we let, for each $i \in I$, $\{\emptyset = N_{i,0}, N_{i,1}, \cdots\}$ be a finite or countable collection of finite subsets of I, and let $f_i(\cdot)$ be a probability distribution on the nonnegative integers. For $\eta \in S$ set

$$C(\eta) = \{i \in I : \eta(1) = 1\}\,. \tag{1.2}$$

Finally let $D(i, \eta) = \{k : N_{i,k} \cap C(\eta) \neq \emptyset\}$. Then

$$\alpha(i, \eta) = \sum_{k \in D(i,\eta)} f_i(k)\,. \tag{1.3}$$

Intuitively what happens is that at time n the process at each site i looks into each of the sets $N_{i,k}$. If $N_{i,k}$ contains at least one site j with $\eta_n(j) = 1$, then the process at site i adds $f_i(k)$ to the probability that at time $n + 1$ it will be a one. Each site does this and then at time $n + 1$ they all change to their new states, each making the choice independently of the others.

As a specific example let $I = Z^2$—the square lattice—and let $N_{i,0} = \emptyset$, $N_{i,1} = \{j : |i - j| = 1\}$—the four nearest neighbors of i. If $f_i(0) = 1 - p$ and $f_i(1) = p$, then before every transition each site looks at its nearest neighbors and if any of them is a one then with probability p it is a one at the next time. If all of its neighbors are zero, then with probability one it is a zero at the next time. It is clear that the configuration which is identically zero is absorbing for this process. If p is small enough the measure concentrated on the configuration which is identically zero is the only stationary measure and the distribution of the process at time t converges weakly to this measure. Corollary (3.1) implies that if $p < \frac{1}{4}$ then this is the case; and by applying the techniques of Section 4, one can easily show that if $p < .31$ it is still the case.

When the coupling program mentioned earlier is carried out, the parameters of the original process determine the parameters of the proximity process. If the proximity process has only one stationary measure, then the interactions in the original process are sufficiently weak that it too has only one stationary measure and the distribution at time t converges weakly to that measure as t goes to infinity.

We study the proximity process by passing to a dual process which we call a branching process with interference. A (discrete time) branching process with interference (b.p.i.) is a Markov chain, A_n, whose state space is the set of all finite subsets of I. The transition function for the branching process with interference is given by

$$\tilde{Q}(A, B) = \sum{}' \prod_{i \in A} f_i(k_i)\,, \tag{1.4}$$

where A and B are finite subsets of I, and $\sum'$ denotes summation over all sequences $\{k_i\}_{i \in A}$ such that $\bigcup_{i \in A} N_{i,k_i} = B$. Here $\{f_i\}$ and $\{N_{i,k}\}$ are as before.

Intuitively what happens in a b.p.i. is that if there is a particle at site i at time

n, then during the transition to time $n+1$ that particle splits, with probability $f_i(k)$, into particles which occupy the sites of $N_{i,k}$. Note that since $N_{i,0} = \emptyset$, the particle just dies with probability $f_i(0)$. Each particle alive at time n does this independently of the others; however, the new particles interfere wiih each other in that if two of them try to occupy the same site only one of them survives. Thus the state at time $n+1$ is the union of a collection of random sets. Each site which is occupied at time n contributes one, possibly empty, randomly chosen set to this union.

We denote the set of all finite subsets of I by $\mathscr{T}$. Let $B(\emptyset) = S$ and for any nonempty element, F, of $\mathscr{T}$ define $B(F) \subset S$ to be

$$B(F) = \{\eta \in S : \eta(i) = 0 \text{ for all } i \in F\}. \tag{1.5}$$

The connection between a proximity process and the corresponding b.p.i. is revealed in Theorem (1.6).

(1.6) THEOREM. *Let η_n be the proximity process determined by the finite sets $\{N_{i,k}\}$ and probability distributions $\{f_i\}$, and let A_n be the* b.p.i. *determined by the same finite sets and distributions. For $F \in \mathscr{T}$ let $B(F)$ be given by* (1.5) *and for $\eta \in S$ let $C(\eta)$ be given by* (1.2). *Then for all $n \geqq 0$, $\eta \in S$, and $F \in \mathscr{T}$ we have*

$$P_\eta(\eta_n \in B(F)) = P_F(A_n \cap C(\eta) = \emptyset). \tag{1.7}$$

The subscripts η and F in $P_\eta(\cdot)$ and $P_F(\cdot)$ indicate the initial state of the proximity process or b.p.i. respectively. We use the letter P for both processes and the initial state indicates which process we are talking about.

Section 2 is devoted to the proof of Theorem (1.6) and its continuous time analogue. Sections three and four contain four simple applications of these theorems. Of these applications only Corollary (4.15) cannot be easily obtained from previously known results and techniques. This paper came about as an attempt to unify and simplify the previous methods. That is our justification for including the other three examples. In Section 5 we give a much more interesting application of the duality theorem to the "voter model".

The term "voter model" will be used to describe the continuous time proximity process in which the sets $N_{i,k}$ are singletons for $k \geqq 1$, which we may as well take to be distinct for each i, and $f_i(0) = 0$ for all $i \in I$. The interpretation of the process which leads to this terminology is that the ith individual periodically reevaluates his position or some issue (the two possible positions on the issue are denoted by 0 and 1), and at each time of reevaluation, he chooses to espouse position 1 with probability $\sum f_i(k)$, where the sum is taken over those k for which the individual $N_{i,k}$ favors position 1. The fact that $N_{i,k} = \{i\}$ for some k is possible permits the individual to let his own previous choice affect his future choice.

Let U_t be the semigroup corresponding to this proximity process, and let $\mathscr{I}$ be the set of invariant probability measures for U_t:

$$\mathscr{I} = \{\mu : \mu U_t = \mu \text{ for all } t \geqq 0\}.$$

Let $\mathscr{I}_e$ denote the set of extreme points of $\mathscr{I}$. The first problem we deal with in Section 5 is to describe $\mathscr{I}_e$. Then for a given $\mu \in \mathscr{I}_e$ we give necessary and sufficient conditions on an initial distribution ν so that $\nu U_t \to \mu$ as $t \to \infty$. The techniques used are similar to those used by Liggett (1973), (1974), and Spitzer (1974) to solve the same problems for a different infinite particle system. The continuous time version of Theorem (1.6) plays the same role in this study that Theorem (1.1) of [6] did there.

In order to state a restricted version of the results of Section 5, let I be the d-dimensional integer lattice, and assume that the process is translation invariant, which means that $N_{i,k} = i + N_{0,k}$ and $f_i(k) = f_0(k)$ for all $i \in I$. Let $X_1(t)$ and $X_2(t)$ be independent copies of the random walk on I which has exponential waiting times with parameter one at each point and transition probabilities $p(i, j) = f_i(k)$ if $N_{i,k} = \{j\}$. Assume that $f_i(k)$ is such that these chains are irreducible. Then $X_1(t) - X_2(t)$ is an irreducible symmetric random walk on I. There are two cases to be considered, depending on whether $X_1(t) - X_2(t)$ is recurrent or transient.

(1.8) THEOREM. *Assume that $X_1(t) - X_2(t)$ is recurrent. Then*

(a) $\mathscr{I}_e = \{\nu_0, \nu_1\}$, *where ν_0 and ν_1 are the point masses on the configurations $\eta \equiv 0$ and $\eta \equiv 1$ respectively.*

(b) *If ν is any translation invariant probability measure on S, then $\nu U_t \to \lambda\nu_0 + (1 - \lambda)\nu_1$, where $\lambda = \nu\{\eta(i) = 0\}$.*

(1.9) THEOREM. *Assume that $X_1(t) - X_2(t)$ is transient. Then*

(a) *For every $\rho \in [0, 1]$, there is a translation invariant and ergodic probability measure μ_ρ on S such that $\mu_\rho \in \mathscr{I}_e$ and $\mu_\rho\{\eta(i) = 1\} = \rho$.*

(b) $\mathscr{I}_e = \{\mu_\rho : 0 \leqq \rho \leqq 1\}$.

(c) *If ν is any translation invariant and ergodic probability measure on S, then $\nu U_t \to \mu_\rho$ where $\rho = \nu\{\eta(i) = 1\}$.*

Of course, Theorem (1.9) holds whenever $d \geqq 3$, while Theorem (1.8) holds if $d = 1$ and $\sum_j p(0, j)|j| < \infty$ or $d = 2$ and $\sum_j p(0, j)|j|^2 < \infty$. Therefore in a one or two dimensional world, a consensus is approached as $t \to \infty$. In higher dimensions, however, differences of opinion tend to persist.

One of the main advantages of introducing the b.p.i. in the study of proximity processes is that it turns questions about stationary measures for the proximity process into questions about harmonic functions for the b.p.i. The latter are considerably easier to handle. In Sections 3 and 4 we essentially find conditions which guarantee that the only bounded harmonic functions for the b.p.i. are constants. In Section 5 we consider situations where there are nonconstant bounded harmonic functions for the b.p.i. (we assume that the one particle process has only constants as its bounded harmonic functions but not the b.p.i.) yet we are still able to analyze the bounded harmonic functions of the b.p.i. well enough to learn a good deal about the stationary measures of the proximity process.

2. The duality theorems.

PROOF OF THEOREM 1.6. Let $\{X_{i,n}: i \in I, n = 1, 2, \cdots\}$ be a set of independent random variables such that $X_{i,n}$ has distribution f_i for all n. Given $\eta \in S$ define η_n inductively by

$$\begin{aligned} \eta_0 &= \eta \\ \eta_{n+1}(i) &= 1 \quad \text{if} \quad N_{i,X_{i,n+1}} \cap C(\eta_n) \neq \emptyset \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{2.1}$$

Since the $X_{i,n}$ are independent, it is clear that η_n is a Markov process and it is easily checked that its transition function is given by (1.1).

Now given $F \in \mathscr{T}$ define A_n inductively by

$$\begin{aligned} A_0 &= F \\ A_{n+1} &= \textstyle\bigcup_{i \in A_n} N_{i,X_{i,n+1}} . \end{aligned} \tag{2.2}$$

Again it is easily checked that A_n is a Markov chain with transition function given by (1.4).

We prove that (1.7) holds by induction on n. For $n = 0$ both sides are either zero or one, and they are clearly equal. For $n = 1$ we have

$$P_\eta(\eta_1 \in B(F)) = \textstyle\sum' \prod_{i \in F} f_i(k_i) , \tag{2.3}$$

where the summation $\sum'$ is over those sequences $\{k_i: i \in F\}$ for which $N_{i,k_i} \cap C(\eta) = \emptyset$. We also have

$$P_F(A_1 \cap C(\eta) = \emptyset) = \textstyle\sum' \prod_{i \in F} f_i(k_i) , \tag{2.4}$$

where the summation $\sum'$ is over the same sequences of k's as in (2.3). Thus the theorem is true for $n = 1$. Assume as the inductive hypothesis that (1.7) is true for n and for all $F \in \mathscr{T}$ and $\eta \in S$. The summations below extend over all sequences $\{k_i: i \in F\}$.

$$\begin{aligned} P_F&(A_{n+1} \cap C(\eta) = \emptyset) \\ &= \textstyle\sum P_F(A_{n+1} \cap C(\eta) = \emptyset \mid X_{i,1} = k_i \text{ for all } i \in F) \\ &\qquad \times P(X_{i,1} = k_i \text{ for all } i \in F) \\ &= \textstyle\sum P_{\bigcup_{i \in F} N_{i,k_i}}(A_n \cap C(\eta) = \emptyset) P(X_{i,1} = k_i \text{ for all } i \in F) . \end{aligned} \tag{2.5}$$

And

$$\begin{aligned} P_\eta(\eta_{n+1} \in B(F)) = \textstyle\sum P_\eta(\eta_{n+1} \in B(F) \mid X_{i,n+1} = k_i \text{ for all } i \in F) \\ \times P(X_{i,n+1} = k_i \text{ for all } i \in F) . \end{aligned} \tag{2.6}$$

From the definition of the η_n we see that $\eta_{n+1} \in B(F)$ if and only if $\eta_n \in B(\bigcup_{i \in F} N_{i,X_{i,n+1}})$. Thus

$$P_\eta(\eta_{n+1} \in B(F) \mid X_{i,n+1} = k_i \text{ for all } i \in F) = P_\eta(\eta_n \in B(\textstyle\bigcup_{i \in F} N_{i,k_i})) . \tag{2.7}$$

Using (2.7) and the assumption that for fixed i the $X_{i,n}$ are identically distributed we see that

$$P_\eta(\eta_{n+1} \in B(F)) = \textstyle\sum P_\eta(\eta_n \in B(\bigcup_{i \in F} N_{i,k_i})) P(X_{i,1} = k_i \text{ for all } i \in F) . \tag{2.8}$$

The proof is completed by applying the inductive hypothesis to (2.8) and (2.5).

We turn to the continuous time version of Theorem (1.6). The continous time proximity process is identified by means of the infinitesimal generator of the corresponding semigroup. We make S into a topological space by giving $\{0, 1\}$ the discrete topology and S the resulting product topology. $\mathscr{C}(S)$ is the set of continuous functions on S and $\mathscr{D} \subset \mathscr{C}(S)$ is the set of functions which depend on only finitely many coordinates. Consider the linear operator, $\mathscr{A}$, on $\mathscr{D}$ given by

$$\mathscr{A}f(\eta) = \textstyle\sum_{i \in I} c(i, \eta)[f({}_i\eta) - f(\eta)] \tag{2.9}$$

where

$$\begin{aligned} {}_i\eta(j) &= \eta(j) && \text{if } j \neq i \\ &= 1 - \eta(j) && \text{if } j = i, \end{aligned}$$

and $c(i, \eta) = c_i[(1 - \eta(i))\alpha(i, \eta) + \eta(i)(1 - \alpha(i, \eta))]$ with $\alpha(i, \eta)$ as in (1.3).

The following theorem is an immediate consequence of Theorems (2.8) and (4.2) of Liggett (1972).

(2.10) THEOREM. *Let $\{N_{i,k} : i \in I, k = 0, 1, \cdots\}$ and $\{f_i\}$ be as in the introduction and denote the number of elements in $N_{i,k}$ by $|N_{i,k}|$. If*

$$\sup_{i \in I} c_i[1 + \textstyle\sum_{k=0}^{\infty} |N_{i,k}| f_i(k)] = \beta < \infty, \tag{2.11}$$

then there is a unique strongly continuous semigroup of positive contractions, U_t: $\mathscr{C}(S) \to \mathscr{C}(S)$, whose infinitesimal generator, $\mathscr{A}$, when restricted to $\mathscr{D}$ is given by (2.9).

The semigroup U_t is the semigroup of the continuous time proximity process. The voter model described in the introduction is an example. Processes with infinitesimal generator of the form (2.9) but without the assumption on the nature of $c(i, \eta)$ have been studied by Dobrushin (1971) and Spitzer (1971). Intuitive descriptions of such processes may be found there.

The continuous time branching process with interference is a pure jump process with state space $\mathscr{T}$. If $A, B \in \mathscr{T}$ and $A \neq B$, then $q_{A,B}$, the infinitesimal rate of transition from A to B, is given by

$$q_{A,B} = \textstyle\sum_{i \in A} \sum_{k \in Q(A,B,i)} c_i f_i(k),$$

where $Q(A, B, i) = \{k : (A \backslash \{i\}) \cup N_{i,k} = B\}$.

We can construct the continuous time branching process with interference as follows. Let $\{R_i(t) : i \in I\}$ be a family of independent Poisson processes, the ith one having intensity c_i, and let $\{X_{i,n} : i \in I, n = 0, 1, 2, \cdots\}$ be a set of random variables independent of each other and of the $\{R_i\}$, with $X_{i,n}$ having distribution f_i. Given $F \in \mathscr{T}$ let $\tilde{A}_0 = F$ and $T_0 = 0$ and define $\tilde{A}_n$ and T_n inductively by

$$T_{n+1} = \inf\{t > T_n : R_i(t) \neq R_i(T_n) \text{ for some } i \in \tilde{A}_n\}$$

and

$$\tilde{A}_{n+1} = (\tilde{A}_n \backslash \{i\}) \cup N_{i, X_{i,n}},$$

where i is the element of $\tilde{A}_n$ for which $R_i(T_{n+1}) \neq R_i(T_n)$. Finally set $A_t = \tilde{A}_n$ for $T_n \leqq t < T_{n+1}$. A_t is the continuous time b.p.i. determined by $\{c_i\}$, $\{N_{i,k}\}$, and $\{f_i\}$. It is easily checked that A_t is a pure jump Markov process whose infinitesimal parameters are $q_{A,B}$. There is the possibility that $\lim_{n\to\infty} T_n < \infty$, in which case A_t is not defined for all $t > 0$; however, we shall show in Lemma (2.19) that if (2.11) holds, this does not happen.

(2.12) THEOREM. *Let* $\{c_i\}$, $\{N_{i,k}\}$, *and* $\{f_i\}$ *satisfy* (2.11) *and let* η_t *and* A_t *be the corresponding continuous time proximity process and* b.p.i. *Then for all* $F \in \mathscr{T}$, $\eta \in S$, *and* $t \geqq 0$

$$P_\eta(\eta_t \in B(F)) = P_F(A_t \cap C(\eta) = \varnothing)\,. \tag{2.13}$$

Conceptually the proof of Theorem (2.12) is merely a passage to a limit using Theorem (1.6); however, there are several technical difficulties which are connected with that a priori $\lim_{n\to\infty} T_n$ may not be infinite. Therefore we begin by proving the theorem in the case that I is a finite set.

(2.14) LEMMA. *If* I *is a finite set, then Theorem* (2.12) *is true.*

PROOF. Let β be as in (2.11) and for each $\lambda > \beta$ let $f_i^{(\lambda)}$ be the probability distribution on $\{-1, 0, 1, \cdots\}$ given by

$$\begin{aligned} f_i^{(\lambda)}(k) &= c_i f_i(k)/\lambda && \text{if } k \geqq 0 \\ &= 1 - c_i/\lambda && \text{if } k = -1\,. \end{aligned}$$

Define

$$\begin{aligned} \tilde{N}_{i,k} &= N_{i,k} && \text{if } k \geqq 0 \\ &= \{i\} && \text{if } k = -1\,. \end{aligned}$$

Let $\sigma_n^{(\lambda)}$ and $B_n^{(\lambda)}$ be the discrete time proximity process and b.p.i. determined by $\{f_i^{(\lambda)}\}$ and $\{\tilde{N}_{i,k}\}$. Finally let $R(t)$ be a Poisson process with intensity λ and set $\eta_t^{(\lambda)} = \sigma_{R(t)}^{(\lambda)}$ and $A_t^{(\lambda)} = B_{R(t)}^{(\lambda)}$. It follows immediately from Theorem (1.6) applied to $\sigma_n^{(\lambda)}$ and $B_n^{(\lambda)}$ that

$$P_\eta(\eta_t^{(\lambda)} \in B(F)) = P_F(A_t^{(\lambda)} \cap C(\eta) = \varnothing)\,. \tag{2.15}$$

Now $\eta_t^{(\lambda)}$, $A_t^{(\lambda)}$, η_t, and A_t are all finite state space Markov processes; and thus to show that the finite dimensional distributions of $\eta_t^{(\lambda)}$ and $A_t^{(\lambda)}$ converge to those of η_t and A_t respectively, it is enough to show that the corresponding infinitesimal parameters converge. This is an elementary computation which is left to the reader. The lemma follows from (2.15) and the convergence of the respective finite dimensional distributions.

In order to pass from the finite to the infinite case we need to use Theorems (2.8) and (4.2) of Liggett (1972) and approximate the infinite processes by finite ones. Thus for each finite subset $J \subset I$ we define $N_{i,k}^{(J)}$ for $i \in J$ by $N_{i,k}^{(J)} = N_{i,k} \cap J$. We then define $\eta_t^{(J)}$ and $A_t^{(J)}$ to be the proximity process and b.p.i. on $\{0, 1\}^J$ and the subsets of J respectively determined by $\{c_i\}$, $\{N_{i,k}^{(J)}\}$ and $\{f_i\}$. The $\eta_t^{(J)}$

process has an infinitesimal generator $\mathscr{A}^{(J)}$ which satisfies

$$\begin{aligned}(2.16)\qquad \mathscr{A}^{(J)}f(\eta) = \textstyle\sum_{i\in J} c_i[(1-\eta(i))\alpha^{(J)}(i,\eta) + \eta(i)(1-\alpha^{(J)}(i,\eta))] \\ \times [f({}_i\eta) - f(\eta)]\,.\end{aligned}$$

The right side of (2.16) makes sense for $f \in \mathscr{C}(S)$ and in fact we want to think of $\eta_t^{(J)}$ as a Markov process on S for which all of the sites outside of J never change.

Let $\{J_n\}$ be a sequence of finite subsets of I with $J_n \subset J_{n+1}$ and $\bigcup_n J_n = 1$. We replace all superscripts (J_n) with (n). It is easily checked that if $f \in \mathscr{D}$, then

$$\lim_{n\to\infty} \sup_{\eta\in S} |\mathscr{A}^{(n)}f(\eta) - \mathscr{A}f(\eta)| = 0$$

Thus if $U_t^{(n)}$ is the semigroup of the process $\eta_t^{(n)}$, Theorems (2.8) and (4.2) of Liggett (1972) imply that when (2.11) is satisfied

$$(2.17)\qquad \lim_{n\to\infty} \sup_{0\le t\le t_0} \sup_{\eta\in S} |U_t^{(n)}f(\eta) - U_t f(\eta)| = 0$$

for all $t_0 > 0$ and all $f \in \mathscr{C}(S)$.

All of the processes $\{A_t^{(m)}\}$ and A_t can be constructed on the same probability space using the same set of Poisson processes and random variables $\{X_{i,n}\}$ for all of the $A_t^{(m)}$ and A_t. We assume that this has been done. If we can show that for all t

$$(2.18)\qquad \lim_{m\to\infty} P_F(A_t^{(m)} = A_t) = 1\,,$$

then Theorem (2.12) will follow from Lemma (2.14), (2.17), and (2.18).

(2.19) LEMMA. *If* (2.11) *is satisfied, then* (2.18) *is true for all* $t \geqq 0$.

Since $A_t^{(m)}$ is a finite state space Markov process there is no possibility of its exploding. Thus we have no problem with the almost sure existence of $A_t^{(m)}$ for all positive t. Implicit in the conclusion of Lemma (2.19) is the almost sure existence of A_t for all positive t, i.e. $P_F(\lim_{n\to\infty} T_n < \infty) = 0$.

PROOF. It is easily seen from the construction that if $A_s^{(n)} = A_s^{(m)}$ for all $s \leqq t$ and all $n > m$, then also $A_t = A_t^{(m)}$. In addition we have for $n > m$

$$(2.20)\qquad P_F(A_s^{(n)} \neq A_s^{(m)} \text{ for some } s \leqq t) \leqq e^{\beta t} P_F(A_t^{(n)} \not\subset J_m)\,.$$

This is because $A_s^{(n)} = A_s^{(m)}$ as long as both are contained in J_m, and if $A_{s_0}^{(n)} \not\subset J_m$ for some $s_0 \leqq t$, then with probability at least $e^{-\beta t}$ at least one of the elements in $A_{s_0}^{(n)}\backslash J_m$ is still in $A_t^{(n)}$. Thus, since $P_F(A_s^{(n)} \not\subset J_m)$ is increasing in n, it suffices to show that

$$(2.21)\qquad \lim_{m\to\infty} \lim_{n\to\infty} P_F(A_t^{(n)} \not\subset J_m) = 0\,.$$

Let σ_m be the element of S which is equal to one on $I\backslash J_m$ and equal to zero on J_m. Then by Lemma (2.14)

$$\begin{aligned}(2.22)\qquad P_F(A_t^{(n)} \not\subset J_m) &= P_F(A_t^{(n)} \cap C(\sigma_m) \neq \varnothing) \\ &= P_{\sigma_m}(\eta_t^{(n)} \notin B(F)) = U_t^{(n)}f(\sigma_m)\,,\end{aligned}$$

where f is the indicator function of the complement of $B(F)$. By (2.17) $\lim_{n\to\infty} U_t^{(n)}f(\sigma_m) = U_t f(\sigma_m)$; and since $U_t f(\eta)$ is continuous as a function of η $\lim_{m\to\infty} U_t f(\sigma_m) = U_t f(\sigma_\infty)$, where σ_∞ is identically zero. But σ_∞ is absorbing for the proximity process η_t. Thus $U_t f(\sigma_\infty) = f(\sigma_\infty) = 0$. This combined with (2.22) proves (2.21) and completes the proof of the lemma.

3. First applications. In this section we apply Theorems (1.6) and (2.12) to the proximity processes that one obtains if he carries out the coupling program mentioned in the introduction in the cases considered by Vasershtein (1969) Dobrushin (1971) and Chover (1974).

In the cases of Vasershtine and Chover the resulting process is a discrete time proximity process which is covered by Corollary (3.1).

(3.1) COROLLARY. *Let η_n be the discrete time proximity process determined by $\{N_{i,k}\}$ and $\{f_i\}$. If there is some constant $\lambda < 1$ such that*

$$\sum_{k=0}^{\infty} |N_{i,k}| f_i(k) \leqq \lambda \qquad \text{for all} \quad i \in I\,, \tag{3.2}$$

then for all $F \in \mathscr{T}$ and all $\eta \in S$

$$P_\eta(\eta_n \in B(F)) \geqq 1 - \lambda^n |F|\,. \tag{3.3}$$

Thus the only stationary distribution for η_n is the one concentrated on η identically zero.

PROOF. By Theorem (1.6)

$$P_\eta(\eta_n \in B(F)) = P_F(A_n \cap C(\eta) = \varnothing) \geqq P_F(A_n = \varnothing)\,. \tag{3.4}$$

But

$$P_F(A_n \neq \varnothing) \leqq E_F\{|A_n|\} \leqq \lambda^n |F|\,. \tag{3.5}$$

The last inequality in (3.5) is a consequence of the Markov property and the following computation in which $\sum'$ denotes summation over all sequences $\{k_i : i \in F\}$.

$$\begin{aligned} E_F\{|A_1|\} &= \textstyle\sum' |\bigcup_{j\in F} N_{j,k_j}| \prod_{i\in F} f_i(k_i) \\ &\leqq \textstyle\sum' \sum_{j\in F} |N_{j,k_j}| \prod_{i\in F} f_i(k_i) \leqq \lambda |F|\,. \end{aligned} \tag{3.6}$$

Note that the interference in the b.p.i. was not taken advantage of in Corollary (3.1). A general theorem which exploited the interference effectively would have rather involved hypotheses; however, in specific examples one can often improve Corollary (3.1). We give such an example in the next section.

As a second application we consider the continuous time proximity process associated with Dobrushin's work (1971). Dobrushin considered quite general processes; however, when the coupling program mentioned in the introduction is applied to these processes the result is a continuous time proximity process with two distinctive features: the rate that the site i changes from one to zero is a constant, $\bar{c}_i d_i$, which does not depend on the configuration, and the rate at which it goes from zero to one is a function of η_t of the form $\tilde{c}_i \sum_{j\in I} g_i(j)\eta_t(j)$, where $g_i(j) \geqq 0$ and $g_i = \sum_{j\in I} g_i(j) < \infty$. Thus the infinitesimal generator, $\mathscr{A}$, of the process is given, for $f \in \mathscr{D}$, by

$$\mathscr{A}f(\eta) = \sum_{i\in I} \tilde{c}_i[d_i\eta(i) + (1-\eta(i)) \sum_{j\in I} g_i(j)\eta(j)][f({}_i\eta) - f(\eta)]\,. \tag{3.7}$$

In order to get (3.7) into the standard form for a proximity process we let $N_{i,0} = \emptyset$ and $\{N_{i,k}\}$ for $k \geqq 1$ be an enumeration of all pairs $\{i, j\}$ with $j \in I$. Let $b_i = d_i + g_i$ and for $k \geqq 1$ set $f_i(k) = g_i(j)/b_i$, where $j \in I$ is the element such that $N_{i,k} = \{i, j\}$. Then $f_i(0) = 1 - \sum_{k=1}^{\infty} f_i(k) = 1 - g_i b_i^{-1} = d_i b_i^{-1}$. Now let $\alpha(i, \eta)$ be defined as in (1.3) using these $\{N_{i,k}\}$ and $\{f_i\}$. It is easily checked that if $\eta(i) = 1$, then $\alpha(i, \eta) = g_i b_i^{-1}$; while if $\eta(i) = 0$, then $\alpha(i, \eta) = b_i^{-1} \sum_{j\in I} g_i(j)\eta(j)$. Thus setting $c_i = \tilde{c}_i b_i$ we have

$$\begin{aligned}\tilde{c}_i[d_i\eta(i) + (1-\eta(i)) \textstyle\sum_{j\in I} g_i(j)\eta(j)]\\ = c_i[(1-\alpha(i,\eta))\eta(i) + (1-\eta(i))\alpha(i,\eta)]\,.\end{aligned} \tag{3.8}$$

To be sure that there is a strong Markov process whose generator is given by (3.7) we must assume that (2.11) is satisfied, which in this case is equivalent to the assumption that $\sup_{i\in I} c_i < \infty$.

(3.9) COROLLARY. *Let η_t be the continuous time proximity process determined by $\{c_i\}$, $\{N_{i,k}\}$, and $\{f_i\}$ as described above. If*

$$\inf_{i\in I} c_i(2f_i(0) - 1) = \chi > 0\,, \tag{3.10}$$

then for all $F \in \mathscr{T}$, all $\eta \in S$ and all $t \geqq 0$

$$P_\eta(\eta_t \in B(F)) \geqq 1 - |F|e^{-\chi t}\,. \tag{3.11}$$

Thus the only stationary distribution for η_t is the one concentrated on the configuration which is identically zero.

Before proving Corollary (3.9) we remark that inequality (4.2) of Dobrushin (1971) implies inequality (3.10) above when all of the couplings mentioned in the introduction are carried out. This corollary also follows from Dobrushin's results. We include a proof here to demonstrate the simplicity of the duality arguments.

PROOF OF COROLLARY (3.9). Let A_t be the continuous time b.p.i. determined by $\{c_i\}$, $\{N_{i,k}\}$ and $\{f_i\}$. By Theorem (2.12)

$$P_\eta(\eta_t \in B(F)) = P_F(A_t \cap C(\eta) = \emptyset) \geqq P_F(A_t = \emptyset)\,. \tag{3.12}$$

Since $|A_t|$ grows no faster than a pure birth process with birth rate proportional to the population size, $E_F\{|A_t|\}$ is finite for all t. A routine argument then yields

$$\frac{d}{dt} E_F\{|A_t|\} = E_F\{\sum_{i\in A_t} c_i \sum_{k=0}^{\infty} f_i(k)[|(A_t\backslash\{i\}) \cup N_{i,k}| - |A_t|]\}\,. \tag{3.13}$$

But $N_{i,0} = \emptyset$ and therefore $[|(A_t\backslash\{i\}) \cup N_{i,0}| - |A_t|] = -1$. Also since $i \in N_{i,k}$

and $|N_{i,k}| = 2$ for all $k \geqq 1$, we have, for $k \geqq 1$, $[|(A_t \backslash \{i\}) \cup N_{i,k}| - |A_t|] \leqq 1$. Thus

$$\sum_{k=0}^{\infty} f_i(k)[|(A_t \backslash \{i\} \cup N_{i,k}| - |A_t|] \leqq -f_i(0) + \sum_{k=1}^{\infty} f_i(k) = 1 - 2f_i(0) . \tag{3.14}$$

Substituting (3.14) into (3.13) and using (3.10) we have

$$\frac{d}{dt} E_F\{|A_t|\} \leqq E_F\{\sum_{i \in A_t} c_i(1 - 2f_i(0))\} \leqq -\chi E_F\{|A_t|\} . \tag{3.15}$$

Combining (3.15) with the initial condition, $E_F\{|A_0|\} = |F|$, we obtain

$$E_F\{|A_t|\} \leqq e^{-\chi t}|F| . \tag{3.16}$$

Since $P_F(A_t \neq \varnothing) \leqq E_F\{|A_t|\}$, (3.11) is a consequence of (3.12) and (3.16).

4. Expoiting the interference. In Section three we never took advantage of the interference in the b.p.i. In specific examples it is often possible to make significant improvements by taking the interference into account. In this section we give two examples where this can be done.

For the first example we consider the discrete time proximity process considered by Stavskaya and Pyatetskii-Shapiro (1971). Our analysis is very similar in spirit to theirs; however, the interpretation is different and inequality (4.5) below seems to be more efficient than their inequality (1). Inequality (4.5) is the main reason for introducing the b.p.i. in this example.

In this example I is the integers, $N_{i,0} = \varnothing$ and $N_{i,1} = \{i - 1, i + 1\}$. We take $f_i(0) = 1 - \lambda$ and $f_i(1) = \lambda$. From Corollary (3.1) we know that if $\lambda < \frac{1}{2}$, then the discrete time proximity process determined by these sets $\{N_{i,k}\}$ and distributions $\{f_i\}$ is ergodic and its distribution tends weakly to the probability measure concentrated on the configuration which is identically zero. We prove the considerably stronger

(4.1) COROLLARY. *If η_t is the discrete time proximity process determined by the $\{N_{i,k}\}$ and $\{f_i\}$ above then for all $F \in \mathscr{T}$ and all $\eta \in S$*

$$P_\eta(\eta_t \in B(F)) \geqq 1 - [h(\lambda)]^{(t-1)/2}|F| , \tag{4.2}$$

where $h(\lambda) = 2\lambda + 2\lambda^2 - 7\lambda^3 + 5\lambda^4 - \lambda^5$.

We remark that $h(\lambda) < 1$ if $\lambda < .6527$.

PROOF. Let A_t be the corresponding b.p.i. Then by (3.4) it is enough to show that

$$P_F(A_t \neq \varnothing) \leqq |F|[h(\lambda)]^{(t-1)/2} . \tag{4.3}$$

Let $\mathbf{1}$ be the configuration $\eta(i) \equiv 1$. Then

$$P_F(A_t \neq \varnothing) = P_F(A_t \cap C(\mathbf{1}) \neq \varnothing) = P_{\mathbf{1}}(\eta_t \notin B(F)) . \tag{4.4}$$

Thus for $F, G \in \mathscr{T}$,

$$
\begin{aligned}
&P_{F\cup G}(A_t \neq \emptyset) \\
&\qquad = P_1(\eta_t \notin B(F \cup G)) \\
&\qquad = P_1((\eta_t \notin B(F)) \cup (\eta_t \notin B(G))) \\
&\qquad = P_1(\eta_t \notin B(F)) + P_1(\eta_t \notin B(G)) - P_1((\eta_t \notin B(F)) \cap (\eta_t \notin B(G))) \\
&\qquad \leqq P_1(\eta_t \notin B(F)) + P_1(\eta_t \notin B(G)) - P_1(\eta_t \notin B(F \cap G)) \\
&\qquad = P_F(A_t \neq \emptyset) + P_G(A_t \neq \emptyset) - P_{F\cap G}(A_t \neq \emptyset)\,.
\end{aligned}
\tag{4.5}
$$

Using the homogeneity of the $\{N_{i,k}\}$ and $\{f_i\}$ we see from (4.5) that

$$P_F(A_t \neq \emptyset) \leqq |F| P_{\{0\}}(A_t \neq \emptyset)\,. \tag{4.6}$$

Now

$$P_{\{0\}}(A_t \neq \emptyset) = \lambda P_{\{-1,1\}}(A_{t-1} \neq \emptyset)\,, \tag{4.7}$$

and again using the homogeneity

$$P_{\{-1,1\}}(A_{t-1} \neq \emptyset) = 2\lambda(1-\lambda)P_{\{-1,1\}}(A_{t-2} \neq \emptyset) + \lambda^2 P_{\{-2,0,2\}}(A_{t-2} \neq \emptyset)\,. \tag{4.8}$$

Since $P_{\{-1,1\}}(A_t \neq \emptyset)$ is clearly decreasing in t we obtain from (4.8), upon replacing t with $t-1$, that

$$\lambda^2 P_{\{-2,0,2\}}(A_{t-3} \neq \emptyset) \leqq [1 - 2\lambda(1-\lambda)]P_{\{-1,1\}}(A_{t-3} \neq \emptyset)\,. \tag{4.9}$$

Also

$$
\begin{aligned}
&P_{\{-2,0,2\}}(A_{t-2} \neq \emptyset) \\
&\qquad = 3\lambda(1-\lambda)^2 P_{\{-1,1\}}(A_{t-3} \neq \emptyset) + 2\lambda^2(1-\lambda)P_{\{-2,0,2\}}(A_{t-3} \neq \emptyset) \\
&\qquad\qquad + \lambda^2 P_{\{-3,-1,1,3\}}(A_{t-3} \neq \emptyset) \\
&\qquad \leqq 3\lambda(1-\lambda)^2 P_{\{-1,1\}}(A_{t-3} \neq \emptyset) + 2\lambda^2(1-\lambda)P_{\{-2,0,2\}}(A_{t-3} \neq \emptyset) \\
&\qquad\qquad + \lambda^2[2P_{\{-2,0,2\}}(A_{t-3} \neq \emptyset) - P_{\{-1,1\}}(A_{t-3} \neq \emptyset)] \\
&\qquad \leqq \{[3\lambda(1-\lambda)^2 - \lambda^2] + [2(1-\lambda) + 2][1 - 2\lambda(1-\lambda)]\} \\
&\qquad\qquad \times P_{\{-1,1\}}(A_{t-3} \neq \emptyset)\,.
\end{aligned}
\tag{4.10}
$$

The first inequality in (4.10) follows from (4.5) by taking $F = \{-3, -1, 1\}$ and $G = \{-1, 1, 3\}$ and then using the homogeneity. The second inequality follows from (4.9). Upon simplifying the right side of (4.10) and substituting back into (4.8) we obtain

$$
\begin{aligned}
P_{\{-1,1\}}(A_{t-1} \neq \emptyset) &\leqq 2\lambda(1-\lambda)P_{\{-1,1\}}(A_{t-2} \neq \emptyset) \\
&\qquad + \lambda^2[4 - 7\lambda + 5\lambda^2 - \lambda^3]P_{\{-1,1\}}(A_{t-3} \neq \emptyset)\,.
\end{aligned}
\tag{4.11}
$$

By multiplying both sides of (4.11) by λ and using (4.7) together with the decrease of $P_{\{0\}}(A_t \neq \emptyset)$ as a function of t we finally arrive at

$$P_{\{0\}}(A_t \neq \emptyset) \leqq h(\lambda)P_{\{0\}}(A_{t-2} \neq \emptyset) \leqq [h(\lambda)]^{(t-1)/2}\,. \tag{4.12}$$

This together with (4.6) completes the proof.

There is no theoretical reason to stop the procedure used in Corollary (4.1) after three steps. One could in principle carry out as many steps as he had the patience for. Indeed a procedure very similar to the one above has been carried out to seven steps on a computer by Scott Brown. He obtained the result that if $\lambda \leqq .6774$ then with probability one the b.p.i. is absorbed at $\varnothing$. Thus if $\lambda \leqq .6774$ there is only one stationary measure for the proximity process of Corollary (4.1).

The above example is presented mainly for its methodological interest. The same technique can be applied to many other problems. For example this technique (using two instead of three steps) was applied to obtain the bound .31 in the example in the introduction.

One could also apply techniques analogous to those in Corollary (4.1) to obtain the results of Harris (1973) on contact processes, again obtaining exponentially fast convergence. In fact although the interpretation is different, the relevant computations are very nicely done by Griffeath (1974). Rather than do that we give an example of another technique which does not rely on any homogeneity. The idea is to find a function on $\mathscr{T}$ which in some sense measures the amount of interference in the b.p.i. As an example let η_t be a continuous time proximity process as in Corollary (3.9). We assume that $f_i(0) < 1$ for all i. Let $r(i,j)$ for all $i, j \in I$ be given by

$$(4.13)\qquad \begin{aligned} r(i, j) &= f_i(k)/(1 - f_i(0)) \qquad \text{if} \quad N_{i,k} = \{i, j\} \\ &= 0 \qquad \text{if there is no} \quad k \quad \text{for which} \quad N_{i,k} = \{i, j\}\,. \end{aligned}$$

Note that since there is no k such that $N_{i,k} = \{i\}$, $r(i, i) = 0$. We also assume that $r(i, j) = r(j, i)$ and that for each i and every finite set A containing i

$$(4.14)\qquad f_i(0)[\textstyle\sum_{j \notin A} r(i, j)\{1 + \sum_{k \notin A} r(j, k)\}] \geqq \sum_{j \notin A} \sum_{k \notin A} r(i, j) r(j, k)\,.$$

The inequality (4.14) is just an ad hoc condition which we need in the proof. Note however that if for all j, $r(i, j)$ is either zero or at least γ_i, then if $f_i(0) \geqq (1 - \gamma_i)/(2 - \gamma_i)$, the inequality (4.14) is satisfied.

For an example where the above conditions are satisfied let, for each $i \in I$, M_i be a finite subset of I. Assume that $i \notin M_i$, each M_i has cardinality d, and that if $j \in M_i$ then $i \in M_j$. For each $i \in I$ let $N_{i,k}$, $k = 1, \cdots, d$, be an enumeration of the pairs $\{\{i, j\} : j \in M_i\}$. Then if $f_i(0) = (d - 1)/(2d - 1)$ and $f_i(k) = 1/(2d - 1)$ for $k = 1, \cdots, d$, the above conditions are satisfied. Here we have $r(i, j) = 1/d$ if $j \in M_i$. Note that this example does not satisfy the hypotheses of Corollary (3.9).

(4.15) COROLLARY. *If $f_i(0) \geqq \delta > 0$ for all i and $r(i, j)$ given by* (4.13) *satisfies $r(i, j) = r(j, i)$ and* (4.14), *then the continuous time proximity process determined by $\{c_i\}$, $\{N_{i,k}\}$, and $\{f_i\}$ satisfies*

$$(4.16)\qquad \lim_{t \to \infty} P_\eta(\eta_t \in B(F)) = 1$$

for all $\eta \in S$ and all $F \in \mathscr{T}$.

PROOF. Let A_t be the continuous time b.p.i. determined by $\{c_i\}$, $\{N_{i,k}\}$, and

$\{f_i\}$; and let $\{T_n\}$ be the branching times as in the construction of A_t given in Section 2. From (3.13) and the construction of A_t we see that it suffices to show that

$$\lim_{n\to\infty} P_F(A_{T_n} = \emptyset) = 1 \,. \tag{4.17}$$

Since $f_i(0) \geqq \delta > 0$ for all i, (4.17) will follow if we can show that $|A_{T_n}|$ stays bounded with probability one.

Let $\mathscr{F}_n$ be the σ-algebra of events prior to T_n and define a function H on $\mathscr{T}$ by the formula

$$H(A) = |A| - \sum_{\{i,j\}\subset A} r(i,j) = \tfrac{1}{2}|A| + \tfrac{1}{2}\sum_{i\in A}\sum_{j\notin A} r(i,j) \,. \tag{4.18}$$

Since $H(A) \geqq \frac{1}{2}|A|$, it will obviously suffice to show that $H(A_{T_n})$ stays bounded with probability one. We do this by showing that $\{H(A_{T_n}), \mathscr{F}_n\}$ is a supermartingale. Toward this end we set $Z(A) = \sum_{i\in A} c_i$ and compute

$$\begin{aligned} &E\{H(A_{T_{n+1}}) \mid A_{T_n} = A\} - H(A) \\ &\quad = Z^{-1}(A)\sum_{i\in A} c_i \sum_{m=0}^{\infty} f_i(m)[H((A\backslash\{i\}) \cup N_{i,m}) - H(A)] \\ &\quad = Z^{-1}(A)\sum_{i\in A} c_i[f_i(0)\{-1 + \textstyle\sum_{j\in A} r(i,j)\} \\ &\qquad + (1 - f_i(0))\textstyle\sum_{j\notin A} r(i,j)\{1 - \sum_{k\in A} r(j,k)\}] \\ &\quad = Z^{-1}(A)\sum_{i\in A} c_i[-f_i(0)\textstyle\sum_{j\notin A} r(i,j)\{1 + \sum_{k\notin A} r(j,k)\} \\ &\qquad + \textstyle\sum_{j\notin A}\sum_{k\notin A} r(i,j)r(j,k)] \,. \end{aligned} \tag{4.19}$$

Inequality (4.14) guarantees that each term on the right side of (4.19) is nonpositive, and thus

$$E\{H(A_{T_{n+1}}) \mid \mathscr{F}_n\} \leqq H(A_{T_n}) \,.$$

5. The voter model. In this section, we consider the continuous time proximity process η_t and corresponding b.p.i. A_t in the case that each set $N_{i,k}$ is a singleton for $k \geqq 1$ and that $f_i(0) = 0$ for all $i \in I$. In this case, A_t possesses the important property that $|A_t|$ is nonincreasing in t, and that $|A_t| \geqq 1$ if $|A_0| \geqq 1$. When restricted to $\{A : |A| = 1\}$, the b.p.i. is simply a Markov chain on I, which will be referred to as the one particle process. Let $p_t(i,j)$ denote its transition probabilities. Its infinitesimal parameters are given by $q_{i,j} = c_i f_i(k)$ if $j \neq i$ and $N_{i,k} = \{j\}$. We will assume that $\sup_i c_i < \infty$ in order to guarantee (2.11), and that the one-particle process is irreducible.

We will make an additional assumption for the sake of simplicity:

(5.1) If α is a bounded function on I which satisfies

$$\sum_j p_t(i,j)\alpha(j) = \alpha(i) \quad \text{for all } i \in I\,, \quad t \geqq 0\,,$$

then α is constant.

The presence of nonconstant bounded harmonic functions for p_t would introduce additional invariant measures for the process η_t in much the same way that it did in [6]. It would also complicate the proofs somewhat, and would blur the distinction between the two cases discussed below, since there would be situations in which the process would behave like Case I on part of the space and like Case II on another part. At the end of the section, however, we will describe briefly the results that can be proved if assumption (5.1) is not made. Recall that by the Choquet–Deny Theorem, assumption (5.1) is satisfied in the translation invariant case discussed in the introduction—the case in which I is the d-dimensional integer lattice and $p_t(i, j) = p_t(0, j - i)$.

Note that for any probability measure ν on S, $\nu[B(F)] = \nu\{\eta : F \cap C(\eta) = \emptyset\}$. Therefore, integration of (2.13) with respect to ν yields

$$\nu U_t[B(F)] = \textstyle\sum_{A \in \mathscr{T}} P_F(A_t = A)\nu[B(A)] \tag{5.2}$$

for any $F \in \mathscr{T}$. This is the form in which Theorem 2.12 will be used.

Two cases arise in the ergodic theory of η_t, which are similar to those that arose in [6], [7] and [9]. In order to describe them, let $X_i(t)$, $X_2(t)$, $\cdots$ be independent copies of the one particle process.

Case I: $P(X_1(t) = X_2(t)$ for some $t > 0) = 1$ for all initial states $X_1(0)$, $X_2(0)$.

Case II: $P(X_1(t) = X_2(t)$ for some $t > 0) < 1$ for all initial states $X_1(0) \neq X_2(0)$.

Since $X_1(t)$ is irreducible, these two cases exhaust all possibilities. In the translation invariant case, they correspond to the recurrence or transience respectively of the chain $X_1(t) - X_2(t)$.

For $A \in \mathscr{T}$, define $g(A) = P_A[|A_t| < |A|$ for some $t > 0]$. This function measures in some sense how far apart the points of A are. Note that $g(A) = 0$ for $|A| \leqq 1$, and that $g(A) \leqq g(B)$ whenever $A \subseteq B$. The latter statement can be obtained by coupling the processes A_t and B_t with initial states A and B respectively in such a way that $A_t \subseteq B_t$ for all t. By the definition of Case I, $g(A) = 1$ if $|A| = 2$, so the motonicity of g gives that $g(A) = 1$ if $|A| \geqq 2$. Therefore in Case I,

$$P_A(|A_t| = 1 \text{ for som } t > 0) = 1 \qquad \text{for all } A \in \mathscr{T}, \quad A \neq \emptyset\,. \tag{5.3}$$

By the definition of Case II, on the other hand, $g(A) < 1$ if $|A| = 2$.

(5.4) LEMMA. *Assume that Case II holds. Then*

(a) $g(A) < 1$ *if* $|A| \geqq 2$.

(b) $P_A[\lim_{t\to\infty} g(A_t) = 0] = 1$ *for all* $A \in \mathscr{T}$.

(c) $P[\lim_{t\to\infty} g(\{X_1(t), \cdots, X_n(t)\}) = 0] = 1$ *for all* $n \geqq 1$ *and all choices of initial points* $X_1(0), \cdots, X_n(0)$.

(d) *In the translation invariant case*, $\lim_{i\to\infty} g(A \cup \{i\}) = g(A)$ *for all* $A \in \mathscr{T}$.

PROOF. Since $g(A) = 0$ for $|A| = 1$, (c) is immediate for $n = 1$. Since $g(A) \leqq \sum_{i\neq j; i,j \in A} g(i, j)$, if we prove (c) for $n = 2$, it will follow for larger n. By assumption (5.1), the process $X_1(t)$ has no nonconstant bounded harmonic functions, so by the simple extension of Lemma (3.14) of [6] to continuous time

Markov chains, neither does the process $(X_1(t), X_2(t))$. Therefore

$$P_{i,j}(\exists\, t_n \to \infty \text{ such that } X_1(t_n) = X_2(t_n)) = 0 \quad \text{or} \quad 1 \tag{5.5}$$

independently of the initial points i, j. But since $g(A) < 1$ for $|A| = 2$, the value of (5.5) must be zero. Define $\tilde{g}$ on I^2 by $\tilde{g}(i,j) = g(\{i,j\})$ if $i \neq j$ and $\tilde{g}(i,i) = 1$. Then by the Markov property, $E_{i,j}\tilde{g}(X_1(t), X_2(t)) = P_{i,j}[\exists\, s \geqq t$ such that $X_1(s) = X_2(s)] \leqq \tilde{g}(i,j)$, so that $\tilde{g}(X_1(t), X_2(t))$ is a bounded supermartingale, which therefore converges with probability one. Furthermore,

$$\lim_{t\to\infty} E_{i,j}\, g(\{X_1(t), X_2(t)\}) = P_{i,j}(\exists\, t_n \to \infty \text{ such that } X_1(t_n) = X_2(t_n)) = 0$$

by (5.5), and so $\lim_{t\to\infty} g(\{X_1(t), X_2(t)\}) = 0$, thus completing the proof of (c). In order to prove (b), let τ_n be the stopping times defined by $\tau_0 = 0$ and $\tau_{n+1} = \inf\{t > \tau_n : |A_t| < |A_{\tau_n}|\}$. Note that if $|A_0| = m$, then $\tau_m = \infty$. On the set $\{\tau_1 = \infty\}$, the process A_t is the same as the process $\{X_1(t), \cdots, X_m(t)\}$, so $\lim_{t\to\infty} g(A_t) = 0$ a.s. on $\{\tau_1 = \infty\}$ by part (c). On the set $\{\tau_1 < \infty, \tau_2 = \infty\}$, A_{τ_1+t} behaves like $\{X_1(t), \cdots, X_{m-1}(t)\}$ started on A_{τ_1}, so the same argument can be applied again to conclude that $\lim_{t\to\infty} g(A_t) = 0$ a.s. on $\{\tau_2 = \infty\}$. Since $\tau_m = \infty$, this argument can be repeated m times to conclude the proof of (b). To prove (a), note that the irreducibility of $X_1(t)$ implies that if $|A| = |B| \geqq 1$, then $P_A(A_t = B$ for some $t) > 0$. (See, for example, the proof of Lemma (2.1) of [7].) Therefore if $g(A) = 1$ for some A, then $g(B) = 1$ for all B with $|A| = |B|$, which contradicts (c). Finally, (d) is proved by observing that $|g(A \cup \{i\} - g(A)| \leqq \sum_{j\in A} g(\{i,j\})$, so it suffices to prove that $\lim_{i\to\infty} g(\{i,j\}) = 0$ for each $j \in I$. But this is equivalent to the fact that for the symmetric, transient random walk $Z(t) = X_1(t) - X_2(t)$ on the d-dimensional integer lattice, $\lim_{k\to\infty} P_k(Z(t) = 0$ for some $t)$ equals 0.

We proceed now to the ergodic theory of the process η_t. The situation is simplest in Case I. Recall that ν_0 and ν_1 are the point masses on the configurations $\eta \equiv 0$ and $\eta \equiv 1$ respectively.

(5.6) THEOREM. *Assume that Case I holds. Then*

(a) $\mathscr{I}_e = \{\nu_0, \nu_1\}$.

(b) *If ν is any probability measure on S, then $\nu U_t \to \lambda\nu_0 + (1-\lambda)\nu_1$ if and only if*

$$\lim_{t\to\infty} \textstyle\sum_j p_t(i,j)\nu\{\eta(j) = 0\} = \lambda \quad \textit{for all} \quad i \in I\,. \tag{5.7}$$

In particular, this is true if $\nu\{\eta(j) = 0\} = \lambda$ for all $j \in I$.

PROOF. It is immediate that $\nu_0, \nu_1 \in \mathscr{I}$. Consider any $\mu \in \mathscr{I}$. Since $\mu U_t = \mu$, applying (5.2) to $F = \{i\}$ gives

$$\mu\{\eta(i) = 0\} = \textstyle\sum_j p_t(i,j)\mu\{\eta(j) = 0\}\,.$$

By assumption (5.1), $\mu\{\eta(i) = 0\}$ is a constant which we will call λ. In order to prove that $\mu = \lambda\nu_0 + (1-\lambda)\nu_1$, it suffices to prove that $\mu[B(F)] = \lambda$ for all $F \neq \varnothing$. Apply (5.2) to such an F to obtain $|\mu[B(F)] - \lambda P_F(|A_t| = 1)| \leqq P_F(|A_t| > 1)$ for all t. By (5.3), $\lim_{t\to\infty} P_F(|A_t| = 1) = 1$, so the result follows.

For the proof of part (b), let ν be any probability measure on S. If $\nu U_t \to \lambda\nu_0 + (1-\lambda)\nu_1$, then in particular, $\nu U_t\{\eta(i) = 0\} \to \lambda$ as $t \to \infty$, so (5.7) follows from (5.2) with $F = \{i\}$. Conversely, suppose (5.7) holds and take $F \in \mathscr{T}$, $F \neq \varnothing$. Let $\tau = \inf\{t > 0 : |A_t| = 1\}$. By (5.3), $\tau < \infty$ with probability one. By (5.2),

$$\nu U_t[B(F)] = \sum_{A\in\mathscr{T}} P_F(\tau > t, A_t = A)\nu[B(A)] + \sum_{i\in I} \int_0^t P_F(\tau \in ds, A_s = \{i\}) \sum_{j\in I} p_{t-s}(i,j)\nu\{\eta(j) = 0\}\,.$$

Therefore $\lim_{t\to\infty} \nu U_t[B(F)] = \lambda$ follows from the bounded convergence theorem. Since this holds for all $F \neq \varnothing$, $\nu U_t \to \lambda\nu_0 + (1-\lambda)\nu_1$.

For the remainder of the section, we will assume that Case II holds. For $0 \leqq \rho \leqq 1$, let ν_ρ be the product measure on S with $\nu_\rho\{\eta(i) = 1\} = \rho$ for all i.

(5.8) THEOREM.

(a) *For* $0 \leqq \rho \leqq 1$, $\mu_\rho = \lim_{t\to\infty} \nu_\rho U_t$ *exists and* $\mu_\rho \in \mathscr{I}$.

(b) $|\mu_\rho[B(F)] - (1-\rho)^{|F|}| \leqq g(F)$ *for all* $F \in \mathscr{T}$, *so in particular* $\mu_\rho\{(i) = 1\} = \rho$ *for all* i.

(c) *In the translation invariant case,* μ_ρ *is translation invariant and ergodic.*

PROOF. By (5.2),

$$\nu_\rho U_t[B(F)] = E_F[(1-\rho)^{|A_t|}] \tag{5.9}$$

for all $F \in \mathscr{T}$. Since $|A_t|$ is nonincreasing, $\lim_{t\to\infty} |A_t|$ exists. Therefore $\lim_{t\to\infty} \nu_\rho U_t[B(F)]$ exists. Since the space of probability measures on S is compact and $\{B(F) : F \in \mathscr{T}\}$ is a determining class, it follows that $\lim_{t\to\infty} \nu_\rho U_t$ exists. By (5.9)

$$|\nu_\rho U_t[B(F)] - (1-\rho)^{|F|}| \leqq P_F(|A_t| < |F|)\,,$$

so (b) follows by letting $t \to \infty$. In the translation invariant case, $\nu_\rho U_t$ is translation invariant for each t, and therefore μ_ρ is also. In order to show that μ_ρ is ergodic, construct b.p.i.'s A_t, A_t^1, and A_t^2 on the same probability space with initial states $F_1 \cup F_2$, F_1, and F_2 respectively, such that A_t^1 and A_t^2 are independent and $A_t = A_t^1 \cup A_t^2$ for $t < \tau = \inf\{t > 0 : A_t^1 \cap A_t^2 \neq \varnothing\}$. Then by (5.9), since $|A_t| = |A_t^1| + |A_t^2|$ for $t < \tau$,

$$\begin{aligned} |\mu_\rho[B(F_1 \cup F_2)] &- \mu_\rho[B(F_1)]\mu_\rho[B(F_2)]| \\ &= \lim_{t\to\infty} |E[(1-\rho)^{|A_t|} - (1-\rho)^{|A_t^1|}(1-\rho)^{|A_t^2|}]| \\ &\leqq P(\tau < \infty) \leqq \textstyle\sum_{i\in F_1, i\in F_2} g(\{i,j\})\,. \end{aligned}$$

Now replace F_2 by $F_2 + i$ and use part (d) of Lemma (5.4) to conclude that $\lim_{i\to\infty} \mu_\rho[B(F_1 \cup (F_2 + i))] = \mu_\rho[B(F_1)]\mu_\rho[B(F_2)]$ and thus complete the proof.

(5.10) THEOREM. $\mathscr{I}$ *is the closed convex hull of* $\{\mu_\rho : 0 \leqq \rho \leqq 1\}$.

PROOF. Let μ be any measure in $\mathscr{I}$, and define h on $\mathscr{T}$ by $h(F) = \mu[B(F)]$. Then by (5.2) and the fact that $\mu \in \mathscr{I}$,

$$h(F) = \sum_{A\in\mathscr{T}} P_F(A_t = A)h(A) \tag{5.11}$$

for all $F \in \mathscr{T}$. We will prove first that h almost depends on F only through $|F|$, in the sense that there is a sequence $\rho_n \geqq 0$ with the property that

$$(5.12) \qquad |h(F) - \rho_n| \leqq g(F)$$

whenever $|F| = n$. By assumption (5.1) and (5.11) applied to $F = \{i\}$, $h(\{i\})$ is independent of i, so ρ_1 can be defined satisfying (5.12). Fix $n > 1$, and consider h to be a function on I^n via $h(i_1, \cdots, i_n) = h(\{i_1, \cdots, i_n\})$. Let V_t be the semi-group corresponding to the Markov chain $(X_1(t), \cdots, X_n(t))$ on I^n. Applying (5.11) to $F = \{i_1, \cdots, i_n\}$ where $n = |F|$ yields

$$(5.13) \qquad |h(F) - V_t h(i_1, \cdots, i_n)| \leqq g(F)$$

for $t \geqq 0$. Therefore $|V_s h - V_{t+s} h| \leqq V_s g$, so since $V_s g \to 0$ by (c) of Lemma (5.4), $\lim_{s\to\infty} V_s h$ exists and is a bounded harmonic function for $(X_1(t), \cdots, X_n(t))$. By assumption (5.1) and the continuous time version of Lemma (3.14) of [6], $\lim_{s\to\infty} V_s h$ is a constant on I^n, which we will call ρ_n. Inequality (5.12) now follows by letting $t \to \infty$ in (5.13). The next step in the proof is to show that there exists a probability measure $\gamma(d\rho)$ on [0, 1] so that

$$(5.14) \qquad \rho_n = \int_0^1 (1 - \rho)^n \gamma(d\rho) .$$

As is well known, a necessary and sufficient condition for this is that for $k, m \geqq 0$, $\sum_{r=0}^{m} \binom{m}{r}(-1)^r \rho_{k+r} \geqq 0$. For such a k and m. By (c) of Lemma (5.4), there exists a sequence F_n of sets of cardinality $k + m$ so that $\lim_{n\to\infty} g(F_n) = 0$. Let G_n be a subset of F_n of cardinality k. By (5.12), $\lim_{n\to\infty} \mu\{\eta : \eta(i) = 0$ for $i \in G_n$ and $\eta(i) = 1$ for $i \in F_n \backslash G_n\} = \sum_{r=0}^{m} \binom{m}{r}(-1)^r \rho_{k+r}$, which gives the required conclusion. By (5.9), $\mu_\rho[B(F)]$ is continuous in ρ for each $F \in \mathscr{T}$, so that a probability measure $\tilde{\mu}$ on S can be defined by $\tilde{\mu} = \int_0^1 \mu_\rho \gamma(d\rho)$. It only remains to show that $\mu = \tilde{\mu}$, since then it follows that μ is in the closed convex hull of $\{\mu_\rho : 0 \leqq \rho \leqq 1\}$. In order to do this, let $\tilde{h}(F) = \tilde{\mu}[B(F)]$, and note that from (b) of Theorem (5.8) and (5.14) it follows that $|\tilde{h}(F) - \rho_n| \leqq g(F)$ whenever $|F| = n$. Therefore by (5.12) $|\tilde{h}(F) - h(F)| \leqq 2g(F)$ for all $F \in \mathscr{T}$. Since both $\tilde{h}$ and h are invariant functions for the b.p.i., $\tilde{h} = h$ follows from this and (b) of Lemma (5.4).

(5.15) COROLLARY. $\mathscr{I}_e = \{\mu_\rho : 0 \leqq \rho \leqq 1\}$.

PROOF. By Theorem (5.10), $\mathscr{I}_e \subset \{\mu_\rho : 0 \leqq \rho \leqq 1\}$. Therefore it suffices to prove that if $\gamma(d\lambda)$ is a probability measure on [0, 1] and $\mu_\rho = \int_0^1 \mu_\lambda \gamma(d\lambda)$, then γ is the point mass at ρ. For any fixed $k \geqq 1$, let F_n be a sequence of sets of cardinality k for which $g(F_n) \to 0$. Then by (b) of Theorem (5.8), $\lim_{n\to\infty} \mu_\lambda[B(F_n)] = (1 - \lambda)^k$, so $(1 - \rho)^k = \int_0^1 (1 - \lambda)^k \gamma(d\lambda)$. Since this is true for all $k \geqq 1$, it follows that γ is the point mass at ρ.

(5.16) THEOREM. *Let ν be any probability measure on S. Then $\nu U_t \to \mu_\rho$ as $t \to \infty$ if and only if ν satisfies*

$$(5.17) \qquad \lim_{t\to\infty} \sum_j p_t(i, j)\nu\{\eta(j) = 0\} = 1 - \rho ,$$

and

$$\lim_{t\to\infty} \textstyle\sum_{j,k} p_t(i,j)p_t(i,k)\nu\{\eta(j)=0, \eta(k)=0\} = (1-\rho)^2 \quad \textit{for all } i \in I\,. \tag{5.18}$$

PROOF. Assume first that $\nu U_t \to \mu_\rho$. An application of (5.2) with $F = \{i\}$ gives $\nu U_t\{\eta(i)\} = 0\} = \sum_j p_t(i,j)\nu\{\eta(j) = 0\}$, so (5.17) follows immediately from the fact that $\mu_\rho\{\eta(i) = 0\} = 1 - \rho$. On the other hand, applying (5.2) to $F = \{n, m\}$ where $n \neq m$ yields $|\nu U_t[B(F)] - \sum_{j,k} p_t(n,j)p_t(m,k)\nu\{\eta(j) = 0, \eta(k) = 0\}| \leqq 2g(F)$, so (b) of Theorem (5.8) gives

$$\begin{aligned}|(1-\rho)^2 - \textstyle\sum_{j,k} p_t(n,j)p_t(m,k)\nu\{\eta(j) = 0, \eta(k) = 0\}|& \\ \leqq 3g(F) + |\nu U_t[B(F)] - \mu_\rho[B(F)]|\,.&\end{aligned}$$

Substituting $X_1(s)$ and $X_2(s)$ for n and m where $X_1(0) = X_2(0) = i$, and then taking expected values yields

$$\begin{aligned}&|(1-\rho)^2 - \textstyle\sum_{j,k} p_{t+s}(i,j)p_{t+s}(i,k)\nu\{\eta(j) = 0, \eta(k) = 0\}| \\ &\quad\leqq 3Eg(\{X_1(s), X_2(s)\}) + P(X_1(s) = X_2(s)) + E|(\nu U_t - \mu_\rho)[B\{X_1(s), X_2(s)\}]|\,.\end{aligned}$$

To complete the proof of (5.18), use (c) of Lemma (5.4) to show that s can be taken so large as to make the first two terms on the right as small as desired, and then use the assumption that $\nu U_t \to \mu_\rho$ to make the last term small by taking t large. For the converse, assume that (5.17) and (5.18) hold. We will show that $\nu U_t[B(F)] \to \mu_\rho[B(F)]$ for every $F \in \mathscr{T}$. By (5.2),

$$\begin{aligned}\nu U_t[B(F)] - \nu_\rho U_t[B(F)] &= E_F\{\nu[B(A_t)] - (1-\rho)^{|A_t|}\} \\ &= \textstyle\sum_{k=1}^{|F|} E_F\{\nu[B(A_t)] - (1-\rho)^k, |A_t| = k\}\,.\end{aligned}$$

In order to prove that this tends to zero as $t \to \infty$, it suffices to prove that

$$\lim_{t\to\infty} E_F\{\nu[B(A_t)], |A_t| = |F|\} = (1-\rho)^{|F|}[1 - g(F)] \tag{5.19}$$

for all $F \in \mathscr{T}$, since then the term corresponding to $k = |F|$ tends to zero, from which it follows by an appropriate application of the strong Markov property that the other terms tend to zero also. From (5.17) and (5.18), it follows that

$$\textstyle\int_S [\sum_j p_t(i,j)\{1 - \eta(j)\} - (1-\rho)]^2\, d\nu \to 0\,,$$

and therefore that $\sum_j p_t(i,j)[1 - \eta(j)] \to (1-\rho)$ in probability relative to ν. Therefore

$$E\{\nu[B\{X_1(t), \cdots, X_n(t)\}]\} = E \textstyle\int \prod_{i=1}^n [1 - \eta(X_i(t))]\, d\nu \to (1-\rho)^n \tag{5.20}$$

for all choices of initial points $X_1(0), \cdots, X_n(0)$. In order to deduce (5.19) from (5.20), write

$$E_F\{\nu[B(A_{t+s})], |A_{t+s}| = |F|\} = \textstyle\sum_{|A_1|=|A_2|=|F|} P_F(A_t = A_1)P_{A_1}(A_s = A_2)\nu[B(A_2)]\,,$$

and use part (b) of Lemma (5.4).

The proof of the following corollary is the same as that of Theorem (5.6) of [6].

(5.21) COROLLARY. *In the translation invariant case, if ν is a translation invariant ergodic probability measure on S, then $\nu U_t \to \mu_\rho$ where $\rho = \nu\{\eta(i) = 1\}$.*

We conclude this section with a description of $\mathscr{I}_e$ when assumption (5.1) does not hold. Let $\mathscr{H}$ be the set of function α on I such that $0 \leqq \alpha \leqq 1$ and $\sum_j p_t(i, j)\alpha(j) = \alpha(i)$. Thus assumption (5.1) is just the statement that $\mathscr{H}$ consists only of the constants between zero and one. For $\alpha \in \mathscr{H}$, $\alpha(X_1(t))$ is a bounded martingale, so $\lim_{t\to\infty} \alpha(X_1(t))$ exists with probability one. Recall that $g(X_1(t), X_2(t))$ is a bounded supermartingale, so its limit exists also. It is not hard to see that $\lim_{t\to\infty} g(X_1(t), X_2(t)) = 0$ or 1 with probability one. Also, $|\alpha(i) - \alpha(j)| \leqq 1 - g(i, j)$ for all $i, j \in I$ and $\alpha \in \mathscr{H}$, so $\lim_{t\to\infty} \alpha(X_1(t)) = \lim_{t\to\infty} \alpha(X_2(t))$ a.s. on $\{\omega: \lim_{t\to\infty} g(X_1(t), X_2(t)) = 1\}$. Let $\mathscr{H}^*$ be the set of all $\alpha \in \mathscr{H}$ for which $\lim_{t\to\infty} \alpha(X_1(t)) = 0$ or 1 on $\{\omega: \lim_{t\to\infty} g(X_1(t), X_2(t) = 1\}$. Then $\mathscr{I}_e$ can be described in terms of $\mathscr{H}^*$ in the following way.

(5.22) THEOREM. *For every $\alpha \in \mathscr{H}^*$, there is a $\mu_\alpha \in \mathscr{I}_e$ such that $\mu_\alpha\{\eta(i) = 1\} = \alpha(i)$. Furthermore, $\mathscr{I}_e = \{\mu_\alpha: \alpha \in \mathscr{H}^*\}$.*

Note that this is consistent with our results under assumption (5.1), since then Case I corresponds to having $\mathscr{H}^* = \{0, 1\}$ and Case II corresponds to having $\mathscr{H}^* = \mathscr{H} = [0, 1]$.

An example of the application of Theorem (5.22) when assumption (5.1) is not satisfied is the following. Let I be the integers, $c_i = 1$ for all i, $N_{i,1} = \{i - 1\}$, $N_{i,2} = \{i + 1\}$, $f_0(1) = f_0(2) = \frac{1}{2}$, $f_i(1) = f_{-i}(2) = q$ for $i > 0$, and $f_i(1) = f_{-i}(2) = p$ for $i < 0$, where $p + q = 1$ and $p > \frac{1}{2}$. Then there are four extreme invariant measures for the proximity process. Two of them are ν_0 and ν_1, a third satisfies $\lim_{i\to-\infty} \mu\{\eta(i) = 0\} = 0$ and $\lim_{i\to+\infty} \mu\{\eta(i) = 0\} = 1$, while the fourth is obtained from the third by reflection about the origin. Moreover, the last two are concentrated on the set $\{\eta: \eta(i) \pm \eta(i + 1)$ for exactly one $i\}$, and the distribution of the i for which $\eta(i) \pm \eta(i + 1)$ is easily computed.

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