WEAK COMPARATIVE PROBABILITY ON INFINITE SETS

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Let $\mathcal{S}$ be a Boolean algebra of subsets of a state space $S$ and let $\succ$ be a binary comparative probability relation on $\mathcal{S}$ with $A \succ B$ interpreted as "$A$ is more probable than $B$." Axioms are given for $\succ$ on $\mathcal{S}$ which are sufficient for the existence of a finitely additive probability measure $P$ on $\mathcal{S}$ which has $P(A) > P(B)$ whenever $A \succ B$. The axioms consist of a necessary cancellation or additivity condition, a simple monotonicity axiom, an axiom for the preservation of $\succ$ under common deletions, and an Archimedean condition.

1. Introduction and main theorem. Throughout, $S$ is a non-empty set of states [13], $\mathcal{S}$ is a Boolean algebra of subsets of $S$ which contains $S$, $\emptyset$ is the empty set, and $\succ$ ("is more probable than") is an asymmetric comparative probability relation on $\mathcal{S}$ with symmetric complement $\sim$, so that $A \sim B$ if neither $A \succ B$ nor $B \succ A$. A finitely additive probability measure $P$ on $\mathcal{S}$

weakly agrees with $\succ$ iff $A \succ B \implies P(A) > P(B)$,

almost agrees with $\succ$ iff $P(A) > P(B) \iff A \succ B$,

for all $A, B \in \mathcal{S}$, and strictly agrees with $\succ$ iff it weakly agrees and almost agrees with $\succ$. The relation $\succ$ is transitive under strict agreement and noncyclic under weak agreement, but it can cycle under almost agreement as when $A \succ B \succ C \succ A$ and $P(A) = P(B) = P(C)$. On the other hand, $\sim$ is transitive (hence an equivalence) under almost agreement or strict agreement, but need not be transitive under weak agreement. Nontransitivity of $\sim$ accommodates Savage’s notion of vagueness in judgments of personal probabilities, as when small successive but accumulating differences between events $A_1, A_2, \ldots, A_n$ give $A_1 \sim A_2 \sim \cdots \sim A_n$, along with $A_n \succ A_1$, and interest in the notion of weak agreement has been expressed by several writers [2, 4, 7, 14, 15, 17]. The purpose of the present paper is to provide axioms for $\succ$ which imply the existence of a weakly agreeing measure when $\mathcal{S}$ is infinite.

Kraft, Pratt and Seidenberg [10] and others [4, 16] present axioms for $\succ$ which are necessary and sufficient for strict agreement when $\mathcal{S}$ is finite, and Fishburn [4] and Domotor and Stelzer [2] axiomatize weak agreement and intermediate cases when $\mathcal{S}$ is finite. Moreover, when $\mathcal{S}$ is finite with atoms $a_1, \ldots, a_n$, so that $A \in \mathcal{S}$ iff $A = \emptyset$ or $A$ is the union of one or more $a_i$, the method of these papers shows that $\mathcal{S}$ has an almost agreeing measure if, and only if, there is no finite sequence $\{(A_k, B_k)\}_{k=1}^n$ of event pairs for which $A_k \succ B_k$ or $A_k \sim B_k$ for all

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$k$ and the number of $B_i$ which include $a_i$ exceeds the number of $A_i$ which include $a_i$ for every $i \in \{1, \cdots, n\}$. Sufficient conditions for strict agreement when $S$ is infinite are given by Koopman [8, 9], Savage [13], Luce [11], Fine [3], and Narens [12], among others, and Savage [13, pages 34–35] gives conditions (including transitivity of $>$) which are sufficient for almost agreement but not for strict agreement when $S$ is infinite. A more recent almost agreeing axiomatization for arbitrary $S$ is given by Narens [12].

An important omission from prior work is the absence of easily interpreted conditions for $>$ which imply the existence of a weakly agreeing probability measure without also implying the existence of a strictly agreeing measure when $\mathcal{S}$ is infinite. The following theorem, proved in the next section, is an attempt to remedy this omission. For every $A \in \mathcal{S}$, $A': S \to \{0, 1\}$ is the indicator function for $A$ with $A'(s) = 1$ iff $s \in A$; $A' \cap B = \{s: s \in A$ and $s \notin B\}$; and a partition of a subset of $S$ is an $\mathcal{S}$ partition iff every set in the partition is in $\mathcal{S}$.

**Theorem 1.** There exists a finitely additive probability measure on $\mathcal{S}$ that weakly agrees with $>$ if the following hold for all $A, B, C, A_i, B_i \in \mathcal{S}$ and all positive integers $n$:

(A1) $(A_i \geq B_i$ and $A_i \cap B_i = \emptyset$ for $i = 1, \cdots, n) \Rightarrow \Sigma_{i=1}^n A'_i \neq \Sigma_{i=1}^n B'_i$.

(A2) $(A \geq B \supseteq C$ or $A \supseteq B \supset C) \Rightarrow A \geq C$.

(A3) $(A \geq B$ and $C \subseteq A \cap B) \Rightarrow A \cap C \geq B \cap C$.

(A4) $A \geq B \Rightarrow$ there is a finite $\mathcal{S}$ partition of $S$ such that $A \geq B \cup C$ for every set $C$ in the partition.

Axiom (A1) is an additivity condition which, since $\Sigma A'_i = \Sigma B'_i = \Sigma P(A_i) = \Sigma P(B_i)$, is necessary for weak agreement: (A1) and (A3) forbid $>$ cycles but do not imply that $>$ is transitive. Axiom (A2) is an appealing monotonicity condition for $>$ preservation under inclusion. Axiom (A3) says that $>$ is preserved under removal of a subset $C$ included in both $A$ and $B$. It seems psychologically realistic since if $A$ is judged to be more probable than $B$ then the bases for this judgment should be even more evident when $C$ is removed from $A$ and $B$. Axioms (A1), (A2) and (A3) are sufficient [4] for weak agreement when $\mathcal{S}$ is finite, but neither (A2) nor (A3) is necessary. Kraft, Pratt and Seidenberg [10] show that some condition like (A1) is required in the general finite context, but some strict-agreement axiomatizations [11, 13] with infinite $\mathcal{S}$ avoid the complexities of (A1) by using weak or simple orders along with strong structural presuppositions.

Axiom (A4), used elsewhere [5, page 195] in a characterization of Savage's strict-agreement axioms, in an Archimedean condition suggested by de Finetti [1] and Savage [13]. It is stronger than necessary since, in conjunction with the other axioms, it requires $A \sim \emptyset$ for every atom $A \in \mathcal{S}$, and when $S \sim \emptyset$ it forces $S$ to be infinite. However, I have not been able to obtain weak agreement under (A1), (A2) and (A3) with the use of a more palatable Archimedean axiom and invite others to attempt to remedy this shortcoming of the axiomatization.
For examples in which the axioms hold but do not imply strict agreement when $S$ is countable, let $S$ be the set of all rational numbers in $[0, 1]$, let $\mathcal{S}$ be the algebra consisting of $\emptyset$ and all finite unions of intervals in $S$, and for each $A \in \mathcal{S}$ let $\mu(A)$ be the Lebesgue measure of the closure of $A$ in $[0, 1]$. Two simple models which satisfy the axioms are $A \succ B$ iff $\mu(A) > \lambda \mu(B)$ with $\lambda \geq 1$, and $A \succ B$ iff $\mu(A) > \mu(B) + \delta$ with $\delta \geq 0$. In the latter case another weakly agreeing measure for $\delta = \frac{1}{2}$ is $P(A) = \frac{2}{3} \mu(A) + \frac{1}{3} A'(s)$ with $s$ any fixed point in $S$.

2. Proof of Theorem 1. My proof of Theorem 1 is based on Hausner and Wendel’s theorem [6] for real lexicographic representations of ordered vector spaces. We call $(V, >)$ an ordered vector space when $V$ is a real vector space with origin $\theta$, $>$ is a linear order (irreflexive, transitive, complete) on $V$ and, for all $x, y \in V$ and $\lambda \in \mathbb{R}$: (i) $x > \theta$ and $\lambda > 0 \Rightarrow \lambda x > \theta$, (ii) $x > \theta$ and $y > \theta \Rightarrow x + y > \theta$, (iii) $x > y$ iff $x - y > \theta$. The positive cone $V^+ = \{x \in V : x > \theta\}$ completely describes $>$. 

Let $(V, >)$ be an ordered vector space and define binary relations $\geq$ and $\approx$ on $V^+$ by $x \geq y$ iff $x - \lambda x > \lambda y$ for all $\lambda > 0$, and $x \approx y$ iff $\lambda x > y > \mu x$ for some $\lambda, \mu > 0$. Then $\approx$ is an equivalence and, with $[x]$ the equivalence class in $V^+/\approx$ which contains $x \in V^+$, the relation $<_{\mathfrak{a}}$ on $V^+/\approx$, defined by $[x] <_{\mathfrak{a}} [y]$ iff $x > y$, is a linear order on $V^+/\approx$. A set $W \subseteq V$ is Archimedean iff $x, y \in W \Rightarrow \lambda x - y \in W$ and $y - \mu x \in W$ for some $\lambda, \mu > 0$. The classes in $V^+/\approx$ are the maximal Archimedean sets in $V^+$.

A function $F : V \rightarrow U$, where $U$ also is a real vector space, is linear iff $F(\lambda x + \mu y) = \lambda F(x) + \mu F(y)$ for all $x, y \in V$ and $\lambda, \mu \in \mathbb{R}$.

Theorem 2 (Hausner and Wendel). Let $(V, >)$ be an ordered vector space with $T = V^+/\approx$ and $[x] <_{\mathfrak{a}} [y]$ iff $x \geq y$. Define $(V_T, >_{\mathfrak{l}})$ as the ordered vector space of all real-valued functions on $T$ which are nonzero on at most a well ordered subset of $(T, <_{\mathfrak{a}})$, with $f >_{\mathfrak{l}} g$ when $f, g \in V_T$ iff $f \neq g$ and $f(t) > g(t)$ for the first $t$ in $T$ at which $f(t) \neq g(t)$. Select $e_t \in t$ for each $t \in T$ and define $f_t \in V_T$ by $f_t(s) = 1$ and $f_t(s) = 0$ for all $s \in T \setminus \{t\}$. Then there exists a linear $F : V \rightarrow V_T$ with $F(e_t) = f_t$ for all $t \in T$ such that $x > y$ iff $F(x) >_{\mathfrak{l}} F(y)$, for all $x, y \in V$.

Henceforth, let $V$ be the real vector space of all real-valued functions on $S$, let $V_0 = \{A' - B' : A, B \in \mathcal{S}$ and $A \succ B\}$ and let $V_1 = \{\sum_{i=1}^n \lambda_i x_i : n \in \{1, 2, \ldots\}, \lambda_i > 0$ and $x_i \in V_0\}$, the convex cone in $V$ generated by $V_0$. We presume axioms (A1) through (A4) and $S \succ \emptyset$, for otherwise $V_0 = \emptyset$ by (A2).

Lemma 1. $\theta \notin V_1$ and $V_1$ is Archimedean.

Proof. Suppose $\theta \in V_1$ with $A_i, B_i \in \mathcal{S}, A_i \succ B_i$ and $\lambda_i > 0$ for $i = 1, \ldots, n$, and $\sum \lambda_i (A'_i - B'_i) = \theta$. Using (A3), $A_i \cap B_i = \emptyset$ can be assumed without loss of generality. Since $A'_i(s) - B'_i(s) \in \{1, 0, -1\}$ for all $i$ and $s$, $\sum \lambda_i (A'_i - B'_i) = \theta$ is tantamount to a finite system $(\lambda_1, \ldots, \lambda_n) \cdot p_i = 0$ for a subset of $p_i$ in $\{1, 0, -1\}^n$. Since the $p_i$ are integral vectors there are integral $\lambda_i^* > 0$ such that $\sum \lambda_i^*(A'_i - B'_i) = \theta$. Then $\lambda_i^*$ replications of $(A_i, B_i)$ gives $\sum_{i=1}^n C'_i = \sum_{i=1}^n D'_i$
with \( C_i > D_i \) for \( i = 1, \ldots, m \) (= \( \sum \lambda_i^x \)) and \( C_i \cap D_i = \emptyset \) for each \( i \). But this contradicts (A1). Hence \( \theta \notin V_1 \).

To show that \( V_1 \) is Archimedean suppose first that \( A > B \). Using (A3), we can presume that \( A \cap B = \emptyset \). Then, using (A2) and (A4), there are partitions \( \{C_i\}_{i=1}^{n} \) of \( A \) and \( \{D_j\}_{j=1}^{m} \) of \( S \setminus B \) such that \( A > B \cup C_i \) and \( A > B \cup D_j \) for all \( i \) and \( j \), so that \( A' - B' - C_i' \in V_1 \) and \( A' - B' - D_j' \in V_1 \) for all \( i \) and \( j \). Addition over all \( i \) and \( j \) then gives \( (n + m)(A' - B') - (S \setminus B)' = (n + m - 1)(A' - B') - S' \in V_1 \) with \( n + m - 1 > 0 \), so that \( N(A' - B') - S' \in V_1 \) for positive \( N \). By an analogous procedure (given \( S \supset \emptyset \)), partitions of \( A \) and \( S \setminus B \) lead to \( MS' - (A' - B') \in V_1 \) for some positive \( M \).

Therefore, if \( A > B \) and \( C > D \), \( N(A' - B') - S' \in V_1 \) and \( S' - M^{-1}(C' - D') \in V_1 \) for some positive \( N \) and \( M \) so that \( NM(A' - B') - (C' - D') \in V_1 \). To complete the Archimedean proof, suppose \( x, y \in V_1 \) with \( x = \sum_{i=1}^{n} \lambda_i (A_i' - B_i') \) and \( y = \sum_{j=1}^{m} \mu_j (C_j' - D_j') \) with \( \lambda_i, \mu_j > 0 \) and \( A_i > B_i \) and \( C_j > D_j \) for all \( i \) and \( j \). Then there exists \( N \) for which \( N(A_i' - B_i') - (C_j' - D_j') \in V_1 \) for all \( i \) and \( j \). Multiplying \( N(A_i' - B_i') - (C_j' - D_j') \) by \( \lambda_i \mu_j \) and double summing over all \( i \) and \( j \), we get \( \sum_{i,j} \lambda_i \mu_j x - y \in V_1 \). This proves that \( V_1 \) is Archimedean.

To complete the proof of Theorem 1 let \( K \) be the set of all convex cones in \( V \) which include \( V_1 \), contain \( A' \) for every nonempty \( A \in \mathcal{S} \), and do not contain \( \theta \). Using (A1), (A2) and (A3) it is easily checked that \( K \neq \emptyset \). Zorn’s lemma then implies that \( K \) contains a maximal such cone, say \( V' \). Defining \( x > y \) iff \( x - y \in V' \), \( (V, >) \) is easily seen to be an ordered vector space. Let \( F : V \to V_T \) be as given by Theorem 2. Since \( V_1 \subseteq V^+ \) and \( V_1 \) is Archimedean by Lemma 1, \( V_1 \) is included in one of the equivalence classes in \( T = V^+ / \approx \), say \( t \in T \). Since \( e_t \in t \) can be chosen as we wish let \( e_t = S' \), with \( F(S') = f_t \). It is readily seen that, with \( F_t(x) \) the value of \( F(x) \) at \( t \) for \( x \in V \), \( F_t(x) > 0 \) for all \( x \in V_1 \), and indeed for all \( x \in t \). Hence if \( A \in \mathcal{S} \) and \( A \neq \emptyset \), then \( F_t(A') > 0 \) if \( A' \in t \). Suppose however that \( A \in \mathcal{S} \), \( A \neq \emptyset \) and \( A' \notin t \). Then, since \( A' \in V^+ \), \( A' \) is in some other class in \( T \), say \( t^* \). Since \( S' - A' \) is the indicator function of \( S_1 \backslash A \), \( S' - A' \in V^+ \). Therefore, the definitions prior to Theorem 2 require \( t <_{t^*} t^* \). It then follows from Theorem 2 that \( F_t(A') = 0 \). Moreover, \( F_t(\theta) = 0 \) by linearity.

A finitely additive probability measure \( P : \mathcal{S} \to \mathbb{R} \) which weakly agrees with \( \geq \) is defined by \( P(A) = F_t(A') \) for all \( A \in \mathcal{S} \). As just noted, \( P(A) \geq 0 \) for all \( A \in \mathcal{S} \), \( P(S) = F_t(S') = f_t(\emptyset) = 1 \) by Theorem 2, and additivity for \( P \) follows from linearity for \( F_t \). Moreover, if \( A, B \in \mathcal{S} \) and \( A \geq B \) then \( A' - B' \in V_1 \) so that \( P(A) - P(B) = F_t(A') - F_t(B') = F_t(A' - B') > 0 \).

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