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EXPLICIT CONSTRUCTION OF INVARIANT MEASURES FOR A CLASS OF CONTINUOUS STATE MARKOV PROCESSES

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An explicit construction of invariant measures for a certain class of continuous state Markov processes is presented. A special version of these processes is of interest in the theory of representation of real numbers (β-expansions). Previous results of Rényi and Parry are generalized, and an open problem of Parry is resolved.

1. The stochastic properties of the Tomlinson (1971) filter in data transmission systems give rise to questions regarding the steady state properties of the process:

$$Y_{n+1} = \beta Y_n + Z_n \pmod L$$

where the Z_n's are independent identically distributed random variables, each attaining a finite number of values, $\beta > 1$ a constant and all the Y_n's assume values in an interval $(a_0, a_0 + L)$.

The process (Y_n) is a Markov process with a continuous state space. By transforming the variables, one obtains the same type of process with the interval $[0, 1]$ and $L = 1$ which will be the assumption used in what follows.

The case where $Z_n \equiv z$ (a constant) is of interest in the theory of representation of real numbers. The case $z = 0$ gives rise to the so-called β-expansions. Rényi (1957) showed in that case there exists an invariant measure for the process which is equivalent to the Lebesgue measure. Parry (1960) gave an explicit expression for that measure, and in a later paper (1964) he derived an expression for an invariant measure for the case $z \neq 0$. He did not, however, prove that his measure is nonnegative, and left this question open.

The process can be regarded as a repeated random choice of functions from a given set. Let $\phi_i(x) = \beta x + z_i \pmod 1$ be a finite set of functions, where z_i are the different values of Z. At each step n a function ϕ_{i_n} is chosen from this set according to a set of predetermined probabilities, and a transition from state x to state $\phi_{i_n}(x)$ occurs. Such a situation, for a general finite set of functions $\phi_1, \cdots, \phi_k$, was treated by Dubins and Freedman (1966). They proved that an invariant measure exists if all the ϕ_i's are continuous. Their result was extended by Yahav (1973) to the case where the ϕ_i's are general concave functions. In the present work we present an explicit formula for a density function of an invariant measure for the process. It turns out to be a Saltus function [5],

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which reduces to Parry's function in the case $k = 1$. Moreover, we show that the function is nonnegative, and thus in particular we answer a question which was posed in Parry (1964).

2. Let K mappings of R into $[0, 1)$ be given:

$$\phi_i(x) = \beta x + z_i \pmod 1 \qquad i = 1, \cdots, k\,,$$

where $z_1, \cdots, z_k, \beta$ are constants. $\beta > 1$.

With each mapping ϕ_i we associate a probability $p_i \geqq 0$, such that $p_1 + \cdots + p_k = 1$.

Let a discrete time Markov process $T(x, A)$ be given by

$$T(x, A) = \textstyle\sum_{i=1}^{k} p_i I_A(\phi_i(x)\} \tag{2.1}$$

for any Borel set A. I_A denotes the indicator function of A. For any measure μ on R,

$$(\mu T)(A) = \textstyle\sum_{i=1}^{k} p_i \mu(\phi_i^{-1}(A))\,. \tag{2.2}$$

We do not require the measures to be nonnegative.

We are interested in finding a T-invariant measure μ. Such a measure would certainly vanish outside $[0, 1)$.

3. Let g be a Saltus function [5] on R, with the sequence $y_0, y_1, \cdots$ as jump points, with jumps $v_0, v_1, \cdots$.

$$\textstyle\sum |v_n| < \infty\,,$$

i.e.

$$g(x) = \textstyle\sum_{x>y_n} v_n\,. \tag{3.1}$$

It is well known that such functions possess left and right limits at each point, and are right continuous.

Next we derive necessary conditions for a Saltus function to be a density function for an invariant measure. We note that one may assume $0 \leqq y_n \leqq 1$ for all n, since otherwise the corresponding v_n must be 0. Without loss of generality we assume $y_0 = 0$, $y_1 = 1$, $0 < y_i < 1$, $i = 2, 3, \cdots$ and $y_i \neq y_j$ for $i \neq j$.

LEMMA 3.1. *If $\mu_g(A) = \int_A g(x)\,dx$ is an invariant measure, then for every $n \geqq 2$ we have*

$$v_n = \frac{1}{\beta} \textstyle\sum_{i=1}^{k} p_i \sum_{\phi_i(y_j)=y_n} v_j\,. \tag{3.2}$$

PROOF. Let A_ε be the interval $[y_n - \varepsilon, y_n)$, where $0 < \varepsilon < y_n$. Then

$$\phi_i^{-1}(A_\varepsilon) = \textstyle\bigcup_{\phi_i(x)=y_n} \left[x - \dfrac{\varepsilon}{\beta}\,,\ x\right).$$

Using (2.2) and dividing by ε we get

$$\frac{1}{\varepsilon}\,\mu_g(A_\varepsilon) = \textstyle\sum_{i=1}^{k} p_i \sum_{\phi_i(x)=y_n} \dfrac{1}{\varepsilon}\,\mu_g\left(\left[x - \dfrac{\varepsilon}{\beta}\,,\ x\right)\right).$$

(We assume that ε is so small that for each union the intervals for different x's do not overlap.)

Sending ε to 0, and using the existence of a left limit we get:

$$g(y_n^-) = \frac{1}{\beta} \sum_{i=1}^k p_i \sum_{\phi_i(x)=y_n} g(x^-) .$$

Similarly, we get that the same formula is satisfied for $g(y_n^+)$ in terms of the $g(x^+)$'s. Finally by subtracting, and noting that

$$\begin{aligned} g(x^+) - g(x^-) &= 0 \qquad \text{if} \quad x \neq y_i \\ &= v_i \qquad \text{if} \quad x = y_i \qquad\qquad i = 0, 1, \cdots \end{aligned}$$

we get (3.2).

Next we rewrite (3.2) in the following form:

$$\beta v_n = \sum_{j=2}^\infty a_{nj} v_j + a_{n0} v_0 + a_{n1} v_1 \qquad n = 2, 3, \cdots . \tag{3.3}$$

LEMMA 3.2. *The coefficients* a_{nj} *satisfy*:

(1) $a_{nj} \geqq 0$ *for all* n *and* j.

(2) $\sum_{n=2}^\infty a_{nj} \leqq 1; j = 0, 1, \cdots$.

PROOF. For each $j, j = 0, 1, \cdots,$

$$\begin{aligned} a_{nj} &= p_i \qquad \text{if} \quad \phi_i(y_j) = y_n \quad \text{for some} \quad i \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Thus the a_{nj} are nonnegative, and

$$\sum_{n=2}^\infty a_{nj} \leqq \sum_{i=1}^k p_i \leqq 1 ,$$

completing the proof.

Lemma 3.2 asserts that the matrix A^{tr}, where $A = (a_{nj})$ $n, j = 2, 3, \cdots$, is substochastic. Hence, regarding A as an operator on the space of absolutely summable sequences, $\beta > 1$ cannot be an eigenvalue. This proves the following theorem:

THEOREM 3.3. *For any given* v_0 *and* v_1, *there exists at most one Saltus function with jumps* v_0 *and* v_1 *at* 0 *and* 1 *respectively, for which* μ_g *is an invariant measure for* $T(x, A)$.

4. In this section we construct a Saltus function which generates a nonnegative finite invariant measure for T. Let $[a, b)$ be any interval such that $0 \leqq a \leqq b \leqq 1$, and let c be any real number.

LEMMA 4.1. *Let* μ *be the measure having the density* $cI_{[a,b)}$, *then* μT *has the density*

$$f_{\mu T} = \frac{c}{\beta} \sum_{i=1}^k p_i \{(m + \varepsilon_i) I_{[0,1)} + (-1)^{\varepsilon_i} I_{\tau(\phi_i(a), \phi_i(b))}\} \tag{4.1}$$

where:

If $\phi_i(a) \leqq \phi_i(b)$ then $\tau(\phi_i(a), \phi_i(b)) = [\phi_i(a), \phi_i(b))$ and $\varepsilon_i = 0$. If $0 < \phi_i(b) < \phi_i(a)$ then $\tau(\phi_i(a), \phi_i(b)) = [\phi_i(b), \phi_i(a))$ and $\varepsilon_i = 1$. If $\phi_i(b) = 0$ then $\tau(\phi_i(a), \phi_i(b)) = [\phi_i(a), 1)$ and $\varepsilon_i = 0$, and finally, m is the integral part of $\beta(b - a)$.

PROOF. Let

$$\begin{aligned} D_i &= [\phi_i(a), \phi_i(b)) && \text{if } \phi_i(a) \leqq \phi_i(b)\,, \\ &= [0, \phi_i(b)) \cup [\phi_i(a), 1) && \text{otherwise.} \end{aligned}$$

One can directly verify that for any $y \in [0, 1)$ the set $\phi_i^{-1}(y) \cap [a, b)$ consists of $m + 1$ or m points, depending on whether $y \in D_i$ or $y \notin D_i$, respectively. Since μ is nonatomic, and ϕ_i^{-1} of a finite set is finite, we can consider only points y which are interior to D_i or interior to $[0, 1) - D_i$. If A_y is a small interval of length δ around y, then $\phi_i^{-1}(A_y) \cap [a, b)$ will consist of $m + 1$ or m intervals of length δ/β each, again depending on whether or not $y \in D_i$, respectively. Applying (2.2) we get $(\mu T)(A_y) = \sum_{i=1}^k p_i \delta(c/\beta)(m + \xi_i)$ where

$$\begin{aligned} \xi_i &= 0 && \text{if } y \notin D \\ &= 1 && \text{if } y \in D\,. \end{aligned}$$

Thus

$$f_{\mu T} = \frac{c}{\beta} \sum_{i=1}^k p_i\{mI_{[0,1)} + I_{D_i}\}\,.$$

The conversion to the form (4.1) is done by replacing I_{D_i} with $I_{[0,1)} - I_{[0,1)-D_i}$ whenever D_i is composed of two disjoint nonvoid intervals. This completes the proof.

Next, we define a sequence of functions:

$$\begin{aligned} (4.2) \qquad f_0 &= I_{[0,1)} \\ f_t &= \frac{1}{\beta^t} \sum_{i_1,\cdots,i_t:1\leq i_j\leq k} p_{i_1} \cdots p_{i_t}(-1)^{\varepsilon_{i_1\cdots i_t}} I_{\tau(\phi_{i_t}\cdots\phi_{i_1}(0),\phi_{i_t}\cdots\phi_{i_1}(1))} \\ & \qquad\qquad t = 1, 2, \cdots\,, \end{aligned}$$

where τ and $\varepsilon_{i_1\cdots i_t}$ are defined the same way as in Lemma 4.1.

Each f_t is a Saltus function with finite number of jumps, whose total of absolute values is bounded by $2/\beta^t$. Also $|f_t| \leqq 1/\beta^t$.

Let

$$f = \sum_{t=0}^\infty f_t$$

then f is a Saltus function for which the total of absolute values of jumps is bounded by

$$\sum_{t=0}^\infty \frac{2}{\beta^t} = \frac{2\beta}{\beta - 1}\,.$$

As in Section 3, let us denote

$$\begin{aligned} v_0 &= f(0^+) - f(0^-) = f(0) \\ v_1 &= f(1^+) - f(1^-) = -f(1^-)\,. \end{aligned}$$

LEMMA 4.2.
$$f(0) \geqq 1\,, \qquad f(1^-) \geqq 1\,.$$

PROOF.
$$f_0(0) = f_0(1^-) = 1\,.$$

For t $\geqq 1$, $f_t(0) \geqq 0$ and $f_t(1^-) \geqq 0$. This follows from the definition of τ. Since the series $\sum f_t$ is uniformly convergent, it follows that $f(x^-) = \sum_{t=0}^{\infty} f_t(x^-)$ for all x. Thus the lemma is proved.

THEOREM 4.3. *Let μ_f be the measure generated by f. Then μ_f is a finite invariant measure for T.*

PROOF. Applying Lemma 4.1 to each component of f_t we get:

$$f_{\mu_t T} = c_t f_0 + f_{t+1} \qquad\qquad c_t \text{ a constant,}$$

where μ_t is the measure generated by f_t. Summing up to N, we get:

$$(4.3)\qquad \begin{aligned} f_{(\sum_{t=0}^{N}\mu_t)T} &= \textstyle\sum_{t=0}^{N} f_{\mu_t T} = (\sum_{t=0}^{N} c_t) f_0 + \sum_{t=1}^{N+1} f_t \\ &= f_{\sum_{t=0}^{N}\mu_t} + (\textstyle\sum_{t=0}^{N} c_t - 1) f_0 + f_{N+1}\,. \end{aligned}$$

Noticing that for any finite measure μ

$$\int f_\mu = \int f_{\mu T}\,,$$

we get by integrating (4.3) that

$$|1 - \textstyle\sum_{t=0}^{N} c_t| = |\int f_{N+1}| \leqq \dfrac{1}{\beta^{N+1}}\,,$$

thus $\sum_{t=0}^{\infty} c_t = 1$. So

$$f_{(\sum_{t=0}^{\infty}\mu_t)T} = f_{\sum_{t=0}^{\infty}\mu_t} = f$$

which completes the proof of the theorem.

Theorem 4.3 yields as a special case ($k = 1$) the result stated in Theorem 6 of Parry (1964). Next we show that f is nonnegative, and thus solve in the affirmative, the question which was posed in Remark 2(a) of Parry (1964).

THEOREM 4.4. $\quad f$ *is a nonnegative function.*

PROOF. Let $f^+ = \max(f, 0)$. It is well known that if the measure generated by f is invariant, so is the measure generated by f^+. Clearly f^+ is also a Saltus function with its set of jump points being a subset of those of f. From Lemma 4.2 it follows that f^+ and f have the same jumps at 0 and at 1. Thus by Theorem 3.3, $f^+ = f$. This completes the proof.

It is worth noting that for $\beta \geqq 2$ Theorem 4.4 is quite trivial, for then:

$$f \geqq f_0 - \textstyle\sum_{t=1}^{\infty} |f_t| = 1 - \sum_{t=1}^{\infty} \dfrac{1}{\beta^t} = 1 - \dfrac{1}{\beta - 1} \geqq 0\,.$$

The same argument shows that:

(a) For $\beta > 2, f > 0$ on $[0, 1)$.
(b) $\lim_{\beta\to\infty} f = f_0$.

Thus for large β the invariant measure is approximately uniform on $[0, 1)$.

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