The Annals of Probability
1975, Vol. 3, No. 5, 815-828

ON ERRORS OF NORMAL APPROXIMATION

By R. N. Bhattacharya

University of Arizona

Let Q_n be the distribution of the normalized sum of n independent random vectors with values in R^k, and Φ the standard normal distribution in R^k. In this article the error $|\int fd(Q_n - \Phi)|$ is estimated (for essentially) all real-valued functions f on R^k which are integrable with respect to Q_n when sth moments are finite, and for which the error may be expected to go to zero. When specialized to known examples, the (main) error bound provides precise rates of convergence.

0. Introduction and summary. In this article we study rates of convergence for the classical central limit theorem. For the sake of simplicity let us assume in this section that $\{X_n : n \geqq 1\}$ is a sequence of i.i.d. random vectors with values in $R^k (k \geqq 1)$ and that

$$(0.1) \qquad EX_1 = 0\,, \qquad \text{Cov}\, X_1 = I\,, \qquad \rho_3 \equiv E\|X_1\|^3 < \infty\,.$$

Here I is the identity matrix. The classical central limit theorem asserts that the distribution Q_n of $n^{-\frac{1}{2}}(X_1 + \cdots + X_n)$ converges weakly to the standard normal distribution Φ on R^k, as $n \to \infty$. This means that

$$(0.2) \qquad \lim_{n\to\infty} |\int_{R^k} fd(Q_n - \Phi)| = 0$$

for every bounded measurable real-valued function f on R^k whose points of discontinuity form a Φ-null set. It is reasonable to expect that the rate of convergence in (0.2) will depend on the range $M_0(f)$ of f (see (1.6)) and on the average oscillation function (see (1.3))

$$(0.3) \qquad \bar{\omega}_f(\varepsilon : \Phi) = \int_{R^k} \omega_f(x, \varepsilon)\Phi(dx) \quad (\varepsilon > 0)\,.$$

Indeed, a variant of a general theorem due to Billingsley and Topsøe [9] (Theorem 1) proved in [3] (Theorem 1′) shows that in order that the relation

$$(0.4) \qquad \lim_n \sup_{f\in\mathscr{F}} |\int_{R^k} fd(P_n - \Phi)| = 0$$

be satisfied for a given class $\mathscr{F}$ of bounded Borel measurable functions on R^k and for every sequence of probability measures $\{P_n : n \geqq 1\}$ converging weakly to Φ, it is necessary as well as sufficient that one has

$$(0.5) \qquad \sup_{f\in\mathscr{F}} M_0(f) < \infty\,, \qquad \lim_{\varepsilon\downarrow 0} \sup_{f\in\mathscr{F}} \bar{\omega}_f(\varepsilon : \Phi) = 0\,.$$

The second inequality (1.11) in our theorem implies, when specialized to $r = 0$,

Received January 8, 1974; revised February 7, 1975.
AMS 1970 *subject classification.* Primary 60F05.
Key words and phrases. Central limit theorem, rates of convergence, average oscillations, Fourier transform.

$s = 3$, that one has

$$(0.6) \qquad |\textstyle\int_{R^k} f d(Q_n - \Phi)| \leqq c_1' M_0(f)\rho_3 n^{-\frac{1}{2}} + c_2' \bar{\omega}_f(c_3'\rho_3 n^{-\frac{1}{2}} \log n : \Phi) .$$

Thus it provides an effective bound for every bounded almost surely (w.r.t. Φ) continuous f (uniformly over every class $\mathscr{F}$ satisfying (0.5)). Further, (1.10) shows that the factor $\log n$ in (0.6) may be removed if one replaces $\bar{\omega}_f$ by the function (of ε)

$$(0.7) \qquad \tilde{\omega}_f(\varepsilon : \Phi) \equiv \sup_{y \in R^k} \omega_{f_y}(\varepsilon : \Phi) ,$$

where f_y is the translate of f by y (see (1.5)), so that one obtains the important inequality

$$(0.8) \qquad |\textstyle\int_{R^k} f d(Q_n - \Phi)| \leqq c_1 M_0(f)\rho_3 n^{-\frac{1}{2}} + c_2 \tilde{\omega}_f(c_3 \rho_3 n^{-\frac{1}{2}} : \Phi) .$$

The applications (2.1), (2.5) follow from (0.8). The inequality (0.6) is still useful in estimating some elusive quantities like the Prokhorov distance between Q_n and Φ (see [5], Application 4.3, pages 472–473), and error bounds for functions f for which $\bar{\omega}_f$ is small and $\tilde{\omega}_f$ is large. As special cases of (0.6), (0.8) (or (2.1), (2.5)) one can obtain virtually all known 'uniform' or Berry–Esseen type bounds. Because $M_0(f) = \infty$ if f is unbounded (and so may be $\tilde{\omega}_f(\varepsilon : \Phi)$), (0.6), (0.8) are unsuitable for unbounded f. It turns out that the proper things to look at are $M_r(f)$, $\tilde{\omega}_g(\varepsilon : \Phi_{r_0})$ defined by (1.4), (1.6), (1.7), (1,9) and (1.13), and one obtains the very general inequalities (1.10), (1.11). This takes care of all functions which are integrable with respect to Q_n under the given moment condition. Application 2 provides the simplest examples of unbounded functions (namely those which are Lipschitzian) to which (1.10) may be applied; however, the same inequality (2.7) would hold if $\tilde{\omega}_g(\varepsilon : \Phi_{r_0}) \leqq d_1 \varepsilon^\alpha (\varepsilon > 0)$, where g, Φ_{r_0} are defined by (1.13), (1.7). Perhaps of greater significance is the fact that (even for bounded f) (1.10) uses different features (of growth and average smoothness) of f for different values of r. This enables one to obtain the very general inequality (2.13). In turn this inequality yields essentially all known 'nonuniform' rates (e.g., (2.16), (2.17)) and the 'mean central limit theorem' (2.18).

References to some earlier work are given in Section 2. It should be mentioned, however, that even for the i.i.d. case and bounded f the present results are significant extensions of corresponding results in [5] (Theorems 4.1, 4.2). For general non-identically distributed random vectors the theorem improves earlier investigations [2]–[4] of the author in two directions. First, with $s = 3$, it relaxes the moment condition assumed earlier (namely, $\rho_{3+\delta} < \infty$ for some $\delta > 0$). Secondly, of course, it is much more general in scope, being able to deal with all integrable functions and yielding existing as well as new nonuniform rates.

The proof of (1.10) is based on a number of technical lemmas which are stated in Section 3 without proof. Some of these are either available in the current literature or easily deduced from them. The other lemmas are new. Detailed proofs of all lemmas will appear in [6]. To facilitate comprehension of the proof of the theorem we briefly sketch the main ideas here. If the distribution Q_1 of X_1 has an integrable characteristic function (ch. f.) $\hat{Q}_1$, then the ch. f. $\hat{Q}_n$ of Q_n is integrable for all n, and one can use Fourier inversion to obtain the density of the signed measure $Q_n - \Phi$ in terms of $\hat{Q}_n - \hat{\Phi}$. To get an estimate of the variation norm $||Q_n - \Phi||$ one may integrate the bound of the density so obtained over R^k. Although precise estimates of $\hat{Q}_n - \hat{\Phi}$ are available, integration over the unbounded domain R^k results in a loss of precision; to overcome this one also incorporates estimates of $D^\alpha(\hat{Q}_n - \hat{\Phi})$ (where α is a nonnegative integer vector and D^α is the αth derivative) in this scheme and uses the powerful Lemma 8. Since this Lemma can be used only if $\int ||x||^{k+1}Q_n(dx)$ is finite, one has to resort to truncation. Lemmas 1, 5, and 6 allow one to take care of the perturbation due to truncation, and a fairly precise estimate of $||Q_n - \Phi||$ is obtained. For integration of unbounded functions, however, one needs to estimate $\int ||x||^r|Q_n - \Phi|(dx)$, where $|Q_n - \Phi|$ is the total variation (measure) of $Q_n - \Phi$. The procedure for this is similar; one looks at the signed measure $||x||^{r_0}(Q_n - \Phi)(dx)$, where r_0 is defined by (1.9), instead of $Q_n - \Phi$. We use r_0 instead of r because $||x||^r$ is not a polynomial for odd r and the Fourier–Stieltjes transform of $||x||^r(Q_n - \Phi)(dx)$ for an odd r is not nearly as well-behaved as that for an even integer r. However, this change from r to r_0 does not entail any essential loss of generality; for one merely changes Φ_r to Φ_{r_0} (see (1.7)) and, the normal density being rapidly decreasing at infinity, this change is insignificant. In the general case (i.e., when X_1 does not have a density) we smoothen Q_n by convolving it with a smooth kernel K_ε, apply the above argument to $(Q_n - \Phi) * K_\varepsilon$ and, for final accounting, use the general Lemma 7. Although in the actual proof one uses expansions of $\hat{Q}_n$ (and $D^\alpha\hat{Q}_n$) beyond the first term $\hat{\Phi}$ for greater precision, the ideas are quite similar to those explained above.

It is noteworthy that the present method allows one to obtain analogous significant extensions of existing results on asymptotic expansions in case Q_1 has a density (as given in Bikjalis [7], Theorem 3) or when Q_1 satisfies the so-called *Cramér's condition* (as given in Bhattacharya [5], Theorem 4.3). Indeed, the derivation of such an extension in the first case using Lemma 3 is simpler (than the present proof), since, as indicated in the sketch above, the smoothing by convolution in the last step may be avoided. These new results and details of their derivations will appear in [6] and will not be discussed any further here.

1. Notation and the main result. Let $X_1, \cdots, X_n$ be n independent random vectors with values in R^k. Throughout this article we assume, without any essential loss of generality,

$$EX_j = 0 \quad (1 \leqq j \leqq n), \qquad n^{-1} \sum_{j=1}^n \operatorname{Cov} X_j = I \tag{1.1}$$

where EX_j is the expectation (vector) and $\operatorname{Cov} X_j$ the covariance matrix of X_j, and I is the $k \times k$ identity matrix. We write

$$\rho_{s,j} = E||X_j||^s \quad (1 \leqq j \leqq n), \qquad \rho_s = n^{-1} \sum_{j=1}^n \rho_{s,j} \quad (s > 0), \tag{1.2}$$

where $||\cdot||$ denotes Euclidean norm in R^k. Let f be a real-valued Borel measurable function on R^k. We define

$$(1.3)\qquad \omega_f(x, \varepsilon) = \sup\{|f(y) - f(x)| : y \in R^k, ||y - x|| < \varepsilon\} \quad (x \in R^k, \varepsilon > 0) .$$

For a given measure ν on R^k (measures and signed measures are defined on the Borel sigma-field) define

$$(1.4)\qquad \begin{aligned} \bar{\omega}_f(\varepsilon : \nu) &= \int_{R_k} \omega_f(x, \varepsilon)\nu(dx) , \\ \tilde{\omega}_f(\varepsilon : \nu) &= \sup_{y \in R^k} \bar{\omega}_{f_y}(\varepsilon : \nu) , \end{aligned}$$

where the translate f_y of f is defined by

$$(1.5)\qquad f_y(x) = f(x + y) \qquad x \in R^k .$$

For a given nonnegative integer r define

$$(1.6)\qquad \begin{aligned} M_r(f) &= \sup_{x \in R^k} (1 + ||x||^r)^{-1}|f(x)| , \qquad r > 0 , \\ M_0(f) &= \sup\{|f(x) - f(y)| : x, y \in R^k\} . \end{aligned}$$

For a given finite (signed) measure ν on R^k and for a given $r_0 \geqq 0$, define a new (signed) measure ν_{r_0} by

$$(1.7)\qquad \begin{aligned} \nu_{r_0}(dx) &= (1 + ||x||^{r_0})\nu(dx) , \qquad r_0 > 0 , \\ \nu_0 &= \nu . \end{aligned}$$

Let Q_n denote the distribution of $n^{-\frac{1}{2}} \sum_{j=1}^n X_j$ and let Φ denote the standard normal distribution on R^k. Our main result is the following.

THEOREM. *Assume*

$$(1.8)\qquad \rho_s < n^{(s-2)/2}/(8k)$$

for some integer $s \geqq 3$. *Let* r *be a nonnegative integer*, $0 \leqq r \leqq s$, *and define*

$$(1.9)\qquad \begin{aligned} r_0 &= r \qquad \text{if } r \text{ is even,} \\ &= r + 1 \qquad \text{if } r \text{ is odd.} \end{aligned}$$

There exist constants c_i, c_i' $(i = 1, 2, 3)$ *depending only on* k, r, s, *such that the inequalities*

$$(1.10)\qquad \begin{aligned} |\textstyle\int_{R^k} fd(Q_n - \Phi)| \leqq{}& c_1 M_r(f) \max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\} \\ &+ c_2 \tilde{\omega}_g(c_3 \rho_3 n^{-\frac{1}{2}} : \Phi_{r_0}) , \end{aligned}$$

and

$$(1.11)\qquad \begin{aligned} |\textstyle\int_{R^k} fd(Q_n - \Phi)| \leqq{}& c_1' M_r(f) \max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\} \\ &+ c_2' \bar{\omega}_f(c_3' \rho_3 n^{-\frac{1}{2}} \log n : \Phi) \end{aligned}$$

hold for every real-valued Borel measurable function f *on* R^k *satisfying*

$$(1.12)\qquad M_r(f) < \infty .$$

Here

$$g(x) = (1 + ||x||^{r_0})^{-1} f(x) \quad \text{if} \quad r > 0\,, \qquad = f(x) \quad \text{if} \quad r = 0\,. \tag{1.13}$$

Assumption (1.8) *may be replaced simply by*

$$\rho_3 < \infty\,, \tag{1.14}$$

if $r = 0$.

2. Applications.

2.1. Let A be a Borel subset of R^k. Take $r = 0$, $s = 3, f = I_A$ (the indicator function of A) in the theorem. Inequality (1.10) then reduces to

$$|Q_n(A) - \Phi(A)| \leqq c_1 \rho_3 n^{-\frac{1}{2}} + c_2 \sup_{y \in R^k} \Phi((\partial A)^{\varepsilon'} + y)\,, \tag{2.1}$$

where

$$\varepsilon' = c_3 \rho_3 n^{-\frac{1}{2}}\,, \tag{2.2}$$

∂A is the topological *boundary* of A and $(\partial A)^{\varepsilon'}$ is the set of all points whose distances from ∂A are less than ε'. This follows from

$$M_0(I_A) = 1\,, \qquad \omega_{I_A}(x, \varepsilon) = I_{(\partial A)^\varepsilon}(x)\,, \qquad x \in R^k\,. \tag{2.3}$$

Denoting by $\mathscr{A}_\alpha^*(d : \Phi)$ the *class of all Borel sets A satisfying*

$$\sup_{y \in R^k} \Phi((\partial A)^\varepsilon + y) \leqq d\varepsilon^\alpha\,, \qquad \varepsilon > 0\,, \tag{2.4}$$

for a given pair of positive numbers α, d, one has (from (2.1))

$$\sup_{A \in \mathscr{A}_\alpha^*(d:\Phi)} |Q_n(A) - \Phi(A)| \leqq c_1 \rho_3 n^{-\frac{1}{2}} + c_2 d(c_3 \rho_3 n^{-\frac{1}{2}})^\alpha\,, \tag{2.5}$$

whenever (1.14) holds. Examples of various classes of sets A satisfying (2.4) uniformly for $\alpha = 1$ and some d are given in [3]. Among these is the class $\mathscr{C}$ *of all Borel measurable convex subsets of* R^k. Inequalities similar to (2.1), (2.5) were first obtained independently by Von Bahr [14] and Bhattacharya [2] under somewhat more stringent moment conditions. For the special class $\mathscr{C}$ (replacing $\mathscr{A}_\alpha^*$ by $\mathscr{C}$ and α by 1) inequality (2.5) was also obtained by Sazonov [13] in the i.i.d. case.

2.2. An immediate application of (1.10) is to a function f satisfying

$$|f(x) - f(y)| \leqq d_1 ||x - y||^\alpha\,, \qquad M_r(f) < \infty\,, \qquad x, y \in R^k\,, \tag{2.6}$$

for some α, $0 < \alpha \leqq 1$, some $d_1 > 0$, and some integer r, $0 \leqq r \leqq s$. For such a function (1.10) yields

$$|\textstyle\int_{R^k} fd(Q_n - \Phi)| \leqq c_1 M_r(f) \max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\} + c_2 d_1 (c_3 \rho_3 n^{-\frac{1}{2}})^\alpha\,. \tag{2.7}$$

2.3. For an application of a different nature, let A be a Borel set and define

$$f(x) = (1 + d^s(0, \partial A)) I_{A'}(x)\,, \qquad x \in R^k\,, \tag{2.8}$$

where

$$A' = A \quad \text{if} \quad 0 \notin R^k, \qquad = R^k \backslash A \quad \text{if} \quad 0 \in R^k, \tag{2.9}$$

and $d(0, \partial A)$ is the *Euclidean distance between* 0 (the origin) *and* ∂A. Note that

$$M_s(f) \leqq 1 . \tag{2.10}$$

Taking $r = s$ in the theorem, one has

$$\begin{aligned} |g(x+y+z) - g(x+y)| &\leqq (1 + \|x+y\|^{s_0})^{-1}|f(x+y+z) - f(x+y)| + c_5\varepsilon \\ &\leqq (1 + \|x+y\|^{s_0})^{-1}(1 + d^s(0, \partial A))I_{(\partial A)^\varepsilon}(x+y) + c_5\varepsilon \\ &\leqq (1 + [d(0, \partial A) - \varepsilon]^{s_0})^{-1}(1 + d^s(0, \partial A))I_{(\partial A)^\varepsilon}(x+y) + c_5\varepsilon \\ &\leqq c_6 I_{(\partial A)^\varepsilon}(x+y) + c_5\varepsilon, \qquad \|z\| < \varepsilon,\ 0 < \varepsilon < c_7, \end{aligned} \tag{2.11}$$

for a suitable constant c_7. The constants c_5, c_6, c_7 as well as c_8—c_{13} below depend only on s and k. On integration with respect to Φ_{s_0}, (2.11) yields

$$\bar{\omega}_g(\varepsilon : \Phi_{s_0}) \leqq c_6 \sup_{y \in R^k} \Phi_{s_0}((\partial A)^\varepsilon + y) + c_8\varepsilon . \tag{2.12}$$

Hence (1.10) reduces to

$$\begin{aligned} (1 + d^s(0, \partial A))|Q_n(A) - \Phi(A)| &= |\textstyle\int_{R^k} f d(Q_n - \Phi)| \\ &\leqq c_1 \max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\} + c_9 \sup_{y \in R^k} \Phi_{s_0}((\partial A)^{\varepsilon'} + y), \end{aligned} \tag{2.13}$$

where

$$\varepsilon' = c_{10}\rho_3 n^{-\frac{1}{2}} . \tag{2.14}$$

For the class $\mathscr{C}$ of convex sets one has (see von Bahr [14], Lemmas 8, 9)

$$\sup_{C \in \mathscr{C}} \Phi_{s_0}((\partial C)^{\varepsilon'} + y) \leqq c_{11}\varepsilon' . \tag{2.15}$$

Using (2.15) in (2.13) one obtains a result announced in Rotar' [12] (Theorem 2):

$$\begin{aligned} \sup_{C \in \mathscr{C}} (1 + d^s(0, \partial C))|Q_n(C) - \Phi(C)| \\ \leqq c_{12} \max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\} . \end{aligned} \tag{2.16}$$

Taking $C = (-\infty, x]$, $x \in R^k$, one obtains

$$\begin{aligned} |F_n(x) - \Phi(x)| &\leqq c_{13}(1 + \min\{|x_i|^s : i = 1, \cdots, k\})^{-1} \\ &\quad \times \max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\}, \qquad x = (x_1, \cdots, x_k) \in R^k , \end{aligned} \tag{2.17}$$

where $F_n(\cdot)$ and $\Phi(\cdot)$ are the distributions of Q_n and Φ, respectively. For $k = 1$, (2.17) was proved by Nagaev [11] in the i.i.d. case. For $k = 1$, (2.17) immediately yields the so-called *mean central limit theorem*:

$$\begin{aligned} \|F_n - \Phi\|_p &\equiv (\textstyle\int_{R^1} |F_n(x) - \Phi(x)|^p)^{1/p} \\ &\leqq c_{14} \max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\} \end{aligned} \tag{2.18}$$

for all $p > 1/s$. Here c_{14} depends only on s and p. Inequalities like (2.18) were first obtained by Agnew [1] and Esseen [10].

3. Proof of the theorem. We shall only give a detailed proof of inequality (1.10), and outline the modifications necessary to prove (1.11). Note that all the applications above stem from (1.10).

We need some additional notation. Let $\chi_{r,j}(t)$ denote the rth *cumulant* of the random variable $\langle t, X_j\rangle$, where $\langle\ ,\ \rangle$ denotes Euclidean inner product, $t \in R^k$, and r is a positive integer. Define

$$\chi_r(it) = n^{-1}i^r \sum_{j=1}^n \chi_{r,j}(t)\,, \tag{3.1}$$
$$\tilde{P}_r(it) = \sum_{m=1}^r \frac{1}{m!}\left\{\sum{}^* \frac{\chi_{r_1+2}(it)}{(r_1+2)!} \cdots \frac{\chi_{r_m+2}(it)}{(r_m+2)!}\right\},$$

where the summation $\sum^*$ is over all m-tuples of positive integers $(r_1, \cdots, r_m)$ satisfying

$$\sum_{l=1}^m r_l = r\,. \tag{3.2}$$

Associated with the polynomials $\tilde{P}_r$ are the functions P_r defined by

$$P_r(x) = (2\pi)^{-k} \int_{R^k} \exp\{-i\langle t, x\rangle - \tfrac{1}{2}||t||^2\}\tilde{P}_r(it)\, dt\,. \tag{3.3}$$

It is easy to show that P_r is a linear combination of the standard normal density on R^k and some of its derivatives. For convenience we write

$$\tilde{P}_0(it) \equiv 1\,, \qquad t \in R^k\,, \tag{3.4}$$
$$P_0(x) = (2\pi)^{-k/2} \exp\{-\tfrac{1}{2}||x||^2\}\,, \qquad x \in R^k\,.$$

We also define *truncated* random vectors

$$Y_j = X_j \quad \text{if} \quad ||X_j|| \leqq n^{\frac{1}{2}}$$
$$= 0 \quad \text{if} \quad ||X_j|| > n^{\frac{1}{2}}\,, \tag{3.5}$$
$$Z_j = Y_j - EY_j\,, \qquad 1 \leqq j \leqq n\,.$$

Write

$$D = n^{-1} \sum_{j=1}^n \operatorname{Cov} Z_j\,, \qquad a_n = n^{-\frac{1}{2}} \sum_{j=1}^n EY_j\,, \tag{3.6}$$

and define *polynomials* $\tilde{P}_r'$ as in (3.1) with $\chi_{r,j}(t)$ replaced by the rth cumulant of $\langle t, Z_j\rangle$. If D is nonsingular, define functions P_r' by

$$P_r'(x) = (2\pi)^{-k} \int_{R^k} \exp\{-i\langle t, x\rangle - \tfrac{1}{2}\langle t, Dt\rangle\}\tilde{P}_r'(it)\, dt\,, \qquad r > 0\,, \tag{3.7}$$
$$P_0'(x) = (2\pi)^{-k/2}(\operatorname{Det} D)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\langle x, D^{-1}x\rangle\}\,, \qquad x \in R^k\,.$$

Let Q_n', Q_n'' denote the distributions of $n^{-\frac{1}{2}}(Z_1 + \cdots + Z_n)$ and $n^{-\frac{1}{2}}(Y_1 + \cdots + Y_n)$, respectively. We also write

$$\rho_r' = n^{-1} \sum_{j=1}^n E||Z_j||^r\,. \tag{3.8}$$

Finally, if D is nonsingular we let B denote the unique symmetric positive definite matrix satisfying

$$B^2 = D^{-1}\,. \tag{3.9}$$

The following series of lemmas will be needed. Detailed proofs of these will appear in the forthcoming monograph [6], although some of them are essentially proved in the literature.

LEMMA 1. *Let $\rho_s < \infty$ for some integer $s > 3$. Then one has*

$$(3.10)\qquad ||a_n|| \leqq k^{\frac{1}{2}} n^{-(s-2)/2}\rho_s\,, \qquad |\langle t, Dt\rangle - ||t||^2| \leqq 2kn^{-(s-2)/2}\rho_s \qquad t \in R^k\,,$$

and

$$(3.11)\qquad \begin{aligned} \rho_r' &\leqq 2^r\rho_r && \text{if}\quad 2 \leqq r \leqq s\,,\\ &\leqq 2^r n^{(r-s)/2}\rho_s && \text{if}\quad r > s\,. \end{aligned}$$

This type of estimate was earlier obtained by Bikjalis [7] (pages 411–412), [8] (Lemma 10).

LEMMA 2. *Let m be an integer not smaller than three. For every integer $r \leqq m$ and every nonnegative integer vector $\alpha = (\alpha_1, \cdots, \alpha_k)$ satisfying $\alpha_1 + \cdots + \alpha_k \leqq 3r$, one has*

$$|(D^\alpha \tilde{P}_r')(it)| \leqq c_{15}(1 + \rho_2'^{\,r(m-3)/(m-2)})(1 + ||t||^{3r-\alpha_1-\cdots-\alpha_k}) \cdot \rho_m'^{\,r/(m-2)}$$

where $D^\alpha = (\partial/\partial t_1)^{\alpha_1} \cdots (\partial/\partial t_k)^{\alpha_k}$ and c_{15} depends only on r, m, k, and α. If $\alpha_1 + \cdots + \alpha_k > 3r$, then $D^\alpha\tilde{P}_r'$ is identically zero.

A special case of Lemma 2 appears in Bikjalis [8] (Lemma 17).

LEMMA 3. *Suppose D is nonsingular. Let*

$$(3.12)\qquad \eta_r \equiv n^{-1}\textstyle\sum_{j=1}^n E||BZ_j||^r\,.$$

Let m be an integer not smaller than three. Then there exist two positive numbers c_{16}, c_{17} depending only on m and k such that if

$$(3.13)\qquad ||t|| \leqq c_{16} n^{(m-2)/2m}/\eta_m^{1/m}\,,$$

then

$$(3.14)\qquad \begin{aligned} &|D^\alpha[\textstyle\prod_{j=1}^n E(\exp\{\langle iBt, n^{-\frac{1}{2}}X_j\rangle\}) - \sum_{r=0}^{m-3} n^{-r/2}\tilde{P}_r'(iBt)\cdot\exp\{-\frac{1}{2}||t||^2\}]|\\ &\quad \leqq c_{17}\eta_m n^{-(m-2)/2}[||t||^{m-\alpha_1-\cdots-\alpha_k} + ||t||^{3(m-2)+\alpha_1+\cdots+\alpha_k}]\cdot\exp\{-\tfrac{1}{4}||t||^2\}\,, \end{aligned}$$

for every nonnegative integer vector $\alpha = (\alpha_1, \cdots, \alpha_k)$ satisfying $\alpha_1 + \cdots + \alpha_k \leqq m$.

Special cases of this lemma appear in Bikjalis [7] (Lemma 8), [8] (Lemma 16).

LEMMA 4. *Suppose (1.8) holds for some integer $s \geqq 3$. Let $\hat{Q}_n'$ denote the characteristic function of Q_n'. If*

$$(3.15)\qquad ||t|| \leqq n^{\frac{1}{2}}/(16\rho_3)\,,$$

then

$$|(D^\alpha\hat{Q}_n')(t)| \leqq c_{18}(1 + ||t||^{\alpha_1+\cdots+\alpha_k})\exp\{-\tfrac{5}{24}||t||^2\}$$

for every nonnegative integer vector $\alpha = (\alpha_1, \cdots, \alpha_k)$. Here c_{18} depends only on α and k.

This result is essentially due to Rotar' [12] (Lemma 7).

LEMMA 5. *Suppose* (1.8) *holds for some integer* $s \geqq 3$. *Then* D *is nonsingular, and for every integer* r, $0 \leqq r \leqq s-2$, *one has*

$$(3.16)\quad \begin{aligned} &n^{-r/2}|P_r(x) - P_r'(x)| \leqq c_{19}\rho_s n^{-(s-2)/2}(1 + ||x||^{3r+2}) \exp\{-\tfrac{1}{6}||x||^2 + ||x||\}, \\ &n^{-r/2}|P_r(x + a_n) - P_r(x)| \\ &\quad \leqq c_{20}\rho_s n^{-(s-2)/2}(1 + ||x||^{3r+1}) \cdot \exp\left\{-\tfrac{1}{2}||x||^2 + \frac{1}{8k^{\frac{1}{2}}}||x||\right\} \qquad x \in R^k, \end{aligned}$$

where c_{19}, c_{20} *depend only on* r, s, k.

LEMMA 6. *Assume* (1.8) *for some integer* $s \geqq 3$. *Recall that* Q_n'' *is the distribution of* $n^{-\frac{1}{2}}(Y_1 + \cdots + Y_n)$. *For every integer* r, $0 \leqq r \leqq s$, *there is a positive number* c_{21} *(depending only on* s, k, *and* r*) such that*

$$\int_{R^k} ||x||^r |Q_n - Q_n''|(dx) \leqq c_{21}\rho_s n^{-(s-2)/2},$$

where $|\mu|$ *denotes the total variation (measure) of a finite signed measure* μ.

LEMMA 7. *Let* μ *be a finite measure and* ν *a finite signed measure on* R^k. *Let* ε *be a positive number and* K_ε *a probability measure on* R^k *satisfying*

$$(3.17)\qquad \beta \equiv K_\varepsilon(\{x : ||x|| < \varepsilon\}) > \tfrac{1}{2}.$$

Then for each real-valued, Borel measurable bounded function f *on* R^k *one has*

$$|\int_{R^k} f d(\mu - \nu)| \leqq (2\beta - 1)^{-1}[||f||_\infty ||(\mu - \nu) * K_\varepsilon|| + \tilde{\omega}_f(2\varepsilon : |\nu|)],$$

where $||f||_\infty = \sup\{|f(x)| : x \in R^k\}$, $|\nu|$ *is the total variation of* ν, *and* $*$ *denotes convolution.*

This is proved in [5] (Lemma 2.2, inequality (2.14)). Finally one has

LEMMA 8. *Let* h *be integrable with respect to Lebesgue measure on* R^k *and satisfy*

$$\int_{R^k} ||x||^{k+1}|h(x)|\, dx < \infty.$$

Then there exists a positive constant c_{22} *depending only on* k *such that*

$$||h||_1 \leqq c_{22} \max\{||D^\beta \hat{h}||_1 : 0 \leqq \beta_1 + \cdots + \beta_k \leqq k + 1\},$$

where $||\ ||_1$ *denotes* L^1*-norm,* $\hat{h}$ *is the Fourier transform of* h *and* $\beta = (\beta_1, \cdots, \beta_k)$ *is a nonnegative integer vector.*

The above lemma is perhaps well known to analysts.

After these preliminaries we proceed to prove (1.10). The constants c_{23}—c_{47} below do not depend on anything other than r, s, k. The symbol $\int h\, d\mu$ denotes integration of h with respect to μ over the whole space R^k. The characteristic function of a probability measure Q is denoted by $\hat{Q}$.

PROOF OF INEQUALITY (1.10). Let Φ', Φ'' denote normal distributions on R^k, Φ' having mean zero and covariance D while Φ'' has mean $-a_n$ and covariance I. One has

$$(3.18)\qquad |\int f d(Q_n - \Phi)| \leqq |\int f d(Q_n - Q_n'')| + |\int f d(Q_n'' - \Phi)|.$$

By Lemma 6,

$$(3.19)\qquad |\textstyle\int fd(Q_n - Q_n'')| \leqq M_r(f) \int (1 - ||x||^r)|Q_n - Q_n''|(dx) \leqq 2c_{23}M_r(f)\rho_s n^{-(s-2)/2} .$$

Also,

$$(3.20)\qquad |\textstyle\int fd(Q_n'' - \Phi)| = |\int f_{a_n} d(Q_n' - \Phi'')| \leqq |\int f_{a_n} d(Q_n' - \Phi')| + |\int f_{a_n} d(\Phi' - \Phi)| + |\int f_{a_n} d(\Phi - \Phi'')| .$$

But, by Lemma 5 (with $r = 0$),

$$(3.21)\qquad \begin{aligned} |\textstyle\int f_{a_n} d(\Phi' - \Phi)| &\leqq M_r(f) \textstyle\int (1 + ||x + a_n||^r)|\Phi' - \Phi|(dx) \\ &\leqq M_r(f) \textstyle\int (1 + 2^r||a_n||^r + 2^r||x||^r)|\Phi' - \Phi|(dx) \\ &\leqq M_r(f)[||\Phi' - \Phi|| + 2^r||a_n||^r||\Phi' - \Phi|| \\ &\quad + 2^r \textstyle\int ||x||^r|\Phi' - \Phi|(dx)] \leqq c_{24}M_r(f)\rho_s n^{-(s-2)/2} , \\ |\textstyle\int f_{a_n} d(\Phi - \Phi'')| &\leqq M_r(f)[||\Phi - \Phi''|| + 2^r||a_n||^r||\Phi - \Phi''|| \\ &\quad + 2^r \textstyle\int ||x||^r|\Phi - \Phi''|(dx)] \leqq c_{25}M_r(f)\rho_s n^{-(s-2)/2} . \end{aligned}$$

Note that $||a_n|| \leqq \rho_s n^{-(s-2)/2} \leqq 1/(8k)$ (by Lemma 1 and (1.8)). Hence (3.18) reduces to

$$(3.22)\qquad |\textstyle\int fd(Q_n - \Phi)| \leqq c_{26}M_r(f)\rho_s n^{-(s-2)/2} + |\int f_{a_n} d(Q_n' - \Phi')| .$$

To estimate the second term on the right side of (3.22) we introduce a kernel probability measure K on R^k satisfying

$$(3.23)\qquad K(\{x : ||x|| < 1\}) \geqq \tfrac{3}{4} , \qquad \textstyle\int ||x||^{k+s+2}K(dx) < \infty , \qquad \hat{K}(t) = 0 \quad \text{if} \quad ||t|| \geqq c_{27} , \qquad t \in R^k .$$

One construction of such a probability measure is given in [5] (Lemma 3.10). For $\varepsilon > 0$ define the probability measure K_ε by

$$(3.24)\qquad K_\varepsilon(B) = K(\varepsilon^{-1}B) \qquad B \in \mathscr{B}^k,\ \varepsilon^{-1}B \doteq \{\varepsilon^{-1}x : x \in B\} .$$

Then one has, by (3.23),

$$(3.25)\qquad K_\varepsilon(\{x : ||x|| < \varepsilon\}) \geqq \tfrac{3}{4} , \qquad \hat{K}_\varepsilon(t) = 0 \quad \text{if} \quad ||t|| \geqq c_{27}/\varepsilon .$$

Now

$$\begin{aligned} &|\textstyle\int f_{a_n} d(Q_n' - \Phi')| \\ &= |\textstyle\int (1 + ||x + a_n||^{r_0})^{-1} f(x + a_n) \cdot (1 + ||x + a_n||^{r_0})(Q_n' - \Phi')(dx)| \\ &\leqq |\textstyle\int (1 + ||x + a_n||^{r_0})^{-1} f(x + a_n)(1 + ||x||^{r_0})(Q_n' - \Phi')(dx)| \\ &\quad + M_{r_0}(f) \textstyle\int |||x + a_n||^{r_0} - ||x||^{r_0}|(Q_n' + \Phi')(dx) , \end{aligned}$$

$$(3.26)\qquad \begin{aligned} &\textstyle\int |||x + a_n||^{r_0} - ||x||^{r_0}|(Q_n' + \Phi')(dx) \\ &\leqq r_0||a_n|| \textstyle\int (||x||^{r_0-1} + ||a_n||^{r_0-1})(Q_n' + \Phi')(dx) \\ &\leqq r_0\rho_s n^{-(s-2)/2}[E||n^{-\frac{1}{2}}(Z_1 + \cdots + Z_n)||^{r_0-1} + (8k)^{-r_0+1} \\ &\quad + \textstyle\int (||x||^{r_0-1} + (8k)^{-r_0+1})\Phi(dx) \\ &\quad + \textstyle\int (||x||^{r_0-1} + (8k)^{-r_0+1})|\Phi' - \Phi|(dx)] \leqq c_{28}\rho_s n^{-(s-2)/2} , \end{aligned}$$

using Lemmas 1, 4, 5 and inequality (1.8). Hence

$$(3.27)\qquad |\textstyle\int f d(Q_n - \Phi)| \leqq c_{29}(M_r(f) + M_{r_0}(f))\rho_s n^{-(s-2)/2} + |\int g_{a_n}(x)(1 + ||x||^{r_0})(Q_n' - \Phi')(dx)|$$

where $g_{a_n}(x) = g(x + a_n)$. By Lemma 7,

$$(3.28)\qquad |\textstyle\int g_{a_n}(x)(1 + ||x||^{r_0})(Q_n' - \Phi')(dx)| \leqq 2(\sup_{x \in R^k} |g(x)|)||(Q_n' - \Phi')_{r_0} * K_\varepsilon|| + 2\tilde{\omega}_g(2\varepsilon : \Phi'_{r_0}),$$

where

$$(3.29)\qquad (Q_n' - \Phi')_{r_0}(dx) = (1 + ||x||^{r_0})(Q_n' - \Phi')(dx).$$

Choose

$$(3.30)\qquad \varepsilon = 16 c_{27} \rho_3 n^{-\frac{1}{2}}.$$

By Lemma 8, writing $|\alpha|$ for the sum of the coordinates of a vector α,

$$(3.31)\qquad ||(Q_n' - \Phi')_{r_0} * K_\varepsilon|| \leqq c_{30} \sum_{|\beta_1+\beta_2| \leqq k+r_0+1} \textstyle\int |D^{\beta_1}(\hat{Q}_n' - \hat{\Phi}')(t) D^{\beta_2}\hat{K}_\varepsilon(t)|\, dt.$$

Since $D^{\beta_2}\hat{K}_\varepsilon(t) = 0$ if $||t|| > n^{\frac{1}{2}}/(16\rho_3)$, and

$$(3.32)\qquad |D^{\beta_2}\hat{K}_\varepsilon(t)| \leqq \textstyle\int \varepsilon^{|\beta_2|}|x^{\beta_2}| K(dx) \leqq c_{31},$$

where one has $x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ for a nonnegative integer vector $\alpha = (\alpha_1, \cdots, \alpha_k)$,

$$(3.33)\qquad \textstyle\int |D^{\beta_1}(\hat{Q}_n' - \hat{\Phi}')(t) \cdot D^{\beta_2}\hat{K}_\varepsilon(t)|\, dt \leqq c_{31} \int_{\{||t|| \leqq n^{\frac{1}{2}}/(16\rho_3)\}} |D^{\beta_1}(\hat{Q}_n' - \Phi')(t)|\, dt.$$

Now

$$(3.34)\qquad \begin{aligned} &\textstyle\int_{\{||t|| \leqq n^{\frac{1}{2}}/(16\rho_3)\}} |D^{\beta_1}(\hat{Q}_n' - \hat{\Phi}')(t)|\, dt \\ &\leqq \textstyle\int_{\{||t|| \leqq A_n\}} |D^{\beta_1}[\hat{Q}_n'(t) - \sum_{r'=0}^{k+s-1} n^{-r'/2}\tilde{P}'_{r'}(it)\exp\{-\frac{1}{2}\langle t, Dt\rangle\}]|\, dt \\ &\quad + \textstyle\int |D^{\beta_1}[\sum_{r'=1}^{k+s-1} n^{-r'/2}\tilde{P}'_{r'}(it)\exp\{-\frac{1}{2}\langle t, Dt\rangle\}]|\, dt \\ &\quad + \textstyle\int_{\{A_n < ||t|| \leqq A_n'\}} |D^{\beta_1}\hat{Q}_n'(t)|\, dt + \int_{\{A_n < ||t|| \leqq A_n'\}} |D^{\beta_1}\hat{\Phi}'(t)|\, dt \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

say, where (using Lemma 1)

$$(3.35)\qquad \begin{aligned} A_n &\equiv c_{32}(n^{(k+s)/2}/\rho'_{k+s+2})^{1/(k+s+2)} \geqq c_{32}(n^{(k+s)/2-(k+2)/2}/\rho_s 2^{k+s+2})^{1/(k+s+2)} \\ &= c_{33}(n^{(s-2)/2}/\rho_s)^{1/(k+s+2)}, \\ A_n' &\equiv n^{\frac{1}{2}}/(16\rho_3). \end{aligned}$$

The positive constant c_{32} is so chosen as to satisfy

$$(3.36)\qquad ||D||^{\frac{1}{2}} A_n \leqq c_{16}[n^{(k+s)/2}/(||B||^{k+s+2}\rho'_{k+s+2})]^{1/(k+s+2)}.$$

Since $||D|| \leqq \frac{5}{4}$ and $||B||^2 = ||D^{-1}|| \geqq \frac{2}{3}$ by (3.10) and (3.11), such a choice is possible (take $c_{32} = (\frac{4}{5})(\frac{2}{3})c_{16}$). By Lemma 3 we then have (using Lemma 1)

$$(3.37)\qquad I_1 \leqq c_{34}||B||^{k+s+2}\rho'_{k+s+2} n^{-(k+s)/2} \leqq c_{35}\rho_s n^{-(s-2)/2}.$$

By Lemmas 1, 2,

$$(3.38)\qquad \textstyle\int |D^{\beta_1}[n^{-r'/2}\tilde{P}'_{r'}(it)\exp\{-\frac{1}{2}\langle t, Dt\rangle\}]|\, dt \leqq c_{36} n^{-r'/2}\rho'_{r'+2} \leqq c_{36} 2^{r'+2} n^{-r'/2}\rho_{r'+2}$$

if $1 \leqq r' \leqq s-2$. If $s-2 < r' < k+s$, then

$$(3.39)\qquad \int |D^{\beta_1}[n^{-r'/2}\tilde{P}'_{r'}(it)\exp\{-\tfrac{1}{2}\langle t, Dt\rangle\}]|\,dt \leqq c_{37}n^{-r'/2}\rho'_{r'+2} \leqq c_{37}2^{r'+2}n^{-r'/2+(r'+2-s)/2}\rho_s = c_{38}n^{-(s-2)/2}\rho_s\,.$$

Hence

$$(3.40)\qquad I_2 \leqq c_{39}\max\{\rho_m n^{-(m-2)/2} : m = 3, \cdots, s\}\,.$$

By Lemma 4 and (3.35)

$$(3.41)\qquad \begin{aligned} I_3 &= \int_{\{A_n<||t||\leqq A_n'\}} |D^{\beta_1}\hat{Q}_n'(t)|\,dt \\ &\leqq c_{40}\int_{\{||t||>A_n\}} (1+||t||^{|\beta_1|})\exp\{-\tfrac{5}{24}||t||^2\}\,dt \\ &\leqq c_{40}A_n^{-(k+s+2)}\int (1+||t||^{|\beta_1|})||t||^{(k+s+2)}\exp\{-\tfrac{5}{24}||t||^2\}\,dt \\ &\leqq c_{41}\rho_s n^{-(s-2)/2}\,. \end{aligned}$$

Finally, again using Lemma 1,

$$(3.42)\qquad \begin{aligned} I_4 &= \int_{\{A_n<||t||\leqq A_n'\}} |D^{\beta_1}\hat{\Phi}'(t)|\,dt \\ &\leqq c_{42}\int_{\{||t||>A_n\}} (1+||t||^{|\beta_1|})\exp\{-\tfrac{3}{8}||t||^2\}\,dt \\ &\leqq c_{42}A_n^{-(k+s+2)}\int (1+||t||^{|\beta_1|})||t||^{k+s+2}\exp\{-\tfrac{3}{8}||t||^2\}\,dt \\ &\leqq c_{43}\rho_s n^{-(s-2)/2}\,. \end{aligned}$$

It follows that

$$(3.43)\qquad ||(Q_n' - \Phi')^* K_\varepsilon|| \leqq c_{44}\max\{\rho_{m+2}n^{-m/2} : m = 1, \cdots, s-2\}\,.$$

Next observe that by Lemma 5,

$$(3.44)\qquad \begin{aligned} &|\tilde{\omega}_g(2\varepsilon : \Phi'_{r_0}) - \tilde{\omega}_g(2\varepsilon : \Phi_{r_0})| \\ &\qquad \leqq \sup_{y\in R^k}\int \omega_{g_y}(x, 2\varepsilon)|\Phi'_{r_0} - \Phi_{r_0}|(dx) \\ &\qquad \leqq 2M_{r_0}(f)||\Phi'_{r_0} - \Phi_{r_0}|| \leqq c_{45}M_{r_0}(f)\rho_s n^{-(s-2)/2}\,. \end{aligned}$$

Using (3.43), (3.44) in (3.28) and noting that

$$(3.45)\qquad \begin{aligned} M_{r_0}(f) &= \sup_{x\in R^k}(1+||x||^{r_0})^{-1}|f(x)| \\ &\leqq M_r(f)\cdot\sup_{x\in R^k}\frac{1+||x||^r}{1+||x||^{r_0}} \leqq 2M_r(f)\,, \end{aligned}$$

we get the desired inequality (1.10). □

The proof of (1.11) differs from that of (1.10) principally in the choice of a kernel probability measure. For (1.11) one needs to choose a probability measure K' (in place of K) with compact support (i.e., assigning probability one to a compact set). This rules out the possibility of $\hat{K}'$ having a compact support (i.e., vanishing outside a compact set). However, it is necessary that $\hat{K}'$ vanishes at infinity rapidly. For such a choice see [5] (Corollary 3.1). By a different smoothing inequality than the one used to obtain (2.12) one obtains (see [5], Corollary 2.1, whose proof extends almost word for word to the present case)

$$(3.46)\qquad \begin{aligned} &\left|\int_{R^k} f_{a_n}\,d(Q_n' - \Phi')\right| \\ &\qquad \leqq \int_{R^k}(|f_{a_n}| + \omega_{f_{a_n}}(\cdot, \varepsilon))\,d|(Q_n' - \Phi') * K_\varepsilon'| + \tilde{\omega}_{f_{a_n}}(2\varepsilon : \Phi')\,, \end{aligned}$$

where K_ε' is obtained on replacing K in (3.24) by K'. One now chooses $\varepsilon = c_{46}\rho_3 n^{-\frac{1}{2}} \log n$ and proceeds with the estimation much the same way as above. One important difference is that $(\hat{Q}_n' - \hat{\Phi}')\hat{K}_\varepsilon'$ does not vanish outside $B_n = \{t : ||t|| \leqq c_{47} n^{\frac{1}{2}}/\rho_3\}$, and since the estimates of $D^\beta(\hat{Q}_n' - \hat{\Phi}')$ are available only in B_n, one has to do some extra estimation outside B_n. It is here that the fast rate of convergence to zero of $\hat{K}'$ at infinity is made use of (see [5], proof of Theorem 4.2, to get an idea of this).

REMARK. By a fairly straightforward truncation argument one can extend the theorem to the case when only ρ_2 is assumed to be finite. This leads to multidimensional extensions and refinements of Liapounov's and Lindeberg's central limit theorems. Although these refinements are new we have not derived them here for fear of overburdening the notation, particularly since the bound would then have to be expressed in terms of the tail behavior of X_j's. This will appear in [6].

Acknowledgment. The author wishes to thank Professor R. Ranga Rao for helpful discussions, and Dr. V. I. Rotar' for suggesting the possibility of deducing the third application from the theorem. Thanks are also due to Professor Charles Stone and the referee for helpful suggestions.

REFERENCES

[1] AGNEW, R. P. (1954). Global versions of the central limit theorem. *Proc. Nat. Acad. Sci.* **40** 800–804.

[2] BHATTACHARYA, R. N. (1968). Berry-Esseen bounds for the multidimensional central limit theorem. *Bull. Amer. Math. Soc.* **74** 285–287.

[3] BHATTACHARYA, N. R. (1970). Rates of weak convergence for the multidimensional central limit theorem. *Theor. Probability Appl.* **15** 68–86.

[4] BHATTACHARYA, R. N. (1971). Rates of weak convergence and asymptotic expansions for classical central limit theorems. *Ann. Math. Statist.* **42** 241–259.

[5] BHATTACHARYA, R. N. (1972). Recent results on refinements of the central limit theorem. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **2** 453–484. Univ. of California Press.

[6] BHATTACHARYA, R. N. and RAO, R. R. (to appear). *Normal Approximation and Asymptotic Expansions.*

[7] BIKJALIS, A. (1968). Asymptotic expansions of the distribution function and the density function for sums of independent identically distributed random vectors. *Litovsk. Mat. Sb.* **8** 405–422 (in Russian).

[8] BIKJALIS, A. (1971). On central limit theorem in R^k. *Litovsk. Mat. Sb.* **11** 27–58 (in Russian).

[9] BILLINGSLEY, P. and TOPSØE, F. (1967). Uniformity in weak convergence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **7** 1–16.

[10] ESSEEN, C.-G. (1958). On mean central limit theorems. *Trans. Roy. Inst. Tech. Stockholm* **121** 1–30.

[11] NAGAEV, S. V. (1965). Some limit theorems for large deviations. *Theor. Probability Appl.* **10** 214–235.

[12] ROTAR', V. I. (1970). A nonuniform estimate for the convergence speed in the multidimensional central limit theorem. *Theor. Probability Appl.* **15** 630–648.

[13] SAZONOV, V. V. (1968). On the multidimensional central limit theorem. *Sankhyā Ser. A* **30** 181–204.

[14] VON BAHR, B. (1967). Multidimensional integral limit theorems. *Ark. Mat.* **7** 71–88.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
TUCSON, ARIZONA 85721