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ON THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM IN TWO DIMENSIONS AND ITS APPLICATION

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This paper provides a generalization of the classical Berry–Esseen theorem in two dimensions. For i.i.d. random variables $\eta_1, \eta_2, \cdots, \eta_r, \cdots$ and real numbers $a_1, a_2, \cdots, a_r, \cdots$ and $b_1, b_2, \cdots, b_r, \cdots$ with $E(\eta_r) = 0$, $E(\eta_r^2) = 1$, $|a_r| \leqq 1$ and $|b_r| \leqq 1$, let $A_n^2 = \sum_{r=1}^n a_r^2$, $B_n^2 = \sum_{r=1}^n b_r^2$ and $S_n = (\sum_{r=1}^n a_r \eta_r / A_n, \sum_{r=1}^n b_r \eta_r / B_n)$.

The main result concerns the rate of convergence of the distribution function of S_n to the corresponding normal distribution function without assuming the existence of third moments. As an application of this result a theorem of P. Erdös and A. C. Offord is generalized.

1. Introduction. W. Feller [6] by using the Berry–Esseen inequality proved a theorem concerning the rate of convergence in the central limit theorem in one dimension in the absence of third moments. In this paper we use Sadikova's inequality to obtain similar results for certain random vectors in two dimensions.

2. Let $\eta_1, \eta_2, \cdots, \eta_r, \cdots$ be a sequence of independent identically distributed random variables with $E(\eta_r) = 0$, and $E(\eta_r^2) = 1$. Let $a_1, a_2, \cdots, a_r, \cdots$ and $b_1, b_2, \cdots, b_r, \cdots$ be real numbers such that $|a_r| \leqq 1$ and $|b_r| \leqq 1$. Also let $A_n^2 = \sum_{r=1}^n a_r^2$, $B_n^2 = \sum_{r=1}^n b_r^2$, $C_n = \sum_{r=1}^n a_r b_r$ and $Z_{r,n} = (X_{r,n}, Y_{r,n})$ where $X_{r,n} = a_r \eta_r / A_n$ and $Y_{r,n} = b_r \eta_r / B_n$. Then $Z_{1,n}, Z_{2,n}, \cdots, Z_{r,n}, \cdots$ is a sequence of independent random vectors in two dimensions with zero first order moments, $E(X_{r,n}^2) = a_r^2/A_n^2$, $E(Y_{r,n}^2) = b_r^2/B_n^2$ and $E(X_{r,n}, Y_{r,n}) = a_r b_r / A_n B_n$. Now consider the normalized sum

$$S_n = \left(\frac{\sum_{r=1}^n a_r \eta_r}{A_n}, \frac{\sum_{r=1}^n b_r \eta_r}{B_n} \right)$$

and let F_n denote the distribution function of S_n and G denote the normal distribution function having the same first and second moments as F_n. In this section we prove the following theorem.

THEOREM 2.1. *Let* $D_n = \min\{A_n, B_n\}$ *and* $0 < \varepsilon < 1$. *Then*

$$\sup_{x,y} |F_n(x, y) - G(x, y)| \leqq \frac{1}{D_n^{1-\varepsilon}} \left[\frac{d_1}{D_n^{1-\varepsilon}[\Lambda(1-\varepsilon)]^{2-\varepsilon}} + \frac{d_2 \delta_n^{1-\varepsilon}}{[\Lambda(1-\varepsilon)]^{\frac{3}{2}-\varepsilon/2}} + \frac{d_3}{\Lambda(1-\varepsilon)} + \frac{d_4 \delta_n + 28}{D_n^{\varepsilon}} \right]$$

where $\Lambda = \frac{1}{2}(1 - C_n^2/A_n^2 B_n^2)$, d_1, d_2, d_3 *are functions of* ε *only*, d_4 *is a constant, and* δ_n *is as in Lemma* 2.2.

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To prove Theorem 2.1 we need some notation. Let $f_r(s, t)$ be the characteristic function of (η_r, η_r). Then the characteristic function of Z_r is $f_r(a_r s/A_n, b_r t/B_n)$ and hence the characteristic function of $\sum_{r=1}^n Z_r$ is

$$\prod_{r=1}^n f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right).$$

Let also $g_r(s, t) = \exp[-\frac{1}{2}(s + t)^2]$. Now to each $Z_{r,n} = (X_{r,n}, Y_{r,n})$ let us correspond the interval $-\infty \leqq -\Gamma_n < 0 < \Gamma_n \leqq \infty$ and define the truncated random variables $\tilde{X}_{r,n}$ and $\tilde{Y}_{r,n}$ as follows.

$$\begin{aligned} \tilde{X}_{r,n} &= X_{r,n} \qquad \text{if} \quad |X_{r,n}| < \Gamma_n |a_r| \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} \tilde{Y}_{r,n} &= Y_{r,n} \qquad \text{if} \quad |Y_{r,n}| < \Gamma_n \frac{|b_r| A_n}{B_n} \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then $\{\omega : |X_{r,n}(\omega)| < \Gamma_n |a_r|\} = \{\omega : |Y_{r,n}(\omega)| < |b_r| \Gamma_n A_n/B_n\} = \{\omega : |\eta_r(\omega)| < A_n \Gamma_n\}$. Hence we define

$$\begin{aligned} \tilde{\eta}_{r,n} &= \eta_r \qquad \text{if} \quad |\eta_r| < A_n \Gamma_n \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then we have

$$\tilde{X}_{r,n} = \frac{a_r \tilde{\eta}_{r,n}}{A_n} \qquad \text{and} \qquad \tilde{Y}_{r,n} = \frac{b_r \tilde{\eta}_{r,n}}{B_n}.$$

Also define $X'_{r,n}$, $Y'_{r,n}$ and $\eta'_{r,n}$ by the following equations: $X_{r,n} = \tilde{X}_{r,n} + X'_{r,n}$, $Y_{r,n} = \tilde{Y}_{r,n} + Y'_{r,n}$ and $\eta_r = \tilde{\eta}_{r,n} + \eta'_{r,n}$. Finally let $\tilde{Z}_{r,n} = (\tilde{X}_{r,n}, \tilde{Y}_{r,n})$ and $Z'_{r,n} = (X'_{r,n}, Y'_{r,n})$. For K, q positive integers we set $E(\tilde{\eta}_{r,n}^K) = \tilde{\gamma}_{K,n}$, $E(\eta'^K_{r,n}) = \gamma'_{K,n}$, $E(|\tilde{\eta}_{r,n}|^K) = \tilde{\delta}_{K,n}$ and $E(|\eta'_{r,n}|^K) = \delta'_{K,n}$. Then we have

$$\begin{aligned} & E(\tilde{X}_{r,n}^K \tilde{Y}_{r,n}) = \frac{a_r^K b_r^q}{A_n^K B_n^q} \tilde{\gamma}_{k+q,n}, \qquad E(X'^K_{r,n} Y'^q_{r,n}) = \frac{a_r^K b_r^q}{A_n^K B_n^q} \gamma'_{K+q,n} \\ (1) \qquad & E(\tilde{X}_{r,n}^K Y'^q_{r,n}) = 0, \qquad E(|\tilde{X}_{r,n}|^K |Y'_{r,n}|^q) = 0, \\ & E(|\tilde{X}_{r,n}|^K |Y_{r,n}|^q) = \frac{|a_r|^K |b_r|^q}{A_n^K B_n^q} \tilde{\delta}_{K+q,n} \qquad \text{and} \\ & E(|X'_{r,n}|^K |Y'_{r,n}|^q) = \frac{|a_r|^K |b_r|^q}{A_n^K B_n^q} \delta'_{K+q,n}. \end{aligned}$$

Before proving Theorem 2.1, we state and prove some lemmas. The proofs of Lemmas 2.1, 2.4 and 2.5 were suggested by the referee.

LEMMA 2.1. *For fixed n and $1 \leqq r \leqq n$, we have*

$$\begin{aligned} & \left| f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right) - g_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right) \right| \\ & \qquad \leqq \frac{1}{8}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^4 + \frac{1}{6}\left|\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)\right|^3 \tilde{\delta}_{3,n} + \left(\frac{a_r s}{A_n} + \frac{a_r t}{B_n}\right)^2 \gamma'_{2,n}. \end{aligned}$$

PROOF. Note that the characteristic function $f_r(u, v)$ of $(c\eta_r, d\eta_r)$ evaluated at (u, v) is the same as the characteristic function of η_r evaluated at $cu + dv$. Therefore if we substitute $\xi/s = a_r s/A_n + b_r t/B_n$ in the inequality (3.3) of [6], the lemma follows.

LEMMA 2.2. *For fixed n, $1 \leqq r \leqq n$, $|s| \leqq T_n$, $|t| \leqq T_n$ and Γ_n large enough we have $|f_r(a_r s/A_n, b_r t/B_n)| \leqq \exp\{-\frac{1}{4}(a_r s/A_n + b_r t/B_n)^2\}$, where $T_n = \min\{A_n, B_n\}/4\delta_n$ and $\delta_n = \max\{1, \tilde{\delta}_{3,n}\}$.*

PROOF. Because of (1) we have the following identity:

$$
\begin{aligned}
f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right) - 1 + \frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 \tilde{\gamma}_{2,n} \\
= E\Big[\exp\Big[i\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)\tilde{\eta}_{r,n}\Big] - 1 - i\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)\tilde{\eta}_{r,n} \\
+ \left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 \tilde{\eta}^2_{r,n}\Big] \\
+ E\Big[\exp\Big[i\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)\eta'_{r,n}\Big] - 1 - i\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)\eta'_{r,n}\Big].
\end{aligned}
\tag{2}
$$

Now by (2) and the inequality

$$
\left|e^{ix} - 1 - \frac{ix}{1!} - \cdots - \frac{(ix)^{n-1}}{(n-1)!}\right| \leqq \frac{|x|^n}{n!}
$$

which holds for $n = 1, 2, \cdots$ and all real x, we get

$$
\begin{aligned}
&\left|f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right) - 1 + \frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\right| \\
&\qquad \leqq \left|\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right|^3 \frac{\tilde{\delta}_{3,n}}{6} + \left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 \gamma'_{2,n}.
\end{aligned}
$$

Hence for fixed n and $1 \leqq r \leqq n$ we have

$$
\begin{aligned}
\left|f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right)\right| &\leqq \left|1 - \frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\right| + \left|f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right)\right. \\
&\qquad \left. - 1 + \frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\right| \\
&\leqq \left|1 - \frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\right| + \frac{1}{6}\left|\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right|^3 \tilde{\delta}_{3,n} \\
&\qquad + \left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 \gamma'_{2,n}.
\end{aligned}
$$

Now if

$$
1 - \frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 > 0, \tag{3}
$$

we can take

$$x = -\frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 + \frac{1}{6}\left|\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right|^3 \tilde{\delta}_{3,n} + \left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 \gamma'_{2,n}$$

in the inequality $1 + x \leqq e^x$ which holds for all x and we get

$$\left|f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right)\right| \leqq \exp\left\{-\frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\left[1 - \frac{\tilde{\delta}_{3,n}}{3}\left|\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right| - 2\gamma'_{2,n}\right]\right\}$$
$$\leqq \exp\left\{-\frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\left[1 - \frac{\tilde{\delta}_{3,n}}{3}\left(\frac{|s|}{A_n} + \frac{|t|}{B_n}\right) - 2\gamma'_{2,n}\right]\right\}.$$

Now choose

$$(4) \qquad T_n = \frac{\min\{A_n, B_n\}}{4\delta_n} \qquad \text{where} \quad \delta_n = \max\{1, \tilde{\delta}_{3,n}\}.$$

Also we can take Γ_n so large that $\gamma'_{2,n} < \frac{1}{24}$, since $\tilde{\gamma}_{2,n} + \gamma'_{2,n} = \gamma_{2,n} = 1$. Thus $(T_n/3)(1/A_n + 1/B_n)\tilde{\delta}_{3,n} + 2\gamma'_{2,n} < \frac{1}{6} + \frac{1}{12} < \frac{1}{2}$. Also (3) holds since

$$\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2 \leqq \left(\frac{|a_r s|}{A_n} + \frac{|b_r t|}{B_n}\right)^2 \leqq T_n{}^2\left(\frac{2}{\min\{A_n, B_n\}}\right)^2$$
$$\leqq \left(\frac{\min\{A_n, B_n\}}{4}\right)^2\left(\frac{2}{\min\{A_n, B_n)}\right)^2 = \frac{1}{4}.$$

Therefore for $|s| \leqq T_n$ and $|t| \leqq T_n$ we get

$$\left|f_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right)\right| \leqq \exp\left\{-\frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\left[1 - \frac{T_n}{3}\tilde{\delta}_{3,n}\left(\frac{1}{A_n} + \frac{1}{B_n}\right) - 2\gamma'_{2,n}\right]\right\}$$
$$\leqq \exp\left\{-\frac{1}{4}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\right\}.$$

LEMMA 2.3. *Let* $\hat{f}_n(s, t) = \prod_{r=1}^n f_r(a_r s/A_n, b_r t/B_n)$, $\hat{g}_n(s, t) = \prod_{r=1}^n g_r(a_r s/A_n, b_r t/B_n)$, *and* $\nabla_n = \hat{f}_n(s, t) - \hat{g}_n(s, t)$. *Then with the same hypotheses as in Lemma 2.2 and* Γ_n *large enough we have*

$$|\nabla_n| \leqq \frac{1}{8D_n}\left\{\exp\left[-\frac{1}{4}\left(s^2 + t^2 + \frac{2C_n st}{A_n B_n}\right)\right]\right\}$$
$$\times\left[\frac{3(s+t)^4}{D_n} + 4|s+t|^3\tilde{\delta}_{3,n} + 24(t+s)^2\right].$$

PROOF. Consider the identity

$$(5) \qquad u_1 u_2 \cdots u_n - v_1 v_2 \cdots v_n = \sum_{\nu=1}^n u_1 u_2 \cdots u_{\nu-1}(u_\nu - v_\nu)v_{\nu+1} \cdots v_n$$

which holds for any u_i, v_i, with $i = 1, \cdots, n$. Letting $u_\nu = f_\nu(a_\nu s/A_n, b_\nu t/B_n)$ and $v_\nu = g_\nu(a_\nu s/A_n, b_\nu t/B_n)$ with $\nu = 1, 2, \cdots, n$ in (5) we get

$$\nabla_n = \sum_{\nu=1}^n f_1\left(\frac{a_1 s}{A_n}, \frac{b_1 s}{B_n}\right) f_2\left(\frac{a_2 s}{A_n}, \frac{b_2 t}{B_n}\right) \cdots f_{\nu-1}\left(\frac{a_{\nu-1}s}{A_n}, \frac{b_{\nu-1}t}{B_n}\right)$$
$$\left(f_\nu\left(\frac{a_\nu s}{A_n}, \frac{b_\nu t}{B_n}\right) - g_\nu\left(\frac{a_\nu s}{A_n}, \frac{b_\nu t}{B_n}\right)\right) g_{\nu+1}\left(\frac{a_{\nu+1}s}{A_n}, \frac{b_{\nu+1}t}{B_n}\right) \cdots g_n\left(\frac{a_n s}{A_n}, \frac{b_n t}{B_n}\right).$$

Now Lemma 2.1 gives us a bound for $|f_\nu(a_\nu s/A_n, b_\nu t/B_n) - g_\nu(a_\nu s/A_n, b_\nu t/B_n)|$ and Lemma 2.2 gives us a bound for $|f_\nu(a_\nu s/A_n, b_\nu t/B_n)|$.
Furthermore

$$g_r\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right) = \exp\left[-\frac{1}{2}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\right] \leqq \exp\left[-\frac{1}{4}\left(\frac{a_r s}{A_n} + \frac{b_r t}{B_n}\right)^2\right]$$

for $r = 1, \cdots, n$. Thus

$$|\nabla_n| \leqq \sum_{\nu=1}^n \left\{\exp\left[-\tfrac{1}{4}\sum_{i=1,i\neq\nu}^n \left(\frac{a_i s}{A_n} + \frac{b_i t}{B_n}\right)^2\right]\right\}\left[\frac{1}{8}\left(\frac{a_\nu s}{A_n} + \frac{b_\nu t}{B_n}\right)^4 + \frac{1}{6}\left|\frac{a_\nu s}{A_n} + \frac{b_\nu t}{B_n}\right|^3 \tilde{\delta}_{3,n} + \left(\frac{a_\nu s}{A_n} + \frac{b_\nu t}{B_n}\right)^2 \gamma'_{2,n}\right].$$

Now by (3) we have $\exp[\frac{1}{4}(a_\nu s/A_n + b_\nu t/B_n)^2] < e < 3$. Also,

$$\sum_{i=1,i\neq\nu}^n \left(\frac{a_i s}{A_n} + \frac{b_i t}{B_n}\right)^2 = \sum_{i=1}^n \left(\frac{a_i s}{A_n} + \frac{b_i t}{B_n}\right)^2 - \left(\frac{a_\nu s}{A_n} + \frac{b_\nu t}{B_n}\right)^2.$$

Hence we have

$$\begin{aligned}\exp&\left[-\tfrac{1}{4}\sum_{i=1,i\neq\nu}^n \left(\frac{a_i s}{A_n} + \frac{b_i t}{B_n}\right)^2\right] \\ &= \exp\left[-\tfrac{1}{4}\sum_{i=1}^n \left(\frac{a_i s}{A_n} + \frac{b_i t}{B_n}\right)^2\right]\exp\left[\frac{1}{4}\left(\frac{a_\nu s}{A_n} + \frac{b_\nu t}{B_n}\right)^2\right] \\ &< 3\exp\left[-\frac{1}{4}\left(s^2 + t^2 + \frac{2C_n}{A_n B_n} st\right)\right].\end{aligned}$$

Therefore

$$\begin{aligned}|\nabla_n| \leqq &\left\{3\exp\left[-\frac{1}{4}\left(s^2 + t^2 + \frac{2C_n}{A_n B_n} st\right)\right]\right\}\left\{\frac{(\sum_{\nu=1}^n a_\nu^4)s^4}{8A_n^4} + \frac{(\sum_{\nu=1}^n |a_\nu^3 b_\nu|)|s^3 t|}{2A_n^3 B_n}\right. \\ &+ 3\,\frac{(\sum_{\nu=1}^n a_\nu^2 b_\nu^2)s^2 t^2}{4A_n^2 B_n^2} + \frac{(\sum_{\nu=1}^n |a_\nu b_\nu|^3)|st|^3}{2A_n B_n^3} + \frac{(\sum_{\nu=1}^n b_\nu^4)s^4}{8B_n^4} \\ &+ \left[\frac{\sum_{\nu=1}^n |a_\nu|^3 |s|^3}{6A_n^3} + \frac{(\sum_{\nu=1}^n a_\nu^2 |b_\nu|)s^2|t|}{2A_n^2 B_n} + \frac{(\sum_{\nu=1}^n |a_\nu| b_\nu^2)|s|t^2}{2A_n B_n^2}\right. \\ &\left.\left. + \frac{(\sum_{\nu=1}^n |b_\nu|^3)|t|^3}{6B_n^3}\right]\tilde{\delta}_{3,n} + \left(s^2 + t^2 + \left|\frac{2C_n}{A_n B_n} st\right|\right)\gamma'_{2,n}\right\}.\end{aligned}$$

Now we may increase Γ_n beyond what was needed in Lemma 2.2 if necessary, to get

(6) $$\gamma'_{2,n} \leqq \frac{1}{D_n} \qquad \text{where} \quad D_n = \min(A_n, B_n\}.$$

Hence we get

$$\begin{aligned}|\nabla_n| \leqq \frac{1}{8D_n}&\left\{\exp\left[-\frac{1}{4}\left(s^2 + t^2 + \frac{2C_n}{A_n B_n} st\right)\right]\right\} \\ &\times\left[\frac{3(|s| + |t|)^4}{D_n} + 4(|s| + |t|)^3\tilde{\delta}_{3,n} + 24(s + t)^2\right],\end{aligned}$$

since $|a_r| \leqq 1$ and $|b_r| \leqq 1$. Now for $1 \leqq r \leqq n$, let $f_r^*(s, t) = f_r(s, 0)f_r(0, t)$ and $g_r^*(s, t) = g_r(s, 0)g_r(0, t)$. Then $f_r^*(s, t)$ is the characteristic function of a random vector (η_r^*, ζ_r^*) where η_r^* and ζ_r^* are independent and the distributions of η_r^* and ζ_r^* are the same as the marginal distributions of (η_r, η_r). Let $X_{r,n}^* = a_r\eta_r^*/A_n$, $Y_{r,n}^* = b_r\zeta_r^*/B_n$ and $Z_{r,n}^* = (X_{r,n}^*, Y_{r,n}^*)$. Then $f_r^*(a_r s/A_n, b_r t/B_n)$ is the characteristic function of $Z_{r,n}^*$, $E(X_{r,n}^{*2}) = a_r^2/A_n^2$, $E(Y_{r,n}^{*2}) = b_r^2/B_n^2$, and $E(X_{r,n}^* Y_{r,n}^*) = 0$. Now we truncate η_r^*, ζ_r^*, $X_{r,n}^*$ and $Y_{r,n}^*$ as follows: define

$$\begin{aligned}
\tilde{\eta}_{r,n}^* &= \eta_r^* \quad &&\text{if } |\eta_r^*| < \Gamma_n A_n \\
&= 0 &&\text{otherwise,} \\
\tilde{\zeta}_{r,n}^* &= \zeta_r^* &&\text{if } |\zeta_r^*| < \Gamma_n A_n \\
&= 0 &&\text{otherwise,} \\
\tilde{X}_{r,n}^* &= X_{r,n}^* &&\text{if } |X_{r,n}^*| < \Gamma_n |a_r| \\
&= 0 &&\text{otherwise}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{Y}_{r,n}^* &= Y_{r,n}^* \quad &&\text{if } |Y_{r,n}^*| < \Gamma_n \frac{|b_r|A_n}{B_n} \\
&= 0 &&\text{otherwise}
\end{aligned}$$

where Γ_n is as before.

Then we have $\tilde{X}_{r,n}^* = a_r\tilde{\eta}_{r,n}^*/A_n$ and $\tilde{Y}_{r,n}^* = b_r\tilde{\zeta}_{r,n}^*/B_n$. Also define $\eta_{r,n}'^*$, $\zeta_{r,n}'^*$, $X_{r,n}'^*$ and $Y_{r,n}'^*$ by the following equations $\eta_r^* = \tilde{\eta}_{r,n}^* + \eta_{r,n}'^*$, $X_{r,n}^* = \tilde{X}_{r,n}^* + X_{r,n}'^*$, $\zeta_r^* = \tilde{\zeta}_{r,n}^* + \zeta_{r,n}'^*$ and $Y_{r,n}^* = \tilde{Y}_{r,n}^* + Y_{r,n}'^*$. Notice that $E[(\eta_{r,n}'^*)^2] = E[(\zeta_{r,n}'^*)^2] = \gamma_{2,n}'$, $E(|\tilde{\eta}_{r,n}^*|^3) = E(|\tilde{\zeta}_{r,n}^*|^3) = \tilde{\delta}_{3,n}$ and $E(|\tilde{\eta}_{r,n}^*|^2) = E(\tilde{\zeta}_{r,n}^{*2}) = \tilde{\gamma}_{2,n}$.

LEMMA 2.4. *For fixed n, and $1 \leqq r \leqq n$ we have*

$$\left|f_r^*\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right) - g_r^*\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right)\right| \leqq \frac{1}{8}\left(\frac{a_r^4 s^4}{A_n^4} + \frac{b_r^4 t^4}{B_n^4}\right) + \frac{1}{6}\left(\frac{|a_r s|^3}{A_n^3} + \frac{|b_r t|^3}{B_n^3}\right)\tilde{\delta}_{3,n} + \left(\frac{s^2 a_r^2}{A_n^2} + \frac{t^2 b_r^2}{B_n^2}\right)\gamma_{2,n}'.$$

PROOF. Note that

$$\begin{aligned}
&\left|f_r^*\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right) - g_r^*\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right)\right| \\
&= \left|f_r\left(\frac{a_r s}{A_n}, 0\right) f_r\left(0, \frac{b_r t}{B_n}\right) - f_r\left(0, \frac{b_r t}{B_n}\right) g_r\left(\frac{a_r s}{A_n}, 0\right)\right. \\
&\qquad \left. + f_r\left(0, \frac{b_r t}{B_n}\right) g_r\left(\frac{a_r s}{A_n}, 0\right) - g_r\left(\frac{a_r s}{A_n}, 0\right) g_r\left(0, \frac{b_r t}{B_n}\right)\right| \\
&\leqq \left|f_r\left(\frac{a_r s}{A_n}, 0\right) - g_r\left(\frac{a_r s}{A_n}, 0\right)\right| + \left|f_r\left(0, \frac{b_r t}{B_n}\right) - g_r\left(0, \frac{b_r t}{B_n}\right)\right|
\end{aligned} \tag{7}$$

The lemma follows from (7) and Lemma 2.1.

LEMMA 2.5. *With the same assumptions as in Lemma* 2.2 *we have*

$$\left|f_r^*\left(\frac{a_r s}{A_n}, \frac{b_r t}{B_n}\right)\right| \leqq \exp\left[-\frac{1}{4}\left(\frac{a_r^2 s^2}{A_n^2} + \frac{b_r^2 t^2}{B_n^2}\right)\right].$$

PROOF. The lemma follows by applying Lemma 2.2 to each term on the right of $|f_r^*(a_r s/A_n, b_r t/B_n)| = |f_r(a_r s/A_n, 0)|\,|f_r(0, b_r t/B_n)|$ separately.

LEMMA 2.6. *Let* $\hat{f}_n^*(s,t) = \prod_{r=1}^n f_r^*(a_r s/A_n, b_r t/B_n)$, $\hat{g}_n^*(s,t) = \prod_{r=1}^n g_r^*(a_r s/A_n, b_r t/B_n)$ *and* $\nabla_n^* = \hat{f}_n^*(s,t) - \hat{g}_n^*(s,t)$. *Then with the same assumption as in Lemma 2.3 we have*

$$|\nabla_n^*| \leqq \frac{1}{8D_n}\{\exp[-\tfrac{1}{4}(s^2+t^2)]\}\left[\frac{3(s^4+t^4)}{D_n} + 4\tilde{\delta}_{3,n}(|s|^3+|t|^3) + 24(s^2+t^2)\right].$$

PROOF. Similar to Lemma 2.3.

Now let $\Omega_n(s,t) = \nabla_n - \nabla_n^*$. Then we have

$$\begin{aligned}|\Omega_n(s,t)| &\leqq |\nabla_n| + |\nabla_n^*| \\ &\leqq \frac{1}{8D_n}\left\{\left[\frac{3(|s|+|t|)^4}{D_n} + 4(|s|+|t|)^3\tilde{\delta}_{3,n} + 24(|s|+|t|)^2\right]\right. \\ &\qquad \times \exp\left[-\frac{1}{4}\left(s^2+t^2+\frac{2C_n}{A_nB_n}st\right)\right] \\ &\qquad + \left[\frac{3(s^4+t^4)}{D_n} + 4\tilde{\delta}_{3,n}(|s|^3+|t|^3) + 24(s^2+t^2)\right] \\ &\qquad \left.\times \exp[-\tfrac{1}{4}(s^2+t^2)]\right\}.\end{aligned}$$

By Holder's inequality for sums, $|C_n| \leqq A_nB_n$. Furthermore, we assume that $|C_n| < A_nB_n$ so that $0 < \Lambda = \frac{1}{2}(1 - C_n^2/A_n^2B_n^2) < 1$. If $|C_n| = A_nB_n$, then $\Lambda = 0$ and in this case Theorem 2.1 is not interesting. Now $s^2 + t^2 + (2C_n/A_nB_n)st \geqq \Lambda(s^2+t^2)$ since the quadratic form $(1-\Lambda)s^2 + (1-\Lambda)t^2 + (2C_n/A_nB_n)st$ is positive definite. Hence

$$\exp\left[-\frac{1}{4}\left(s^2+t^2+\frac{2C_n}{A_nB_n}st\right)\right] \leqq \exp\left[\frac{-\Lambda}{4}(s^2+t^2)\right].$$

Therefore we have

$$\begin{aligned}|\Omega_n(s,t)| &\leqq \frac{1}{8D_n}\left\{\exp\left[-\frac{\Lambda}{4}(s^2+t^2)\right]\right\} \\ &\qquad \times \left[\frac{3(2s^4+2t^4+4|s^3t|+6s^2t^2+4|st^3|)}{D_n}\right. \\ &\qquad \left. + 4\tilde{\delta}_{3,n}(2|s|^3+2|t|^3+3|t|s^2+3|s|t^2) + 48(s^2+t^2+|st|)\right].\end{aligned} \tag{8}$$

LEMMA 2.7. *For all* s, t *and for fixed* n *we have*

$$|\Omega_n(s,t)| = |\nabla_n - \nabla_n^*| \leqq 4|st|. \tag{9}$$

PROOF. This lemma follows from Lemma 1 in [8].

Now let ε be any number satisfying $0 < \varepsilon < 1$. We can write $|\Omega_n(s,t)| = |\Omega_n(s,t)|^{\varepsilon}|\Omega_n(s,t)|^{1-\varepsilon}$. Hence by (8) and (9) with the assumptions of Lemma 2.3

we have

$$\left|\frac{\Omega_n(s,t)}{st}\right| = \left|\frac{\Omega_n(s,t)}{st}\right|^{\varepsilon}\left|\frac{\Omega_n(s,t)}{st}\right|^{1-\varepsilon}$$

$$\leqq \frac{N_1}{D_n^{1-\varepsilon}}\Bigg[\frac{6s^4+6t^4+2|ts|^3+18s^2t^2+2|st|^3}{D_n|st|} \tag{10}$$

$$+\frac{4\tilde{\delta}_{3,n}}{|st|}(2|s|^3+2|t|^3+3s^2|t|+3|s|t^2)$$

$$+\frac{48}{|st|}(s^2+t^2+|st|)\Bigg]^{1-\varepsilon}\exp\left\{\frac{-\Lambda(1-\varepsilon)}{4}(s^2+t^2)\right\}$$

where $N_1 = 4^{\varepsilon}/8^{1-\varepsilon}$.

Finally we use Sadikova's inequality [8] to prove Theorem 2.1. Let (ζ, η) and (ζ', η') be two-dimensional random vectors. Let F (respectively H) be distribution functions, f (respectively h) the characteristic functions of the first and (respectively second) vectors. H is assumed to be differentiable as a function of two variables. Let also

$$\tilde{f}(s,t) = f(s,t) - f(s,0)f(0,t)\,, \qquad \tilde{h}(s,t) = h(s,t) - h(s,0)h(0,t)\,,$$

$$M_1 = \sup_{x,y}\frac{\partial H(x,y)}{\partial x} \qquad \text{and} \qquad M_2 = \sup_{x,y}\frac{\partial H(x,y)}{\partial y}\,.$$

THEOREM 2.2. (*Sadikova's inequality*). *For any $T > 0$ and all x and y we have*

$$|F(x,y) - H(x,y)|$$

$$\leqq \frac{2}{(2\pi)^2}\int_{-T}^{T}\int_{-T}^{T}\left|\frac{\tilde{f}(s,t)-\tilde{h}(s,t)}{st}\right|ds\,dt + 2\sup_x|F(x,\infty) - H(x,\infty)|$$

$$+ 2\sup_y|F(\infty,y) - H(\infty,y)| + 2[3(2^{\frac{1}{2}}) + 4(3^{\frac{1}{2}})]\frac{(M_1+M_2)}{T}\,.$$

In our application of Sadikova's inequality we will take $F(x,y) = F_n(x,y)$ and $H(x,y) = G(x,y)$, where $F_n(x,y)$ and $G(x,y)$ are as in Section 2. In this case we have $M_1 = M_2 = (2\pi)^{-\frac{1}{2}}$. Next we use Feller's one-dimensional bound [6], to establish a bound for the second and third terms on the right hand side of Sadikova's inequality. From Theorem 1 in [6] and (6) it follows that

$$2\sup_x|F_n(x,\infty) - G(x,\infty)| \leqq 12\left(\frac{\tilde{\delta}_{3,n}(\sum_{r=1}^{n}|a_r|^3)}{A_n^3} + \frac{\gamma'_{2,n}(\sum_{r=1}^{n}a_r^2)}{A_n^2}\right) \tag{11}$$

$$\leqq 12\left(\frac{\tilde{\delta}_{3,n}}{D_n} + \gamma'_{2,n}\right) \leqq 12\left(\frac{\tilde{\delta}_{3,n}+1}{D_n}\right),$$

and similarly

$$2\sup_y|F_n(\infty,y) - G(\infty,y)| \tag{12}$$

$$\leqq 12\left(\frac{\tilde{\delta}_{3,n}(\sum_{r=1}^{n}|b_r|^3)}{B_n^3} + \frac{\gamma'_{2,n}(\sum_{r=1}^{n}b_r^2)}{B_n^2}\right) \leqq 12\left(\frac{\tilde{\delta}_{3,n}+1}{D_n}\right).$$

Also by (4) we have

$$(13) \qquad 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cdot \frac{[3(2^{\frac{1}{2}}) + 4(3^{\frac{1}{2}})]}{T_n} = 8\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{[3(2^{\frac{1}{2}}) + 4(3^{\frac{1}{2}})]}{D_n} \delta_n .$$

Therefore by (10) and the inequality

$$(14) \qquad (\textstyle\sum_{i=1}^n x_i)^\alpha \leqq \sum_{i=1}^n x_i^\alpha$$

which holds for $0 \leqq \alpha \leqq 1$ and $x_i \geqq 0$ we get

$$\frac{1}{2\pi^2} \int_{-T_n}^{T_n} \int_{-T_n}^{T_n} \left|\frac{\Omega_{n(s,t)}}{st}\right| ds\,dt$$

$$\leqq \frac{N_1}{\pi^2 D_n^{1-\varepsilon}} \int_0^\infty \int_0^\infty \left[\frac{6s^4 + 6t^4 + 12s^3t + 18s^2t^2 + 12st^2}{D_n st} + \frac{4\delta_n}{st}(2s^3 + 2t^3 + 3s^2t + 3st^2) + \frac{48}{st}(s^2 + t^2 + st)\right]^{1-\varepsilon} \times \exp\left[\frac{-\Lambda(1-\varepsilon)(s^2+t^2)}{4}\right] ds\,dt$$

$$(15) \qquad \leqq \frac{N_1}{\pi^2 D_n^{1-\varepsilon}} \int_0^\infty \int_0^\infty \left\{\frac{1}{D_n^{1-\varepsilon}} [(6s^3t^{-1})^{1-\varepsilon} + (6t^3s^{-1})^{1-\varepsilon} + (12s^2)^{1-\varepsilon} + (18st)^{1-\varepsilon} + (12t^2)^{1-\varepsilon}] + (4\delta_n)^{1-\varepsilon}[(2s^2t^{-1})^{1-\varepsilon} + (2t^2s^{-1})^{1-\varepsilon} + (3s)^{1-\varepsilon} + (3t)^{1-\varepsilon}] + (48)^{1-\varepsilon}[(st^{-1})^{1-\varepsilon} + (ts^{-1})^{1-\varepsilon} + 1]\right\} \times \exp\left[\frac{-\Lambda(1-\varepsilon)(s^2+t^2)}{4}\right] ds\,dt .$$

Now letting $u = \frac{1}{4}(1-\varepsilon)s^2$ and $v = \frac{1}{4}(1-\varepsilon)t^2$, we get

$$\int_0^\infty \int_0^\infty (s^3t^{-1})^{1-\varepsilon} \exp\left[\frac{-\Lambda(1-\varepsilon)}{4}(s^2+t^2)\right] ds\,dt = \frac{2^{2-2\varepsilon}}{[\Lambda(1-\varepsilon)]^{2-\varepsilon}} \left(\int_0^\infty u^{\varepsilon/2-1}e^{-u}\,du\right)\left(\int_0^\infty v^{1-3\varepsilon/2}e^{-v}\,dv\right) .$$

Similarly,

$$\int_0^\infty \int_0^\infty (s^2)^{1-\varepsilon} \exp\left[\frac{-\Lambda(1-\varepsilon)}{4}(s^2+t^2)\right] ds\,dt = \frac{\pi^{\frac{1}{2}}2^{2(1-\varepsilon)}(\int_0^\infty u^{\frac{1}{2}-\varepsilon}e^{-u}\,du)}{[\Lambda(1-\varepsilon)]^{2-\varepsilon}} ,$$

$$\int_0^\infty \int_0^\infty (st)^{1-\varepsilon} \exp\left[\frac{-\Lambda(1-\varepsilon)(s^2+t^2)}{4}\right] ds\,dt = \frac{2^{2(1-\varepsilon)}(\int_0^\infty u^{-\varepsilon/2}e^{-u}\,du)^2}{[\Lambda(1-\varepsilon)]^{2-\varepsilon}} ,$$

$$\int_0^\infty \int_0^\infty (s^2t^{-1})^{1-\varepsilon} \exp\left[\frac{-\Lambda(1-\varepsilon)}{4}(s^2+t^2)\right] ds\,dt = \frac{2^{1-\varepsilon}}{[\Lambda(1-\varepsilon)]^{\frac{3}{2}-\varepsilon/2}} \left(\int_0^\infty u^{\varepsilon/2-1}e^{-u}\,du\right)\left(\int_0^\infty v^{\frac{1}{2}-\varepsilon}e^{-v}\,dv\right) ,$$

$$\int_0^\infty \int_0^\infty s^{1-\varepsilon} \exp\left[\frac{-\Lambda(1-\varepsilon)}{4}(s^2+t^2)\right] ds\,dt = \frac{\pi^{\frac{1}{2}}2^{1-\varepsilon}}{[\Lambda(1-\varepsilon)]^{\frac{3}{2}-\varepsilon/2}} \left(\int_0^\infty u^{-\varepsilon/2}e^{-u}\,du\right) ,$$

$$\int_0^\infty \int_0^\infty (st^{-1})^{1-\varepsilon} \exp\left[\frac{-\Lambda(1-\varepsilon)}{4}(s^2+t^2)\right] ds\, dt$$
$$= \frac{1}{\Lambda(1-\varepsilon)} \left(\int_0^\infty u^{\varepsilon/2-1} e^{-u}\, du\right)\left(\int_0^\infty v^{-\varepsilon/2} e^{-v}\, dv\right),$$

and finally,

$$\int_0^\infty \int_0^\infty \exp\left[\frac{-\Lambda(1-\varepsilon)}{4}(s^2+t^2)\right] ds\, dt = \frac{\pi}{\Lambda(1-\varepsilon)}.$$

Notice that $\int_0^\infty e^{-x} x^{p-1}\, dx$ converges if $p > 0$. Hence the above integrals converge and by (15) we have

$$(16) \qquad \frac{1}{2\pi^2} \int_{-T_n}^{T_n} \int_{-T_n}^{T_n} \left|\frac{\Omega_n(s,t)}{st}\right| ds\, dt$$
$$\leqq \frac{N_1}{\pi^2 D_n^{1-\varepsilon}} \left(\frac{N_2}{D_n^{1-\varepsilon}[\Lambda(1-\varepsilon)]^{2-\varepsilon}} + \frac{N_3 \delta_n^{1-\varepsilon}}{[\Lambda(1-\varepsilon)]^{\frac{3}{2}-\varepsilon/2}} + \frac{N_4}{\Lambda(1-\varepsilon)}\right),$$

where

$$N_2 = 6^{1-\varepsilon} 2^{2-2\varepsilon}\left[\left(2 \int_0^\infty u^{\varepsilon/2-1} e^{-u}\, du\right)\left(\int_0^\infty u^{1-3\varepsilon/2} e^{-u}\, du\right) + 2^{2-\varepsilon}\pi^{\frac{1}{2}}\left(\int_0^\infty u^{1/2-\varepsilon} e^{-u}\, du\right) + 3^{1-\varepsilon}\left(\int_0^\infty u^{-\varepsilon/2} e^{-u}\, du\right)^2\right],$$

$$N_3 = 2^{4-3\varepsilon}\left[\left(\int_0^\infty u^{\varepsilon/2-1} e^{-u}\, du\right)\left(\int_0^\infty u^{\frac{1}{2}-\varepsilon} e^{-u}\, du\right) + 3^{1-\varepsilon}\pi^{\frac{1}{2}} \int_0^\infty u^{-\varepsilon/2} e^{-u}\, du\right]$$

and

$$N_4 = (48)^{1-\varepsilon}\left[2\left(\int_0^\infty u^{\varepsilon/2-1} e^{-u}\, du\right)\left(\int_0^\infty u^{-\varepsilon/2} e^{-u}\, du\right) + \pi\right].$$

Now let $d_1 = N_1 N_2/2$, $d_2 = N_1 N_3/2$, $d_3 = N_1 N_4/2$, and $d_4 = 24 + 8(2/\pi)^{\frac{1}{2}}[3(2^{\frac{1}{2}}) + 4(3^{\frac{1}{2}})]$. Then Theorem 2.1 follows from (11), (12), (13) and (16).

3. Some remarks.

REMARK 1. Theorem 2.1 gives us a useful result when either a) $\lim_{n\to\infty} \delta_n < \infty$ or b) $\lim_{n\to\infty} \delta_n / D_n = 0$. If $E(|\eta_r|^3) < \infty$: then a) holds and even in this case Theorem 2.1 generalizes Sadikova and Dunnage's results. But if $E(|\eta_r|^3) = \infty$, then Theorem 2.1 gives a result when b) holds. If $\lim_{n\to\infty} \delta_n / D_n \neq 0$ then Theorem 2.1 doesn't say anything about the speed of convergence of $F_n(x, y)$ to $G(x, y)$. We now give examples where (b) holds and where (b) does not hold.

EXAMPLE 1. Let $\eta_1, \eta_2, \cdots, \eta_r, \cdots$ be a sequence of independent identically distributed random variables with density

$$f(x) = \frac{8}{3^{\frac{1}{2}}(x + 3^{\frac{1}{2}})^4}, \quad \text{if} \quad \frac{-1}{3} \leqq x < \infty$$
$$= 0 \qquad \text{otherwise.}$$

Then it is easy to see that $E(\eta_r) = 0$ and $E(\eta_r^2) = 1$. Let also

$$a_r = 1 \quad \text{if} \quad r \neq 2 \qquad \text{and} \qquad b_r = 1 \quad \text{if} \quad r \geqq 2$$
$$= \tfrac{1}{2} \quad \text{if} \quad r = 2 \qquad\qquad = \tfrac{1}{2} \quad \text{if} \quad r = 1.$$

Then $A_n = B_n = D_n = (n - \frac{3}{4})^{\frac{1}{2}}$. In this example we can see that (6) holds and

$\gamma'_{2,n} < \frac{1}{24}$ with $\Gamma_n \geqq 192$. Furthermore $\lim_{n\to\infty} \delta_n/D_n = 0$. Hence Theorem 2.1 is applicable.

EXAMPLE 2. Consider Example 1 with $a_r = 1$ and $b_r = 1/r^{\frac{1}{2}}$, $r = 1, 2, \cdots$, then $D_n = (1 + \frac{1}{2} + \cdots + 1/n)^{\frac{1}{2}}$. In this example (6) holds and $\gamma'_{2,n} < \frac{1}{24}$ with $\Gamma_n \geqq 192$, but $\lim_{n\to\infty} \delta_n/D_n \neq 0$. Hence Theorem 2.1 is not applicable.

REMARK 2. The d_1, d_2, d_3 of Theorem 2.1 are functions of ε and although are finite for every $0 < \varepsilon < 1$, $\lim_{\varepsilon\to 0} d_i = \infty$ for $i = 1, 2$ and 3.

4. Application of Theorem 2.1. In this section we apply Theorem 2.1 to generalize the following theorem due to P. Erdös and A. C. Offord [5]. Let $\eta_1, \eta_2, \cdots, \eta_r, \cdots$ be a sequence of independent random variables which take the values $+1$ and -1 each with probability $\frac{1}{2}$. Let also $a_1, a_2, \cdots, a_r, \cdots$ and $b_1, b_2, \cdots, b_r, \cdots$ be real numbers satisfying $0 \leqq a_r \leqq 1$, $0 \leqq b_r \leqq 1$ and let

$$A_n^2 = \sum_{r=1}^n a_r^2, \qquad B_n^2 = \sum_{r=1}^n b_r^2, \qquad C_n = \sum_{r=1}^n a_r b_r \qquad \text{and}$$
$$\Lambda = \frac{1}{2}\left(1 - \frac{C_n^2}{A_n^2 B_n^2}\right).$$

Then Erdös and Offord proved ([5], Lemma 2) that if $1 < A_n \leqq B_n$, then the probability that the two sums $\sum_{r=1}^n a_r\eta_r$ and $\sum_{r=1}^n b_r\eta_r$ differ in sign is

$$\frac{1}{\pi}\sin^{-1}(2\Lambda)^{\frac{1}{2}} + O\left(\frac{(\log A_n)^{\frac{1}{2}}}{\Lambda^{\frac{1}{2}} A_n^{\frac{2}{3}}}\right), \qquad \text{i.e.}$$

$$P[\omega : \textstyle\sum_{r=1}^n a_r\eta_r(\omega) \text{ and } \sum_{r=1}^n b_r\eta_r(\omega) \text{ differ in sign}] = \frac{1}{\pi}\sin^{-1}(2\Lambda)^{\frac{1}{2}} + O\left(\frac{(\log A_n)^{\frac{1}{2}}}{\Lambda^{\frac{1}{2}} A_n^{\frac{2}{3}}}\right).$$

J. E. A. Dunnage [3] generalized this theorem in the following way.

THEORY 4.1. *Let $\eta_1, \eta_2, \cdots, \eta_r, \cdots$ be a sequence of independent, identically distributed random variables such that $E(\eta_r) = 0$, $E(\eta_r^2) = 1$, $E(|\eta_r|^3) = m$ and let $a_1, a_2, \cdots, a_r, \cdots$ and $b_1, b_2, \cdots, b_r, \cdots$ be real numbers such that $|a_r| \leqq 1$ and $|b_r| \leqq 1$, $r = 1, 2, \cdots$. Then*

$$P[\omega : \textstyle\sum_{r=1}^n a_r\eta_r \text{ and } \sum_{r=1}^n b_r\eta_r \text{ differ in sign}] = \frac{1}{\pi}\sin^{-1}(2\Lambda)^{\frac{1}{2}} + O\left[\frac{m}{A_n^{\frac{1}{2}}\Lambda^{\frac{3}{4}}}\left(1 + \frac{\log n}{A_n^{\frac{1}{2}}\Lambda^{\frac{3}{4}}}\right)\right]$$

where Λ, A_n, B_n and C_n have the same meaning as before and it is assumed that $A_n \leqq B_n$.

Now we generalize this theorem in the following way.

THEOREM 4.2. *Let $\eta_1, \eta_2, \cdots, \eta_r, \cdots$ be a sequence of independent identically distributed random variables with $E(\eta_r) = 0$ and $E(\eta_r^2) = 1$. Let $a_1, a_2, \cdots, a_r, \cdots$ and $b_1, b_2, \cdots, b_r, \cdots, b_r, \cdots$ be real numbers such that $|a_r| \leqq 1$ and $|b_r| \leqq 1$ for*

$r = 1, 2, \cdots$. *Then*

$$P_1 = P[\omega : \textstyle\sum_{r=1}^n a_r \eta_r(\omega) \text{ and } \sum_{r=1}^n b_r \eta_r(\omega) \text{ differ in sign}]$$
$$\leqq \frac{1}{\pi} \sin^{-1} (2\Lambda)^{\frac{1}{2}} + \frac{1}{D_n^{1-\varepsilon}} \left[\frac{d_1}{D_n^{1-\varepsilon}[\Lambda(1-\varepsilon)]^{2-\varepsilon}} + \frac{d_2 \delta_n^{1-\varepsilon}}{[\Lambda(1-\varepsilon)]^{\frac{3}{2}-\varepsilon/2}} + \frac{d_3}{\Lambda(1-\varepsilon)} + \frac{d_4 \delta_n + 28}{D_n^{\varepsilon}} \right],$$

where δ_n, d_1, d_2, d_3, d_4, Λ, *and* D_n *are as in Theorem* 2.1 *and* $0 < \varepsilon < 1$.

REMARK. Theorem 4.2 is a generalization of Theorem 4.1, because we do not require the existence of third moments. Further even if the third moments exist, Theorem 4.2 gives a better bound for P_1 than Theorem 4.1.

PROOF OF THEOREM 4.2. As in Theorem 2.1, let $F_n(x, y)$ be the distribution function of $S_n = (\sum_{r=1}^n a_r \eta_r / A_n, \sum_{r=1}^n b_r \eta_r / B_n)$ and let $G(x, y)$ be the corresponding normal distribution function, i.e. $G(x, y)$ is the distribution function of the sum $(\sum_{r=1}^n a_r \eta_r / A_n, \sum_{r=1}^n b_r \eta_r / B_n)$ assuming that η_r's have the standard normal distribution function. Then by Theorem 2.1 for every x and y we have

$$(17) \qquad |F_n(x, y) - G(x, y)| \leqq \frac{1}{D_n^{1-\varepsilon}} \left[\frac{d_1}{D_n^{1-\varepsilon}[\Lambda(1-\varepsilon)]^{2-\varepsilon}} + \frac{d_2 \delta_n^{1-\varepsilon}}{[\Lambda(1-\varepsilon)]^{\frac{3}{2}-\varepsilon/2}} + \frac{d_3}{\Lambda(1-\varepsilon)} + \frac{d_4 \delta_n + 28}{D_n^{\varepsilon}} \right].$$

Dunnage [1] has shown that

$$P_2 = P[\omega : \textstyle\sum_{r=1}^n a_r \eta_r(\omega) \text{ and } \sum_{r=1}^n b_r \eta_r \text{ differ in sign when } \eta_r\text{'s are normally distributed}] = \frac{1}{\pi} \sin^{-1} (2\Lambda)^{\frac{1}{2}}.$$

Therefore by (17) we have

$$P_1 = |(P_1 - P_2) + P_2| \leqq |P_1 - P_2| + P_2 \leqq \sup_{x,y} |F_n(x, y) - G(x, y)| + P_2$$
$$\leqq \frac{1}{\pi} \sin^{-1} (2\Lambda)^{\frac{1}{2}} + \frac{1}{D_n^{1-\varepsilon}} \left[\frac{d_1}{D_n^{1-\varepsilon}[\Lambda(1-\varepsilon)]^{2-\varepsilon}} + \frac{d_2 \delta_n^{1-\varepsilon}}{[\Lambda(1-\varepsilon)]^{\frac{3}{2}-\varepsilon/2}} + \frac{d_3}{\Lambda(1-\varepsilon)} + \frac{d_4 \delta_n + 28}{D_n^{\varepsilon}} \right]$$

and the theorem is proven.

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