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A FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR WEIGHTED EMPIRICAL DISTRIBUTIONS[1]

BY BARRY R. JAMES

Instituto de Matematica Pura e Aplicada

Finkelstein's (1971) functional law of the iterated logarithm for empirical distributions is extended to cases where the empirical distribution is multiplied by a weight function, w. We let $X_1, X_2, \cdots$ be independent random variables each having the uniform distribution on [0, 1], with F_n the empirical df at stage n. The weight function w, defined on [0, 1], is assumed to be bounded on interior intervals and to satisfy some smoothness conditions. Then convergence of the integral $\int_0^1 w^2(t)/\log\log(t^{-1}(1-t)^{-1})\,dt$ is seen to be a necessary and sufficient condition for the sequence $\{U_n : n \geqq 3\}$, defined by

$$U_n(t) = \frac{n^{\frac{1}{2}} w(t)(F_n(t) - t)}{(2 \log \log n)^{\frac{1}{2}}}$$

to be uniformly compact on a set of probability one, with set of limit points

$$K_w = \{wf : f \in K\} .$$

K is the set set of absolutely continuous functions on [0, 1] with $f(0) = 0 = f(1)$ and

$$\int_0^1 [f'(t)]^2\, dt \leqq 1 .$$

1. Introduction. Finkelstein's (1971) functional law of the iterated logarithm for empirical distributions, which was inspired by Strassen's (1964) invariance principle, is as follows:

Suppose $X_1, X_2, \cdots$ are independent random variables having the uniform distribution on [0, 1], with F_n the empirical distribution function at stage n (i.e., for $0 \leqq t \leqq 1$, $nF_n(t)$ is the number of $X_1, X_2, \cdots, X_n$ which are less than or equal to t). For $n \geqq 3$ and $0 \leqq t \leqq 1$, set

$$G_n(t) = \frac{n^{\frac{1}{2}}[F_n(t) - t]}{(2 \log \log n)^{\frac{1}{2}}} .$$

Let $B = B[0, 1]$ be the space of bounded real-valued functions on [0, 1], with the supremum norm $||f|| = \sup\{|f(t)| : 0 \leqq t \leqq 1\}$, and let K be the set of absolutely continuous functions f in B such that $f(0) = 0 = f(1)$ and $\int_0^1 [f'(t)]^2\, dt \leqq 1$. (Note that this K is the tied-down version of Strassen's invariance-principle K). Then Finkelstein says that on a set of probability one, the sequence $\{G_n\}_{n=3,4,\cdots}$ is relatively compact in B and the set of its limit points is K.

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In this paper, we will investigate sequences of the form $\{wG_n\}_{n\geqq 3}$, where w is a weight function defined on [0, 1]. It is natural to expect that if w is not "too large," the latter sequence will also be relatively compact, with set of limit points equal to $K_w = \{wf : f \in K\}$. This is clearly the case if w is bounded. But it should also be clear from Finkelstein's result that such sequences can be relatively compact only if the weight function w is bounded on interior intervals. In fact, if $\sup\{|w(t)| : \varepsilon \leqq t \leqq 1 - \varepsilon\} = \infty$ for some $\varepsilon > 0$, then on the set of probability one given us by Finkelstein,

$$\limsup_{n\to\infty} ||wG_n|| = \infty .$$

Therefore, our attention will be focussed on the behavior of the weighted functions near 0 and 1. The main results concerning this may be found in Section 3, where we derive some almost sure analogs of certain theorems of Chibisov (1964, Theorems 1 and 2). These results are then used in section 4 to prove our theorem, which states that the extension of Finkelstein's result holds if and only if w satisfies an integrability condition slightly weaker than square integrability (with certain regularity assumptions on w).

Section 2 is introductory, containing the basic assumptions and notation and some preliminary lemmas.

2. Preliminaries. Let $X_1, X_2, \cdots$ be independent random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, each having the uniform distribution on [0, 1]. For each positive integer n, let F_n be the empirical distribution function at stage n, and for $0 \leqq t \leqq 1$ set

$$S_n(t) = \sum_{i=1}^n [I_{\{X_i \leq t\}} - t] = n[F_n(t) - t]$$

and

$$B_n(t) = n^{\frac{1}{2}}[F_n(t) - t] .$$

For $n \geqq 3$, set

$$\lambda(n) = (\log\log n)^{\frac{1}{2}} .$$

Observe that Finkelstein's G_n is $B_n/[2^{\frac{1}{2}}\lambda(n)]$ in our notation.

The lemmas which follow will be referred to frequently in the proofs of Section 3. The notation, especially the use of $1/\phi$ in place of w, has been chosen to accord with that of Chibisov (1964).

LEMMA 2.1. *If, for some $\delta > 0$, f is positive and nonincreasing on $(0, \delta]$ and*

$$\int_0^\delta \frac{1}{tf(t)}\, dt < \infty ,$$

then

(i) $$\lim_{t\downarrow 0} \frac{\log(1/t)}{f(at)} = 0 \qquad \textit{for any} \quad a > 0 ,$$

(ii) $$\lim_{t\downarrow 0} \frac{f(t)}{\log\log(1/t)} = \infty ,$$

(iii) $$\sum_{n>N(\delta)} \frac{1}{nf(1/n)} < \infty \qquad \text{for some positive} \quad N(\delta)\,,$$

(iv) $$\sum_{n>M(\delta)} \frac{1}{f(1/2^n)} < \infty \qquad \text{for some positive} \quad M(\delta)\,,$$

and

(v) *if* $h(t) = (f(t)/\log\log(1/t))^{\frac{1}{2}}$ *then there exist* $\tau_0 > 0$ *and* $C > 0$ *such that for any* $0 < \tau < \tau_0$, $2^{i/2}h(\tau 2^{i+1-n}) > Cn^{\frac{1}{4}}$ *for all* $i = 0, 1, \cdots, n-1$, *eventually.*

PROOF. (i) through (iv) are straightforward. For (v), part (i) implies the existence of $\tau_1 > 0$ such that $h^2(t) > \log(1/t)/\log\log(1/t) > [\log(1/t)]^{\frac{1}{2}}$ for $t < \tau_1$. Thus if $\tau < \tau_1$ and $i \leqq n-1$,

$$2^i h^2(\tau 2^{i+1-n}) > 2^i[(n-i-1)\log 2 - \log\tau]^{\frac{1}{2}}\,.$$

Now set $f_n(x) = 2^x[(n-x-1)\log 2 - \log\tau]^{\frac{1}{2}}$. It can easily be checked that $f_n'(x) > 0$ for $0 \leqq x \leqq n-1$ (if $\tau < e^{-\frac{1}{2}}$), which implies $2^i h^2(\tau 2^{i+1-n}) > f_n(0) = [(n-1)\log 2 - \log\tau]^{\frac{1}{2}} > (n\log 2)^{\frac{1}{2}}/2$ for all $i \leqq n-1$ and all n sufficiently large. This proves (v) with $\tau_0 = \min(\tau_1, e^{-\frac{1}{2}})$. □

LEMMA 2.2. *If, for some* $\delta > 0$, f *is positive and nonincreasing on* $(0, \delta]$ *and*

$$\int_0^\delta \frac{1}{tf(t)}\,dt = \infty\,,$$

then

$$\sum_{n\geqq n(\delta)} \frac{1}{nf[1/(n\log^2 n)]} = \infty\,.$$

LEMMA 2.3. ("*Reflection*" *inequality.*) *Let* ϕ *be a positive function on* $(0, \delta]$, *for some* $0 < \delta \leqq 2$, *such that* $h(t) = t^{-\frac{1}{2}}\phi(t)$ *is monotone decreasing on* $(0, \delta]$ *and satisfies* $\lim_{t\downarrow 0} h(t) = \infty$. *Then there exists a positive nondecreasing function* τ *on* $(0, \infty)$ *such that, for* $\varepsilon > 0$, $n \geqq 1$, *and* $0 < a < b \leqq \tau(\varepsilon)$,

$$P\left(\sup_{a\leqq t<b,\, n<i\leqq 2n} \frac{|B_i(t)|}{\phi(t)} > \varepsilon\right) \leqq 2P\left(\sup_{a\leqq t<b} \frac{|B_{2n}(t)|}{\phi(t)} > \varepsilon 8^{-\frac{1}{2}}\right). \tag{2.1}$$

PROOF. Let $\tau(\varepsilon) = \frac{1}{2}\sup\{t : t < \delta \text{ and } h(t) \geqq 8^{\frac{1}{2}}/\varepsilon\}$. The conditions on h imply that τ is positive (it is clearly nondecreasing). Note that τ is bounded above by $\delta/2$ and that $h[\tau(\varepsilon)] \geqq 8^{\frac{1}{2}}/\varepsilon$.

To prove (2.1), choose $\varepsilon > 0$, $n \geqq 1$, and $0 < a < b \leqq \tau(\varepsilon)$. Let $\{r_k : k \geqq 1\}$ be an ordering of the rationals in (a, b), so that

$$\text{LHS (2.1)} \leqq P\left(\sup_{k\geqq 1,\, n<j\leqq 2n} \frac{|S_j(r_k)|}{\phi(r_k)} > \varepsilon n^{\frac{1}{2}}\right). \tag{2.2}$$

For $n < j \leqq 2n$ and $k \geqq 1$ set

$$A_{j,k} = \left\{\sup_{n<i\leqq j,\, l<k} \frac{|S_i(r_l)|}{\phi(r_l)} \leqq \varepsilon n^{\frac{1}{2}}\,,\ \frac{|S_j(r_k)|}{\phi(r_k)} > \varepsilon n^{\frac{1}{2}}\right\}$$

and

$$B_{j,k} = \{|S_{2n}(r_k) - S_j(r_k)| \leqq \phi(r_k)\varepsilon n^{\frac{1}{2}}/2\} .$$

The $A_{j,k}$ are disjoint in k for fixed j, and the proof of Loève's "Lemma for events" ([5], page 246), generalized for the countable index in k, yields

$$\inf P(B_{j,k}) \cdot \text{RHS (2.2)} \leqq P\left\{\sup_{k\geqq 1} \frac{|S_{2n}(r_k)|}{\phi(r_k)} > \frac{\varepsilon n^{\frac{1}{2}}}{2}\right\} .$$

The proof is completed by applying Chebyshev's inequality to obtain $P(B_{j,k}) \geqq \frac{1}{2}$ for all j, k. □

LEMMA 2.4. *For* $\varepsilon > 0$, $n \geqq 1$, *and* $0 < a < b < 1$,

$$\text{(2.3)} \qquad P\left(\sup_{a\leqq t<b} \frac{|S_n(t)|}{1-t} > \frac{\varepsilon}{1-b}\right)$$
$$\leqq 2 \exp\left\{-\frac{\varepsilon}{\beta}\left[\left(1 + \frac{\sigma^2}{\beta\varepsilon}\right)\log\left(1 + \frac{\beta\varepsilon}{\sigma^2}\right) - 1\right]\right\}$$
$$\leqq 2 \exp\left\{-\frac{\varepsilon^2}{2(\sigma^2 + \beta\varepsilon/3)}\right\} ,$$

where $\beta = \max(b, 1-b)$ *and* $\sigma^2 = nb(1-b)$.

PROOF. It is easily checked that for each $n \geqq 1$, $\{S_n(t)/(1-t) : 0 < t < 1\}$ is a Martingale (for the case $n = 1$, see Kiefer (1972), page 7). The lemma then follows from Hoeffding's (1963) formulae (2.12), (2.13), and (2.18), with the factor 2 appearing upon application of the same procedure to $-S_n(t)$. We mention here that the bounds in (2.3) are due to G. Bennett and S. N. Bernstein, respectively. □

LEMMA 2.5. *Let* $g(\lambda) = (1 + 1/\lambda)\log(1+\lambda) - 1$ *for* $\lambda > 0$. *Then*

(i) *g is positive and strictly increasing on* $(0, \infty)$, *and*
(ii) $g(\lambda) \sim \log \lambda$ *as* $\lambda \to \infty$.

PROOF. Use calculus for (i), inspection for (ii). □

3. Behavior near 0 and 1. We shall see that convergence of a certain integral is a necessary and sufficient condition for ϕ to belong to a type of upper class (Lemma 3.3).

LEMMA 3.1. *Let* ϕ *be a positive function on* $(0, \delta]$, *for some* $0 < \delta < 1/e$, *such that* $h(t) = t^{-\frac{1}{2}}\phi(t)$ *is monotone decreasing on* $(0, \delta]$. *If*

$$\int_0^\delta \frac{1}{\phi^2(t)\log\log\frac{1}{t}}\,dt < \infty ,$$

then for any $a > 0$

$$\sup_{0<t<a/n} \frac{|B_n(t)|}{\lambda(n)\phi(t)} \to 0 \quad \text{a.s.}$$

and

$$\sup_{0<t<a/n} \frac{|B_n(1-t)|}{\lambda(n)\phi(t)} \to 0 \quad \text{a.s.}$$

PROOF. Fix $a > 0$, and set

$$U_n(t) = \frac{B_n(t)}{\lambda(n)\phi(t)} .$$

Let m be a positive integer, with $M = 256m^2$, and set

$$a_n = \frac{M}{nh^2(1/n)\lambda^2(n)}, \qquad b_n = a_{2^n}, \qquad c_n = \frac{1}{n2^n}, \qquad d_n = \frac{a}{2^{n-1}} .$$

Note that Lemma 2.1 (i) implies $b_n < c_n < d_n < \delta$ for n sufficiently large.

The lemma will be proved in four stages:

(i) $\sup_{0<t<a_n} |U_n(t)| \to 0$ a.s.

(ii) Let

$$p_n = P(\sup_{b_n \leqq t < d_n, 2^{n-1}<k\leqq 2^n} |U_k(t)| > 1/m).$$

Then

$$p_n \leqq 2P(\sup_{b_n \leqq t < d_n} |U_{2^n}(t)| > 1/4m)$$

for n sufficiently large.

(iii) If $q_n = P(\sup_{b_n \leqq t < c_n} |U_{2^n}(t)| > 1/4m)$, then $\sum q_n < \infty$.

(iv) If $r_n = P(\sup_{c_n \leqq t < d_n} |U_{2^n}(t)| > 1/4m)$, then $\sum r_n < \infty$.

We note here that the lemma clearly follows from (i) through (iv) and Borel-Cantelli.

PROOF OF (i). $\sum_{n>N(\delta)} a_n < \infty$ (Lemma 2.1 (iii)), so that $X_n > a_n$ eventually a.s. (Borel–Cantelli). Because the a_n are decreasing, it follows that, eventually a.s., $\min(X_1, \cdots, X_n) > a_n$, $F_n(a_n) = 0$, and

$$\sup_{0<t<a_n} |U_n(t)| = \sup_{0<t<a_n} \frac{(nt)^{\frac{1}{2}}}{\lambda(n)h(t)} \leqq \frac{(na_n)^{\frac{1}{2}}}{\lambda(n)h(a_n)} .$$

A glance at the definition of a_n now yields (i).

PROOF OF (ii). A consquence of Lemma 2.3.

PROOF OF (iii). For n large (so that $c_n < \delta$),

$$q_n \leqq P\left(\sup_{b_n \leqq t < c_n} \frac{|S_{2^n}(t)|}{(1-t)\phi(t)} > \frac{2^{n/2}\lambda(2^n)}{4m}\right)$$
$$\leqq P\left(\sup_{b_n \leqq t < c_n} \frac{|S_{2^n}(t)|}{1-t} > \frac{2^{n/2}\lambda(2^n)b_n^{\frac{1}{2}}h(c_n)}{4m}\right).$$

By Lemma 2.4,

$$q_n \leqq 2\exp\left\{-\gamma_n\left[\left(1 + \frac{2^n c_n}{\gamma_n(1-c_n)}\right)\log\left(1 + \frac{\gamma_n(1-c_n)}{2^n c_n}\right) - 1\right]\right\},$$

where

$$\gamma_n = \frac{2^{n/2}\lambda(2^n)b_n^{\frac{1}{2}}h(c_n)}{4m} = \frac{M^{\frac{1}{2}}h(c_n)}{4mh(2^{-n})} \geqq \frac{M^{\frac{1}{2}}}{4m} = 4 .$$

Since $\gamma_n(1 - c_n)2^{-n}/c_n \geqq \frac{1}{2}n\gamma_n \geqq 2n$, Lemma 2.5 implies

$$\begin{aligned} q_n &\leqq 2\exp\{-4g(2n)\} \\ &\leqq 2\exp\{-2\log(2n)\} = 2\left(\frac{1}{2n}\right)^2 , \end{aligned}$$

eventually.

Therefore, $\sum q_n < \infty$.

PROOF OF (iv). For n large (so that $ed_n < \delta$),

$$\begin{aligned} r_n &\leqq \textstyle\sum_{i=0}^{[\log 2an]} P\left(\sup_{c_n e^i \leq t < c_n e^{i+1}} \frac{|S_{2^n}(t)|}{(1-t)\phi(t)} > \frac{2^{n/2}\lambda(2^n)}{4m}\right) \\ &\leqq \textstyle\sum_{i=0}^{[\log 2an]} P\left(\sup_{c_n e^i \leq t < c_n e^{i+1}} \frac{|S_{2^n}(t)|}{1-t} > \frac{2^{n/2}\lambda(2^n)c_n^{\frac{1}{2}}e^{i/2}h(ed_n)}{4m}\right) . \end{aligned}$$

Apply Lemma 2.4 to get (for n such that $ed_n < \frac{1}{2}$)

$$r_n \leqq 2\textstyle\sum_{i=0}^{[\log 2an]} \exp\left\{-\gamma_{ni}\left[\left(1 + \frac{1}{\alpha_{ni}}\right)\log(1+\alpha_{ni}) - 1\right]\right\} ,$$

where

$$\begin{aligned} \gamma_{ni} &= \frac{2^{n/2}\lambda(2^n)c_n^{\frac{1}{2}}e^{i/2}h(ed_n)}{4m} \\ &= \frac{\lambda(2^n)e^{i/2}h(ed_n)}{4mn^{\frac{1}{2}}} \geqq \frac{\lambda(2^n)h(ed_n)}{4mn^{\frac{1}{2}}} \end{aligned}$$

and

$$\alpha_{ni} = \frac{\gamma_{ni}(1 - c_n e^{i+1})}{2^n c_n e^{i+1}} > \frac{n^{\frac{1}{2}}\lambda(2^n)h(ed_n)}{8me^{i/2+1}} .$$

Since $\log\log 1/2aet \sim \log\log 1/t$ as $t \downarrow 0$, Lemma 2.1 (i) yields

$$\frac{\lambda(2^n)h(ed_n)}{n^{\frac{1}{2}}} > 2M^{\frac{1}{2}} \quad \text{eventually.}$$

Using this and the fact that $e^{i/2} \leqq e^{(\log 2an)/2} = (2an)^{\frac{1}{2}}$, we see that

$$\gamma_{ni} \geqq \frac{M^{\frac{1}{2}}}{2m} = 8$$

and

$$\alpha_{ni} \geqq \frac{(Mn)^{\frac{1}{2}}}{4me(2a)^{\frac{1}{2}}} = \frac{4n^{\frac{1}{2}}}{e(2a)^{\frac{1}{2}}} ,$$

uniformly in i for large n.

Lemma 2.5 now gives $\sum r_n < \infty$. □

LEMMA 3.2. *Let ϕ and h be as in Lemma* 3.1, *but this time assume*

$$\int_0^\delta \frac{1}{\phi^2(t)\log\log\frac{1}{t}}\,dt = \infty . \tag{3.1}$$

Then for any $a > 0$

$$\limsup_{n\to\infty} \left\{ \sup_{0<t<a/n} \frac{|B_n(t)|}{\lambda(n)\phi(t)} \right\} = \infty \quad \text{a.s.}$$

PROOF. Fix $a > 0$ and let m be a positive integer. Set

$$U_n(t) = \frac{B_n(t)}{\lambda(n)\phi(t)}, \qquad a_n = \frac{1}{n \log^2 n}, \qquad b_n = \frac{1}{mnh^2(a_n) \log\log n},$$

and $Z_n = \min(X_1, \cdots, X_n)$.

We will first show that

$$a_n < Z_n < b_n < \frac{\min(a, 1)}{2n} \tag{3.2}$$

infinitely often, a.s.

Since $\sum a_n < \infty$, we have eventually a.s. $X_n > a_n$ (Borel–Cantelli) and $Z_n > a_n$ (the a_n are decreasing). By (3.1) and Lemma 2.2,

$$\sum \frac{1}{nh^2(a_n) \log\log \frac{1}{a_n}} = \infty .$$

But $\log\log(1/a_n) \sim \log\log n$, so that $\sum b_n = \infty$. An application of Borel–Cantelli yields $X_n < b_n$ (and hence $Z_n < b_n$) infinitely often, a.s., proving (3.2).

Now apply some algebra. Whenever (3.2) holds,

$$\sup_{0<t<a/n} |U_n(t)| \geqq U_n(Z_n) = \frac{n^{\frac{1}{2}}(1/n - Z_n)}{\lambda(n) Z_n^{\frac{1}{2}} h(Z_n)}$$

$$\geqq \frac{1}{2\lambda(n)(nb_n)^{\frac{1}{2}} h(a_n)} = \frac{m^{\frac{1}{2}}}{2}.$$

Therefore,

$$\limsup_{n\to\infty} (\sup_{0<t<a/n} |U_n(t)|) \geqq \frac{m^{\frac{1}{2}}}{2} \quad \text{a.s.},$$

and the lemma is proved. □

LEMMA 3.3. *Let* ϕ *be a positive function on* $(0, \delta]$, *for some* $0 < \delta < 1/e$, *such that* $h(t) = t^{-\frac{1}{2}}\phi(t)$ *is monotone decreasing on* $(0, \delta]$. *If*

$$\int_0^\delta \frac{1}{\phi^2(t) \log\log \frac{1}{t}}\, dt < \infty, \tag{3.3}$$

then for each $\varepsilon > 0$ *there exists* $\tau > 0$ *such that*

$$\sup_{0<t<\tau} \frac{|B_n(t)|}{\lambda(n)\phi(t)} < \varepsilon \quad \textit{eventually} \text{ a.s.}$$

Conversely, if the integral in (3.3) *diverges, then*

$$\limsup_{n\to\infty} \left\{ \sup_{0<t<\tau} \frac{|B_n(t)|}{\lambda(n)\phi(t)} \right\} = \infty \quad \text{a.s.}$$

for each $0 < \tau \leqq \delta$.

(*And, of course, the same conclusions hold if $B_n(t)$ is replaced by $B_n(1-t)$.*)

PROOF. The converse follows immediately from Lemma 3.2. So, taking account of Lemma 3.1, we need only show that if (3.3) holds, there exists $\tau > 0$ for which

$$\sup_{\tau/n \leqq t < \tau} \frac{|B_n(t)|}{\lambda(n)\phi(t)} < \varepsilon \quad \text{eventually a.s.}$$

For $\tau \leqq \tau(\varepsilon)$ (as defined in Lemma 2.3) set $\alpha_{ni} = \tau 2^{i-n}$ for $i = 0, 1, \cdots, n$. For convenience, the dependence of the α's on τ will not be expressed in the notation. By methods similar to those used in parts (ii) and (iv) of the proof of Lemma 3.1 (using, however, the Bernstein bound of Lemma 2.4), we obtain

$$(3.4) \quad P\left(\sup_{\tau/k \leqq t < \tau, 2^{n-1} < k \leqq 2^n} \frac{|B_k(t)|}{\lambda(t)\phi(t)} \geqq \varepsilon\right)$$
$$\leqq P\left(\sup_{\tau 2^{-n} \leqq t < \tau, 2^{n-1} < k \leqq 2^n} \frac{|B_k(t)|}{\lambda(k)\phi(t)} > \frac{\varepsilon}{2}\right)$$
$$\leqq 4 \sum_{i=0}^{n-1} \exp\left\{-\frac{\beta_{ni}^2}{2[\sigma_{ni}^2 + (1-\alpha_{n,i+1})\beta_{ni}/3]}\right\},$$

where $\sigma_{ni}^2 = 2^n \alpha_{n,i+1}(1-\alpha_{n,i+1})$ and

$$\beta_{ni} = \frac{\varepsilon 2^{n/2}\lambda(2^n)\alpha_{ni}^{\frac{1}{2}} h(\alpha_{n,i+1})(1-\alpha_{n,i+1})}{8}.$$

Set LHS (3.4) $= p_n(\tau)$. To prove the lemma, Borel–Cantelli says we need only find $0 < \tau \leqq \tau(\varepsilon)$ such that

$$(3.5) \quad \sum_{n \text{ large}} p_n(\tau) < \infty .$$

Observe that for $a, b, c > 0$, we clearly have $\exp(-a/(b+c)) \leqq \exp[-\min(a/2b, a/2c)]$. This and (3.4) imply

$$(3.6) \quad p_n(\tau) \leqq 4 \sum_{i=0}^{n-1} \exp\left[-\min\left(\frac{\beta_{ni}^2}{4\sigma_{ni}^2}, \frac{3\beta_{ni}}{4(1-\alpha_{n,i+1})}\right)\right].$$

Now do the algebra (the inequalities below are necessarily true only for large n):

$$(3.7) \quad \frac{\beta_{ni}^2}{4\sigma_{ni}^2} = \frac{\varepsilon^2\lambda^2(2^n)h^2(\alpha_{n,i+1})(1-\alpha_{n,i+1})}{512} \geqq \frac{\varepsilon^2 \log n h^2(\tau)}{2048}$$

for all $i \leqq n-1$, and

$$(3.8) \quad \frac{3\beta_{ni}}{4(1-\alpha_{n,i+1})} = \frac{3\varepsilon\lambda(2^n)\tau^{\frac{1}{2}}2^{i/2}h(\alpha_{n,i+1})}{32}$$
$$\geqq \frac{3\varepsilon(\tau \log n)^{\frac{1}{2}}2^{i/2}h(\alpha_{n,i+1})}{64}$$
$$\geqq \frac{3\varepsilon C(\tau \log n)^{\frac{1}{2}} n^{\frac{1}{2}}}{64}$$

uniformly in i, where the last inequality follows from Lemma 2.1 (v) for τ sufficiently small.

Now (3.6), (3.7), and (3.8) give (3.5). □

4. The theorem. Recall that $B = B[0, 1]$ is the space of bounded real-valued functions on $[0, 1]$ with the supremum norm, and K is the set of absolutely continuous $f \in B$ for which $f(0) = 0 = f(1)$ and $\int_0^1 [f'(t)]^2\,dt \leqq 1$.

THEOREM. *Let w be a positive real-valued function on* $[0, 1]$ *such that for some* $0 < \delta \leqq \frac{1}{2}$, $t^{\frac{1}{2}}w(t)$ *is monotone increasing on* $(0, \delta]$, $(1-t)^{\frac{1}{2}}w(t)$ *is monotone decreasing on* $[1-\delta, 1)$, *and w is bounded on* $[\delta, 1-\delta]$. *For* $n \geqq 3$, *set*

$$U_n = \frac{wB_n}{2^{\frac{1}{2}}\lambda(n)}\,.$$

If

$$\int_0^1 \frac{w^2(t)}{\log\log \dfrac{1}{t(1-t)}}\,dt < \infty\,, \tag{4.1}$$

then there is a set Ω_0 of probability one, such that on Ω_0 the sequence $\{U_n\}_{n\geqq 3}$ is relatively compact in B and the set of its limit points is

$$K_w = \{wf : f \in K\}\,.$$

Conversely, if the integral in (4.1) *diverges,*

$$\limsup_{n\to\infty} \|U_n\| = \infty \quad \text{a.s.}$$

(We emphasize, for the sake of clarity, that the monotoneity conditions are not required to hold at 0 and 1, making the (finite) values of w at these points arbitrary and irrelevant.)

PROOF. Let $\phi_1(t) = 1/w(t)$ and $\phi_2(t) = 1/w(1-t)$. If LHS (4.1) $= \infty$, then, since the integrand is bounded on $[\delta, 1-\delta]$, the integral diverges when taken over $(0, \delta] \,\dot\cup\, [1-\delta, 1)$. But this means

$$\int_0^\delta \frac{1}{\phi_1^2(t)\log\log \dfrac{1}{t}}\,dt = \infty$$

or

$$\int_0^\delta \frac{1}{\phi_2^2(t)\log\log \dfrac{1}{t}}\,dt = \infty\,,$$

so that the converse follows from Lemma 3.3.

So suppose (4.1) holds. Lemma 2.1 (ii) and the assumptions on w imply that

(*) $[t(1-t)]^{\frac{1}{2}}w(t)$ is bounded on $[0, 1]$ and tends to 0 as $t \to 0$ or 1.

Therefore, $U_n \in B$ for all n, a.s. The theorem now follows from Finkelstein's result (quoted earlier), together with (*), Lemma 3.3 (applied to ϕ_1 and ϕ_2), and the fact that $f(t) \leqq [t(1-t)]^{\frac{1}{2}}$ for $f \in K$ and $0 \leqq t \leqq 1$ (cf. Finkelstein (1971), inequality (3)). □

REMARK 1. Let w be any real-valued function of $[0, 1]$. Then it is clear that a sufficient condition for the existence of a set Ω_0 of probability one, on which $\{wB_n/[2^{\frac{1}{2}}\lambda(n)]\}_{n\geqq 3}$ is relatively compact in B with set of limit points K_w, is that $|w|$ be bounded by a function satisfying the conditions of the theorem (including (4.1)).

The theorem can be easily generalized to the case of an arbitrary common continuous distribution:

COROLLARY 1. *Let $Y_1, Y_2, \cdots$ be independent random variables with common continuous* df F, *and let F_n be the empirical* df *at stage n. Let $B(\mathbb{R})$ be the space of bounded real-valued functions on the real line, with the* sup *norm. Suppose w satisfies the conditions of the theorem or of Remark* 1 *(including* (4.1)), *and for $n \geqq 3$ and $x \in \mathbb{R}$ set*

$$V_n(x) = \frac{n^{\frac{1}{2}}[F_n(x) - F(x)]w[F(x)]}{2^{\frac{1}{2}}\lambda(n)}.$$

Then there is a set of probability one on which the sequence $\{V_n\}_{n\geqq 3}$ is relatively compact in $B(\mathbb{R})$ with set of limit points

$$K_{w,F} = \{w(F)f(F) : f \in K\}.$$

PROOF. The variables $X_n \equiv F(Y_n)$, $n = 1, 2, \cdots$, form a sequence of independent random variables, each distributed uniformly on $[0, 1]$, and the result can be obtained by applying the theorem (or Remark 1) to the X_n. □

REMARK 2. The corollary above is also valid when the Y_n have any common df F. The proof, which we omit, can be handled by the method above plus randomization.

As an application of the theorem, we have

COROLLARY 2. *Suppose w satisfies the condtions of the theorem (or Remark* 1), *including* (4.1). *Set $r(t) = [t(1-t)]^{\frac{1}{2}}$ for $0 \leqq t \leqq 1$. Then*

$$\limsup_{n\to\infty} \left\|\frac{wB_n}{2^{\frac{1}{2}}\lambda(n)}\right\| = \|wr\| \quad \text{a.s.}$$

PROOF. The theorem implies that on a set of probability one,

$$\limsup_{n\to\infty} \left\|\frac{wB_n}{2^{\frac{1}{2}}\lambda(n)}\right\| = \sup\{\|wf\| : f \in K\}. \tag{4.2}$$

Since $|f(t)| \leqq r(t)$ for $f \in K$ and $0 \leqq t \leqq 1$ (cf. Finkelstein (1971) inequality (3)), it follows that

$$\text{RHS (4.2)} \leqq \|wr\|.$$

For each $0 < s < 1$, define the function f_s by $f_s(0) = 0$, $f_s(s) = r(s)$, $f_s(1) = 0$, f_s linear in between. It is easy to check that $f_s \in K$. Therefore,

$$\begin{aligned}\text{RHS (4.2)} &\geqq \sup\{\|wf_s\| : 0 < s < 1\}\\ &\geqq \sup\{w(s)f_s(s) : 0 < s < 1\} = \|wr\|.\end{aligned}$$ □

REMARK 3. Lemmas 3.1 through 3.3 remain true if the assumption that h is monotone decreasing on $(0, \delta]$ is replaced by the slightly weaker assumption that hl is monotone decreasing on $(0, \delta]$, where

$$l(t) = \left(\log\log\frac{1}{t}\right)^{\frac{1}{2}}.$$

The theorem remains true if the monotonicity assumptions on $t^{\frac{1}{2}}w(t)$ and $(1-t)^{\frac{1}{2}}w(t)$ are replaced by similar assumptions on $t^{\frac{1}{2}}w(t)/l(t)$ and $(1-t)^{\frac{1}{2}}w(t)/l(1-t)$. The proofs will not be given here. (Actually, the proofs already given can be followed step by step, with only minor adjustments due to the fact that h need no longer be monotone. The main reason the proofs still work is that the function l varies little on the intervals over which the supremum is taken. However, the definition of a_n in the proof of Lemma 3.1 has to be changed somewhat, in order to guarantee that $l(c_n)/l(b_n) \to 1$. This is accomplished by, say, letting $\tilde{a}_n = \max(a_n, 1/n^2)$.)

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INSTITUTO DE MATEMATICA PURA E APLICADA
RUA LUIZ DE CAMOES, 68; ZC-58
RIO DE JANEIRO, GB
BRAZIL