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ON EXISTENCE AND NON-EXISTENCE OF PROPER, REGULAR, CONDITIONAL DISTRIBUTIONS[1]

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If $\mathscr{A}$ is the tail, invariant, or symmetric field for discrete-time processes, or a field of the form $\mathscr{F}_{t+}$ for continuous-time processes, then no countably additive, regular, conditional distribution given $\mathscr{A}$ is proper. A notion of normal conditional distributions is given, and there always exist countably additive normal conditional distributions if $\mathscr{A}$ is a countably generated sub σ-field of a standard space. The study incidentally shows that the Borel-measurable axiom of choice is false. Classically interesting subfields $\mathscr{A}$ of $\mathscr{B}$ possess certain desirable properties which are the defining properties for $\mathscr{A}$ to be "regular" in $\mathscr{B}$.

1. Whenever stronger conditions are not explicitly imposed, $(\Omega, \mathscr{B})$ is a *measurable* space, that is, Ω is a nonempty set, $\mathscr{B}$ is a σ-field of its subsets; and $\mathscr{A}$ is a sub σ-field of $\mathscr{B}$.

A *regular conditional distribution* (r.c.d.) given $\mathscr{A}$ on $\mathscr{B}$ is a function Q defined on $\Omega \times \mathscr{B}$ which satisfies for all $\omega \in \Omega$ and $B \in \mathscr{B}$:

(a-1) $Q(\omega, \cdot)$ is a probability measure on $\mathscr{B}$,
(a-2) $Q(\omega, \cdot)$ is countably additive;
(b) For each $B \in \mathscr{B}$, $Q(\cdot, B)$ is $\mathscr{A}$-measurable.

For a number of decades it has generally been considered appropriate to establish for a countably additive probability measure P on $\mathscr{B}$ the *existence for* P of a r.c.d. given $\mathscr{A}$, which has meant a Q satisfying (a) and (b) and related to P via:

$$\int_A Q(\omega, B)\, dP(\omega) = P(A \cap B), \qquad \text{for } A \in \mathscr{A} \text{ and } B \in \mathscr{B}. \tag{1.1}$$

However, regular conditional distributions may not be the appropriate concept on which to focus primary attention. As was shown in [3], and as will be shown below, for some classically interesting pairs of σ-fields $(\mathscr{A}, \mathscr{B})$, regular conditional distributions never satisfy the intuitive desideratum introduced in [3] of being *proper*, that is, of satisfying this requirement:

$$Q(\omega, A) = 1 \qquad \text{whenever } \omega \in A \in \mathscr{A}. \tag{1.2}$$

Note that for proper Q, Condition (1.1) is implied by the weaker, and simpler

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condition:

$$\int Q(\omega, B)\, dP(\omega) = P(B) \qquad \text{for } B \in \mathscr{B}. \tag{1.3}$$

A probability measure P on $\mathscr{A}$ is *extreme* if $P(A) = 0$ or 1 for all $A \in \mathscr{A}$. An *$\mathscr{A}$-atom* is the intersection of all elements of $\mathscr{A}$ that contain a given point of Ω. If for an $A \in \mathscr{A}$, $P(A) = 1$, P is *supported* by A.

THEOREM 1. *Suppose that $\mathscr{B}$ is countably generated. Then each of these conditions implies its successor.*

(a) *There exists an extreme probability measure on $\mathscr{A}$ which is countably additive and which is supported by no $\mathscr{A}$-atom belonging to $\mathscr{A}$;*
(b) *$\mathscr{A}$ is not countably generated;*
(c) *No regular conditional distribution given $\mathscr{A}$ is proper.*

PROOF. Suppose (c) fails, and let Q be a r.c.d. given $\mathscr{A}$ which is proper. Let $\mathscr{F}$ be a countable field which generates $\mathscr{B}$, and let $\mathscr{A}^*$ be the smallest σ-field with respect to which $Q(\cdot, F)$ is measurable for all $F \in \mathscr{F}$. Plainly, $\mathscr{A}^*$ is countably generated. So (b) will be seen to fail once it is shown that $\mathscr{A} = \mathscr{A}^*$. Obviously, $\mathscr{A}^* \subset \mathscr{A}$. For the reverse inclusion, verify that the set of B for which $Q(\cdot, B)$ is $\mathscr{A}^*$-measurable includes $\mathscr{F}$ and is closed under complements, countably monotone unions, and countably monotone intersections. Hence, $Q(\cdot, B)$ is $\mathscr{A}^*$-measurable for all $B \in \mathscr{B}$. In particular, for $A \in \mathscr{A}$, the event, $Q(\cdot, A) = 1$, namely A itself, is an element of $\mathscr{A}^*$. That is, $\mathscr{A} \subset \mathscr{A}^*$. So $\mathscr{A} = \mathscr{A}^*$ is countably generated, which contradicts (b).

Suppose next that (b) is false, that is, that $\mathscr{A}$ is countably generated, and let P be an extreme probability measure on $\mathscr{A}$ which is countably additive. Let $A_1, A_2, \cdots$ be an enumeration of a countable system of generators for $\mathscr{A}$, and let B_i equal A_i or $\Omega - A_i$ according as $P(A_i) = 1$ or 0. The intersection of the B_i has P-probability 1 and is an atom of $\mathscr{A}$, which contradicts (a). This completes the proof.

In view of Kolmogoroff's zero-one law, the Hewitt–Savage zero-one law, and the ergodicity of the shift, Theorem 1 implies that *there exists no proper, regular, conditional distribution given the tail field, the field of symmetric events, nor the invariant field, for one-sided, as well as for two-sided, discrete-time processes, where the states are elements of, say, a separable metric space with at least two points.*

Likewise, let Ω be the space C of continuous, real-valued functions ω defined on $[0, \infty)$ for which $\omega(0) = 0$, let $\mathscr{F}_t$ be, as usual, the least σ-field with respect to which all evaluation maps for $t' \leqq t$ are measurable and, let $\mathscr{F}_{t+} = \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon}$. *There exists no proper, regular conditional distribution given $\mathscr{F}_{t+}$.* For this, letting $\mathscr{B}$ be the σ-field generated by all the evaluation mappings and letting $\mathscr{A}_t = \mathscr{F}_{t+}$, is suffices to see that $\mathscr{A}_t$ satisfies Condition (a) of Theorem 1. As is well known, and easily verified, the restriction to $\mathscr{A}_0$—call it μ—of the distribution of a standard Brownian motion on C assigns to every element of $\mathscr{A}_0$ a probability of zero or 1. Now for each $t > 0$, it is plain that $\mathscr{A}_t$, too, satisfies Condition (a). For let $\mathscr{B}_t$ be the collection of all elements of $\mathscr{A}_t$ which are subsets of

$$B = [\omega : \omega(s) = 0 \text{ for } 0 \leqq s \leqq t] .$$

Then the σ-field $\mathscr{B}_t$ is isomorphic to $\mathscr{A}_0$, so on $\mathscr{B}_t$ there is a measure μ_t isomorphic to μ. The extension $\mu_t{}^*$ of μ_t to all of $\mathscr{A}_t$ obtained by letting $\mu_t{}^*(A) = \mu_t(A \cap B)$ for all $A \in \mathscr{A}_t$ is a remote, countably additive measure on $\mathscr{A}_t$ which is supported by no $\mathscr{A}_t$-atom. Hence, $\mathscr{A}_t$ satisfies Condition (a).

Similarly, *if* Ω *is the space* D *of right-continuous, with left limits, real-valued functions defined on* $[0, \infty)$, *no regular conditional distribution given* $\mathscr{F}_{t+}$ *is proper*, for the same argument, with Brownian motion replaced by a standard stable process, applies.

Since properness is a fundamentally desirable property for conditional distributions, it is important to relax one or more of the defining conditions for regularity.

Let P be a probability measure defined for $\mathscr{B}$. Consonant with a definition given in [5], a function Q defined on $\Omega \times \mathscr{B}$ is a *normal* conditional distribution given $\mathscr{A}$ for P if these conditions are satisfied:

(i) For each ω, $Q(\omega, \cdot)$ is a finitely additive probability on $\mathscr{B}$.

(ii) For each $B \in \mathscr{B}$, $Q(\cdot, B)$ is constant on each $\mathscr{A}$-atom.

(iii) For each $B \in \mathscr{B}$, $Q(\cdot, B)$ is measurable with respect to the completion of P, and (1.3) holds.

(iv) $Q(\omega, A) = 1$ if $\omega \in A \in \mathscr{A}$.

Plainly, if Q is a regular, proper distribution for P, then Q is normal and countably additive.

If Ω is a Borel subset of a complete separable metric space and $\mathscr{B}$ is the set of all its Borel subsets, then $(\Omega, \mathscr{B})$ is a *standard space*.

Even if $\mathscr{A}$ is a countably generated sub σ-field of a standard space, there may exist no proper, regular, conditional distributions as was shown in [3]. But normal conditional distributions do then exist:

THEOREM 2. *If* $(\Omega, \mathscr{B})$ *is a standard space, or, more generally, a Lusin space, and* $\mathscr{A}$ *is a countably generated sub sigma-field of* $\mathscr{B}$, *then, for every countably additive* P *on* $\mathscr{B}$, *there is a countably additive, normal, conditional distribution* Q *given* $\mathscr{A}$.

As defined in [1], $(\Omega, \mathscr{B})$ is a *Lusin* space if $\mathscr{B}$ is countably generated and the range of every $\mathscr{B}$-measurable, real-valued function defined on Ω is an analytic set.

PROOF OF THEOREM 2. According to [1] (Theorem 5), there is an $N \in \mathscr{A}$ with $P(N) = 0$ and a countably additive Q that satisfies all the conditions for normality except that (iv) may fail for $\omega \in N$. Let ϕ by any mapping of Ω into Ω which maps each $\mathscr{A}$-atom, A, into a single point of A. Now define a new Q

thus. Outside of N, the new Q is the same as the old. But for $\omega \in N$, let $Q(\omega, \cdot)$ be the one-point delta-measure that assigns probability one to $\phi(\omega)$. As redefined, Q is normal.

When $\mathscr{A}$ is not countably generated, it might sometimes be useful, in view of Theorem 2, to know that for a class $\mathscr{P}$ of P's there is a countably generated $\mathscr{A}^* \subset \mathscr{A}$ equivalent to $\mathscr{A}$. The latter clause means that, for every $P \in \mathscr{P}$ and every $A \in \mathscr{A}$, there is an $A^* \in \mathscr{A}^*$ such that A and A^* differ by an event of P-measure zero.

For discrete-time i.i.d. processes, P, the zero-one laws replace the nonseparable tail or symmetric fields by the trivial field. For stationary processes, the nonseparable field of invariant events is equivalent to the separable $\mathscr{A}^*$ determined by the variables

$$Y_n = \limsup_{k\to 0} \frac{B_n + \cdots + B_n(T^k)}{k+1} \tag{1.4}$$

where B_n is a sequence of events generating $\mathscr{B}$, T is the usual shift, and the convenient notational device suggested by de Finetti [4] of using the same letter to designate an event as well as its indicator has been employed.

That there is no countably generated $\mathscr{A}^*$ equivalent to the tail σ-field $\mathscr{A}$ for Markov processes, it is the purpose of the next theorem to demonstrate.

THEOREM 3. *Let Ω be the set of infinite sequences $\omega = x_1, x_2, \cdots$ with each x_i equal to 0 or 1. Let $X_n(\omega) = x_n$, $n = 1, 2, \cdots$, let $\mathscr{B}_n = \mathscr{B}(X_n, X_{n+1}, \cdots)$ be the σ-field generated by the X_j for $j \geqq n$, and let $T = \bigcap_{n=1}^{\infty} \mathscr{B}_n$ be the tail field. Then, for any countably generated sub-sigma-field $\mathscr{S}$ of $\mathscr{T}$, there is a P on Ω under which X_n is Markov and for which $\mathscr{S}$ is not equivalent to $\mathscr{T}$.*

PROOF OF THEOREM 3. Let P^* be fair coin measure. There is an $\mathscr{S}$-atom S with $P^*(S) = 1$. For any $\omega \in \Omega$, denote by $\bar{\omega}$ the (unique) point such that $X_n(\bar{\omega}) = 1 - X_n(\omega)$ for all n. Since $\omega \rightarrow \bar{\omega}$ is P^* preserving, $\exists\, \omega_0 \in S$ with $\bar{\omega}_0 \in S$. Consider the P with $P(\omega_0) = P(\bar{\omega}_0) = \frac{1}{2}$. It is Markov; in fact, given any X_n, one can produce the entire process. And $\mathscr{S}$ is not P-equivalent to $\mathscr{T}$, since the tail atoms containing ω_0 and $\bar{\omega}_0$ have P-measure $\frac{1}{2}$ each are contained in the $\mathscr{S}$-atom S. This ends the proof.

If attention is restricted to stationary Markov processes, or Markov processes with stationary transitions, perhaps, in contrast to Theorem 3, there is a countably generated $\mathscr{S}$ equivalent to $\mathscr{T}$. For the finite state case this is certainly the case, for the indecomposable set of states in which the process is, and the phase, are a tail variable that determines the tail field up to equivalence.

2. That no regular, conditional distribution given the tail field of the usual coin-tossing space Ω is proper can be established by an argument somewhat different from that of Theorem 1. Indeed, this also follows, as in [5], from the non-existence of an analytic set that has precisely one point in common with each tail atom, an *analytic $\mathscr{A}$-section*, where $\mathscr{A}$ is the tail field. And this latter non-existence holds for, by a Vitali-like argument relying on the group-theoretic properties enjoyed by a product of two-point groups, and in particular, because of the existence on it of an invariant Haar measure, there exists no Lebesgue measurable $\mathscr{A}$-selection, and, a fortiori, no analytic $\mathscr{A}$-selection.

Though we see no way to show the non-existence of Lebesgue-like selections for σ-fields $\mathscr{A}$ where there is no obvious group structure, the non-existence of analytic $\mathscr{A}$-selections is indeed a fairly general phenomenon, as Theorem 4, and, to some extent, as Theorem 5, suggests.

A countably generated sub σ-field $\mathscr{A}$ of the Borel subsets, $\mathscr{B}$, of the real line which separates points is $\mathscr{B}$, where $\mathscr{A}$ *separates* points if, for x different from y, there is an $A \in \mathscr{A}$ with $x \in A$ and $y \in A^c$. Slightly generalized, this fact becomes this lemma, previously reported in ([1] Theorem 3).

LEMMA 1. *Let* $(\Omega, \mathscr{B})$ *be a Lusin space and* $\mathscr{A}$ *a countably generated sub* σ*-field of* $\mathscr{B}$. *Then every* $B \in \mathscr{B}$ *that is a union of elements of* $\mathscr{A}$ *is itself an element of* $\mathscr{A}$.

THEOREM 4. *Let* $\mathscr{A}$ *be a sub* σ*-field of a standard space* $(\Omega, \mathscr{B})$. *Then each of the following conditions implies its successors.*

(a) $\mathscr{A}$ *has a proper, regular conditional distribution.*

(b) $\mathscr{A}$ *has a selection function*, i.e., *there exists a function* $f\colon \Omega \to \Omega$ *such that* $f^{-1}(\mathscr{B}) \subset \mathscr{A}$, *and* $x \in A \in \mathscr{A}$ *implies* $f(x) \in A$.

(c) $\mathscr{A}$ *has a separating function*, i.e., $\exists\, \mathscr{A}$*-measurable* $g : \Omega \to$ *some complete, separable, metric space such that* $x \in A \in \mathscr{A}$ *and* $y \notin A \to f(y) \neq f(x)$.

(d) $\mathscr{A}$ *is countably generated.*

(e) *Every* 0–1 *countably additive* P *on* $\mathscr{A}$ *concentrates on an atom of* $\mathscr{A}$.

PROOF. Assume (a). By Theorem 1, $\mathscr{A}$ is countably generated, so [3] applies to yield (b). If (b) holds, (c) is immediate since all selection functions are separating functions. Suppose that (c) holds, and let g be a separating function for $\mathscr{A}$. *Then* $\mathscr{B}(g) \subset \mathscr{A}$. So every atom of $\mathscr{B}(g)$ is a union of $\mathscr{A}$-atoms. On distinct $\mathscr{A}$-atoms, g assumes distinct values, and on each $\mathscr{B}(g)$-atom, g is constant. This implies that every atom of $\mathscr{B}(g)$ is an $\mathscr{A}$-atom, and conversely. In particular, for $A \in \mathscr{A}$, A is a union of $\mathscr{B}(g)$-atoms and $A \in \mathscr{B}$. In view of Lemma 1, $A \in \mathscr{B}(g)$. So $\mathscr{A} = \mathscr{B}(g)$, which yields (d). That (d) $\to$ (e) is part of Theorem 1. □

COROLLARY 1. *Let* Ω *be the space of continuous functions* ω *on* $0 \leqq t < \infty$. *If* $\phi : \Omega \to \Omega$ *is tail measurable, or equivalently, is Borel measurable and satisfies* $\phi(\omega) = \phi(\omega^*)$ *if* $\omega(t) = \omega^*(t)$ *for all sufficiently large* t, *then* $\exists\, \omega$ *for which* $\phi(\omega)$ *is not in the same tail atom as is* ω.

Let $\mathscr{S}$ be the set of all nonempty, finite or denumerable, sets of real numbers. A real-valued function h defined on $\mathscr{S}$ is a *choice function* if $h(S) \in S$ for all $S \in \mathscr{S}$. That there is no Borel measurable choice function is perhaps already known to logicians, but it is related to Theorem 4, thus.

COROLLARY 2. *The Borel-measurable axiom of choice is false, that is, there is no Borel-measurable function g defined on the space Ω of sequences of real numbers such that $g(\omega_1) = g(\omega_2)$ whenever ω_1 and ω_2 have the same range and such that, for all ω, $g(\omega)$ is an element of the range of ω.*

Of course, if $\omega = x_1, x_2, \cdots$, then the *range*, $\rho(\omega)$, of ω is the set of real numbers x such that, for some n, $x_n = x$.

PROOF. Let $\mathscr{A}$ be the least σ-field with respect to which g is Borel measurable, let μ be a countably additive probability on the Borel subsets of the real line such that $\mu(S) = 0$ for all countable S, and let P be that measure on Ω which is the product of infinitely many copies of μ. As the Hewitt–Savage zero-one law implies, restricted to the Borel subsets of Ω invariant under finite permutations, and a fortiori restricted to $\mathscr{A}$, P is a 0–1 measure.

To each real number, s, there is associated the atom A of $\mathscr{A}$ consisting of all ω such that $\rho(\omega) = s$, and every atom of $\mathscr{A}$ is of this form. Therefore, for all $\mathscr{A}$-atoms A,

$$\begin{aligned} P(A) &\leqq P(\textstyle\bigcup_n (X_n = s)) \\ &\leqq \textstyle\sum P(X_n = s) \\ &\leqq \textstyle\sum \mu\{s\} \\ &= 0\,. \end{aligned} \tag{2.1}$$

Since $P(A)$ is 0, Condition (e), and hence Condition (c), of Theorem 4 fails to hold. Since there is no $\mathscr{A}$-separating function, there is no g with the properties stated in Corollary 2. □

As is easily seen, the implications (b) $\rightarrow$ (c) $\rightarrow$ (d) $\rightarrow$ (e) hold if $(\Omega, \mathscr{B})$ is only assumed to be a Lusin space. Perhaps the implication (a) $\rightarrow$ (b), and hence, Theorem 4 itself, holds under this weakened hypothesis.

REMARK. As for the reverse implications, obviously (b) implies (a) and (d) implies (c). Whether (e) implies (d) we do not know. The remaining implication, (c) implies (b), is false even if $(\Omega, \mathscr{B})$ is a standard space, as can be seen thus. There exists a Borel subset, B, of the unit square whose projection on the x axis is the unit interval I but which contains no Borel graph over I (e.g., see [3] and its references). Let II be the projection onto I and let $\mathscr{A} = \mathscr{B}(\text{II})$. As is evident, II is a separating function for $\mathscr{A}$, but there exists no selection function for $\mathscr{A}$.

Of course, Condition (b) of Theorem 4 implies:

(b′) $\mathscr{A}$ has an analytic selection-set S, that is, there is an analytic subset S of Ω which intersects each atom of $\mathscr{A}$ in exactly one point.

We suppose that (b′) is weaker than (b), but in the presence of modest conditions on $\mathscr{A}$, (b′) implies (b), as the next theorem shows.

THEOREM 5. *Let $(\Omega, \mathscr{B})$ be a standard space, and let $\mathscr{A}$ be a sub σ-field of $\mathscr{B}$ such that:*

(i) *Every $\mathscr{A}$-atom is an element of $\mathscr{A}$;*
(ii) *If $B \in \mathscr{B}$ is a union of elements of $\mathscr{A}$, then $B \in \mathscr{A}$;*
(iii) *$Q = \{(x, y) \in \Omega \times \Omega : x$ and y are in the same $\mathscr{A}$-atom$\}$ is analytic.*

Then, for any analytic subset S of Ω which intersects each $\mathscr{A}$-atom in exactly one point, if $f(x)$ is the unique $y \in S$ which is in the same $\mathscr{A}$-atom as is x, then f is $\mathscr{A}$-measurable.

PROOF. The graph G of f is analytic, for $G = Q \cap \{y \in S\}$ is the intersection of two analytic sets. The following argument leans on that in ([6] page 398). Note that G^c, namely $\Omega \times \Omega - G$, is the projection on the first two coordinates of

$$A = \{(x, y, z) \in \Omega \times \Omega \times \Omega : z \neq y \text{ and } (x, z) \in G\} . \tag{2.2}$$

Obviously, A, and hence also G^c, is analytic. Since both G and its complement are analytic, G is Borel. Because $(\Omega, \mathscr{B})$ is a standard space and the graph of f is Borel, f itself is Borel. Since f is constant on $\mathscr{A}$-atoms, (i) and (ii) together yield the $\mathscr{A}$-measurability of f.

3. This section and those that follow are concerned to show that various well known sub σ-fields $\mathscr{A}$ of standard spaces $(\Omega, \mathscr{B})$ possess properties (i), (ii) and (iii) of Theorem 5.

Call $\mathscr{A}$ *regular* if (i), (ii) and this strengthening of (iii) of Theorem 5 hold:

(iv) $Q = Q(\mathscr{A})$ is a Borel subset of $\Omega \times \Omega$.

As is easily verified with the help of Lemma 1, *if $\mathscr{A}$ is countably generated, then it is regular.* With countably generated $\mathscr{A}$'s as building blocks, new $\mathscr{A}$'s that are regular can be constructed.

For each $x \in \Omega$, let $\mathscr{A}(x)$ be the $\mathscr{A}$-atom containing x. If, for each x, $\mathscr{A}(x) \in \mathscr{A}$, $\mathscr{A}$ is *atomic* and $Q(\mathscr{A})$ is the set of (x, y) such that $\mathscr{A}(x) = \mathscr{A}(y)$.

FACT 1. $\mathscr{A}_1 \subset \mathscr{A}_2$ implies: $Q(\mathscr{A}_2) \subset Q(\mathscr{A}_1)$, and $\mathscr{A}_2(x) \subset \mathscr{A}_1(x)$ for all x.

FACT 2. $Q(\mathscr{A}) = Q(\mathscr{A}^*)$, where $\mathscr{A}^*$ is the least field, including $\mathscr{A}$, which satisfies (ii) of Theorem 5.

FACT 3. If $\mathscr{A}_1 \supset \mathscr{A}_2 \supset \cdots$, then $Q(\bigcap \mathscr{A}_j) = \bigcup Q(\mathscr{A}_j)$.

PROOF. Since $\bigcap \mathscr{A}_j \subset \mathscr{A}_j$, $Q(\bigcap \mathscr{A}_j) \supset Q(\mathscr{A}_j)$ for all j, so

$$Q(\bigcap \mathscr{A}_j) \supset \bigcup Q(\mathscr{A}_j) . \tag{3.1}$$

For the reverse inclusion, suppose (x, y) is not in $\bigcup Q(\mathscr{A}_j)$. Then, for all j, $\exists\, A_j \in \mathscr{A}_j$ such that, $x \in A_j$ and $y \in A_j^c$. So $x \in A = \limsup A_j$ and $y \in A^c$. Since $A \in \bigcap \mathscr{A}_j$, (x, y) is not in $Q(\bigcap \mathscr{A}_j)$, which completes the proof.

For $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots$, let $\bigcup \mathscr{A}_i$ be the set theoretic union of the $\mathscr{A}_i$, which is a field, and let $\bigvee \mathscr{A}_i$ be the σ-field generated by $\bigcup \mathscr{A}_i$.

FACT 4. If $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots$, then

$$Q(\bigvee \mathscr{A}_i) = Q(\bigcup \mathscr{A}_i) = \bigcap_i Q(\mathscr{A}_i) . \tag{3.2}$$

PROOF. The first equality obtains essentially by Fact 2. And, by Fact 1,

$$Q(\bigcup \mathscr{A}_i) \subset \bigcap Q(\mathscr{A}_i) . \tag{3.3}$$

For the reverse inclusion, suppose $(x, y) \notin Q(\bigcup \mathscr{A}_i)$. Then $\exists\, A \in \bigcup \mathscr{A}_i$, such that $x \in A$, $y \in A^c$. Since $A \in \mathscr{A}_i$ for some i, $(x, y) \notin Q(\mathscr{A}_i)$. That is, $(x, y) \notin \bigcap Q(\mathscr{A}_i)$. This completes the proof.

Several of the above facts are summarized, thus:

PROPOSITION 1. *Suppose $\mathscr{A}_i$ is a monotone sequence of σ fields. Then*

(a) *$Q(\mathscr{A}_i)$ is monotone in the reverse order;*

(b) $Q(\lim \mathscr{A}_i) = \lim Q(\mathscr{A}_i)$;

(c) *If each $Q(\mathscr{A}_i)$ is Borel or analytic, so is $Q(\lim \mathscr{A}_i)$.*

(d) *For each $x \in \Omega$, the sequence of atoms $\mathscr{A}_i(x)$ is monotone in the reverse order, and*

$$(\lim \mathscr{A}_i)(x) = \lim (\mathscr{A}_i(x)) ; \tag{3.4}$$

(e) *If also each $\mathscr{A}_i$ is atomic, so is $\lim \mathscr{A}_i$.*

For $\mathscr{A} \subset \mathscr{B}$, $\mathscr{A}$ is *saturated* (in $\mathscr{B}$) if every element of $\mathscr{B}$ which is a union of elements of $\mathscr{A}$ is itself an element of $\mathscr{A}$. Slightly more stringent would be to require that every element of $\mathscr{B}$ which is a union of atoms of $\mathscr{A}$ is in $\mathscr{A}$. If $\mathscr{A}$ is atomic, these notions coalesce, of course.

PROPOSITION 2. *Suppose $\mathscr{A}_i$ is a monotone decreasing sequence of σ-fields each of which is saturated in $\mathscr{B}$. Then $\lim \mathscr{A}_i$ is saturated, too.*

PROOF. Let $B \in \mathscr{B}$ be a union of elements of $\lim \mathscr{A}_i$. Then, for each i, B is a union of elements of $\mathscr{A}_i$, and hence, is itself an element of $\mathscr{A}_i$, for $\mathscr{A}_i$ is saturated. That is, $B \in \lim \mathscr{A}_i$. This completes the proof.

We do not know whether the same conclusion holds for increasing sequences of σ-fields, though we doubt it.

COROLLARY 3. *The tail σ-field of a product of denumerably many standard spaces is regular. Also the usual fields $\mathscr{F}_{t-}$, $\mathscr{F}_t$, and $\mathscr{F}_{t+}$ for continuous time processes with paths in C or D are regular.*

COROLLARY 4. *Let Ω be the space of infinite sequences of real numbers, and let $S \subset \Omega$ have one, and only one, element in common with each tail atom. Then S is not analytic.*

4. This section shows that the symmetric and invariant fields are regular with "saturated" replaced by a somewhat stronger property.

For each subset E of Ω, let $\mathscr{A}(E)$ be the union of all $\mathscr{A}$-atoms whose intersection with E is nonempty.

PROPOSITION 3. *The operation $E \to \mathscr{A}(E)$ enjoys these properties:*

(a) $E \subset \mathscr{A}(E)$;

(b) $E_1 \subset E_2 \to \mathscr{A}(E_1) \subset \mathscr{A}(E_2)$;

(c) $\mathscr{A}(\mathscr{A}(E)) = \mathscr{A}(E)$;

(d) $\mathscr{A}_1 \subset \mathscr{A}_2 \to \mathscr{A}_2(E) \subset \mathscr{A}_1(E)$;

(e) *If* $\mathscr{A}_i$ *is a monotone decreasing sequence of* σ*-fields, then* $\mathscr{A}_i(E)$ *is monotone in the reverse order, and*

(f) $(\lim \mathscr{A}_i)(E) = \lim \mathscr{A}_i(E)$.

For $\mathscr{A} \subset \mathscr{B}$, $\mathscr{A}$ *is* strongly saturated *in* $\mathscr{B}$ *if, whenever* $B \in \mathscr{B}$, $\mathscr{A}(B) \in \mathscr{A}$.

PROPOSITION 4. *Let* $\mathscr{A}_i$ *be a monotone decreasing sequence of* σ*-fields each of which is strongly saturated in* $\mathscr{B}$. *Then* $\lim \mathscr{A}_i$ *is strongly saturated, too.*

PROOF. Let $B \in \mathscr{B}$. Then $(\lim \mathscr{A}_i)(B) = \lim \mathscr{A}_i(B)$, and $\mathscr{A}_i(B) \in \mathscr{A}_i$ for all i. As is easily seen, $\lim \mathscr{A}_i(B) \in \lim \mathscr{A}_i$, which completes the proof.

Let $(\Omega, \mathscr{B})$ be a measurable space and G a group of $(\mathscr{B}, \mathscr{B})$-measurable transformations of Ω onto itself, that is, each $g \in G$ is a 1–1 map of Ω onto Ω such that

$$g^{-1}(B) \in \mathscr{B} \qquad \text{for each} \quad B \in \mathscr{B}. \tag{4.1}$$

A subset A of Ω is *G-invariant* if $g^{-1}(A) = A$ for all $g \in G$. Plainly, the G-invariant sets form a field closed under the formation of arbitrary unions, so if $\mathscr{A}$ designates the collection of G-invariant sets that are elements of $\mathscr{B}$, then $\mathscr{A}$ is a saturated, sub σ-field of $\mathscr{B}$. For $E \in \mathscr{B}$ and G countable, $\mathscr{A}(E)$, the union of the $\mathscr{A}$-atoms that intersect E, is simply the union over all $g \in G$ of $g(E)$. This implies:

LEMMA 2. *If* G *is a (countable) group of measurable transformations of the measurable space* $(\Omega, \mathscr{B})$, *then the* G*-invariant,* $\mathscr{B}$*-measurable sets constitute a (strongly) saturated, sub* σ*-field of* $\mathscr{B}$.

As is evident, $Q = Q(\mathscr{A})$, the set of (x, y) such that x and y cannot be separated by elements of $\mathscr{A}$, is simply the set of (x, y) such that x and y are in the same G-orbit, and the $\mathscr{A}$-atom that contains x is the orbit of x. In particular, $(x, y) \in Q$ if, and only if, for some $g \in G$, $g(x) = y$, that is, if and only if, for some $g \in G$, (x, y) is in the inverse image of the main diagonal D of $\Omega \times \Omega$ under the map $(x, y) \to (g(x), y)$. So if D is an element of the product σ-field, $\mathscr{B} \times \mathscr{B}$, and G is countable, then Q, too, is an element of $\mathscr{B} \times \mathscr{B}$ in which event, $\mathscr{A}$ is atomic. For D to be an element of $\mathscr{B} \times \mathscr{B}$ it suffices that some countable subset of $\mathscr{B}$ separates points, as is easily verified. These remarks, together with Lemma 2, imply:

PROPOSITION 5. *Let* G *be a countable group of measurable transformations acting on a standard space* $(\Omega, \mathscr{B})$ *and let* $\mathscr{A}$ *be the* G*-invariant,* $\mathscr{B}$*-measurable sets. Then* $\mathscr{A}$ *is regular and strongly saturated in* $\mathscr{B}$.

Suppose $(\Omega, \mathscr{B})$ is the product of a denumerable number of copies of the same standard space. Then, as usual, *the symmetric sets* are the $\mathscr{B}$-sets that are invariant under all permutations of the coordinates that keep all but a finite number of coordinates fixed, and the *invariant* sets are those $\mathscr{B}$-sets that are invariant under the shift operation.

COROLLARY 5. *Let* $(\Omega, \mathscr{B})$ *be the space of doubly infinite sequences* $\cdots x_{-1}, x_0, x_1, \cdots$ *of elements of a standard space. Then the symmetric sets, as well as the sets invariant under the shift, form a regular, strongly saturated, sub* σ*-field of* $\mathscr{B}$.

REMARK. As is now evident, "symmetric" or "invariant under invertible shifts" can replace "tail" in Corollary 4.

Of course, if $Q(\mathscr{A})$ is Borel, then every $\mathscr{A}$-atom is Borel. And, as the referee pointed out to us, if $Q(\mathscr{A})$ is analytic, then every $\mathscr{A}$-atom is analytic, because every section of Q is analytic. Perhaps $Q(\mathscr{A})$ is analytic only if it is Borel?

5. Let $\mathscr{F}_t : t \geqq 0$, be an increasing family of regular, sub σ-fields of a standard space $(\Omega, \mathscr{B})$ which generates $\mathscr{B}$, that is, for which

$$V_t \mathscr{F}_t = \mathscr{B}. \tag{5.1}$$

To avoid difficulties later, a *stopping* time τ in this paper means a real-valued function with domain *all* of Ω and with values in the half-line $[0 \leqq t < \infty)$ which satisfies

$$(\tau \leqq t) \in \mathscr{F}_t \qquad \text{for all } t. \tag{5.2}$$

As usual, $\mathscr{F}_\tau$, the field of measurable events determined by time τ, is defined by

$$A \in \mathscr{F}_\tau \leftrightarrow A \cap (\tau \leqq t) \in \mathscr{F}_t \qquad \text{for all } t. \tag{5.3}$$

As will soon be evident, $\mathscr{F}_\tau$ is atomic and saturated. Perhaps $Q(\mathscr{F}_\tau)$ is Borel, in which event $\mathscr{F}_\tau$ is also regular, but we have established this only in the presence of additional assumptions.

THEOREM 6. *Under any of the following additional hypotheses,* $\mathscr{F}_\tau$ *is regular*:

(a) τ *assumes only a countable number of values*;

(b) *The family* $\mathscr{F}_t$: $t \geqq 0$ *is continuous on the right*;

(c) Ω *is either* C, *the space of continuous, or* D, *the space of right-continuous with left-limits,* R*-valued functions defined on* $[0, \infty)$, *and* $\mathscr{F}_t$ *is the* σ*-field generated by the evaluation maps for moments of time* $s \leqq t$.

LEMMA 3. *If each* $\mathscr{F}_t$ *is a saturated sub* σ*-field of* $\mathscr{B}$, *then so is* $\mathscr{F}_\tau$.

LEMMA 4. *For* x *in the domain of* τ,

$$\mathscr{F}_\tau(x) = \mathscr{F}_{\tau(x)}(x). \tag{5.4}$$

PROOF. Suppose y is not in $\mathscr{F}_{\tau(x)}(x)$. Then $\exists\, A \in \mathscr{F}_{\tau(x)}$ such that $x \in A$ and $y \in A^c$. Let $B = A \cap \{\tau = \tau(x)\}$, and verify that $B \in \mathscr{F}_\tau$, $x \in B$ and $y \in B^c$, so y is not in $\mathscr{F}_\tau(x)$. Conversely, if y is not in $\mathscr{F}_\tau(x)$, $\exists\, B \in \mathscr{F}_\tau$ such that $x \in B$ and $y \in B_c$. Let $t = \tau(x)$ and let $A = B \cap \{\tau \leqq t\}$. Since $A \in \mathscr{F}_t$, $x \in A$ and $y \in A^c$, y is not in $\mathscr{F}_t(x)$. That is, y is not in $\mathscr{F}_{\tau(x)}(x)$. This completes the proof.

LEMMA 5. *If each* $\mathscr{F}_t$ *is atomic, then* $\mathscr{F}_\tau$ *is also atomic.*

PROOF. In view of Lemma 4, all that need be verified is that $\mathscr{F}_{\tau(x)}(x) \in \mathscr{F}_\tau$, or, equivalently, that

$$(\tau \leqq t) \cap \mathscr{F}_{\tau(x)}(x) \in \mathscr{F}_t \qquad \text{for all } t . \tag{5.5}$$

Case (1). $t \geqq \tau(x)$. Then $\mathscr{F}_{\tau(x)}(x) \in \mathscr{F}_{\tau(x)} \subset \mathscr{F}_t$, since $\mathscr{F}_{\tau(x)}$ is atomic and $\tau(x) \leqq t$. So (5.5) holds.

Case (2). $t < \tau(x)$. Since $x \in (\tau = \tau(x)) \in \mathscr{F}_{\tau(x)}$, $\mathscr{F}_{\tau(x)}(x) \subset (\tau = \tau(x))$. Hence the left-hand side of (5.5) is empty, which proves Lemma 5.

The facts and lemmas above, and their proofs, are valid without assumptions (a), (b) or (c) and, indeed, without the assumption that $(\Omega, \mathscr{B})$ be a standard space.

LEMMA 6. $Q(\mathscr{F}_\tau)$ *is Borel if* (a), (b) *or* (c) *obtain.*

PROOF. Case (a). Let A_t be the set of $(x, y) \in \Omega \times \Omega$ such that $\tau(x) = \tau(y) = t$, and verify that

$$Q(\mathscr{F}_\tau) = \bigcup_t A(t) \cap Q(\mathscr{F}_t) . \tag{5.6}$$

Since each $Q(\mathscr{F}_t)$ is Borel, as is each $A(t)$, and the union is over only a countable set of t's, the proof is complete.

Case (b). Let $\tau(n) \searrow \tau$, each $\tau(n)$ countably valued. So $\mathscr{F}_\tau$ is the intersection of the decreasing sequence $\mathscr{F}_{\tau(n)}$, and $Q(\mathscr{F}_{\tau(n)})$ is Borel for all n. An application of Proposition 1(c) completes the proof for this case.

Case (c). $Q(\mathscr{F}_\tau)$ is the set of (x, y) such that

$$\tau(x) = \tau(y) ; \qquad x(\tau(x)) = y(\tau(y)) ; \tag{5.7}$$

and

$$x(t) = y(t) \qquad \text{for all rational } \ t \leqq \tau(x) . \tag{5.8}$$

That this is a Borel set is easily confirmed. This completes the proof of Lemma 6, which, together with Lemmas 3 and 5, proves Theorem 6. Possibly the hypothesis (c) of Theorem 6 implies not only that $\mathscr{F}_\tau$ is regular, but also that it is countably generated.

Here is a final, though less useful, observation.

PROPOSITION 6. *If* $(\Omega, \mathscr{B})$ *is a measurable space,* $\mathscr{F}_t \colon t \geqq 0$ *is an increasing family of strongly saturated sub* σ*-fields of* $\mathscr{B}$, *then* $\mathscr{F}_\tau$ *is strongly saturated if Conditions* (a) *or* (b) *of Theorem* 6 *hold.*

PROOF. What must be seen is that for $B \in \mathscr{B}$, $\mathscr{F}_\tau(B) \in \mathscr{F}_\tau$. If τ assumes only a countable number of values, it suffices that, for each t,

$$A = (\tau = t) \cap \mathscr{F}_\tau(B) \in \mathscr{F}_t . \tag{5.9}$$

To verify (5.9), note first that

$$\mathscr{F}_\tau(B) = \bigcup_t \mathscr{F}_\tau(B \cap (\tau = t)) = \bigcup_t \mathscr{F}_t(B \cap (\tau = t)), \tag{5.10}$$

and that

$$\mathscr{F}_t(B \cap (\tau = t)) \subset (\tau = t). \tag{5.11}$$

Together, (5.10) and (5.11) imply:

$$A = \mathscr{F}_t(B \cap (\tau = t)). \tag{5.12}$$

Because $\mathscr{F}_t$ is saturated A is therefore an element of $\mathscr{F}_t$.

Suppose now that $\mathscr{F}_t$ is continuous on the right and let $\tau(1) \geqq \tau(2) \geqq \cdots$ be a decreasing sequence of stopping times each of which assumes only a countable number of values and $\tau(n) \to \tau$. Since $\mathscr{F}_{\tau(n)}$ is a decreasing sequence of strongly saturated fields which converges to $\mathscr{F}_\tau$, Proposition 4 implies that $\mathscr{F}_\tau$ is strongly saturated, too. □

We suppose that even in the absence of one or both of the assumptions: $\bigvee \mathscr{F}_t = B$, and the domain of τ is *all* of Ω, the results of this section suitably formulated remain valid. But there may be some difficulties in demonstrating this, as a careful and helpful referee called to our attention.

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