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THE JOINT DISTRIBUTION OF RECORD VALUES AND INTER-RECORD TIMES

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The joint distribution of the nth record value and the nth inter-record time is considered to obtain a transformation which has nondegenerate limit distribution.

Let X_n, $n = 0, 1, \cdots$ be i.i.d. random variables with a common absolutely continuous df F. By convention X_0 is a record value. X_k, $k \geqq 1$, is an upper record value if $X_k > \max(X_0, \cdots, X_{k-1})$.

Define $M_0 = 0$, $M_k = \min\{n : n > M_{k-1}, X_n > X_{M_{k-1}}\}$, $k \geqq 1$; $\Delta_0 = 0$, $\Delta_k = M_k - M_{k-1}$, $k \geqq 1$; and $Z_k = X_{M_k}$, $k \geqq 0$. $\{Z_k, k \geqq 0\}$ is the record value sequence and $\{\Delta_k, k \geqq 1\}$ are the inter-record times. Under the transformation $T_n = -\ln(1 - F(Z_n))$, $n \geqq 0$, the problem becomes that of exponential (1) rv's for which it is known that

$$P(\Delta_n > \delta) = \int_0^\infty \frac{x^{n-1}}{(n-1)!} e^{-x}(1 - e^{-x})^{\delta^*}\, dx\,, \qquad \delta > 0\,, \tag{1}$$

where δ^* is the largest integer $\leqq \delta$. It is also known that T_n has the gamma $(n + 1, 1)$ distribution with the density $e^{-t}t^n/n!$, $t > 0$. Moreover, from the asymptotic $(n \to \infty)$ normality of the gamma distribution it follows that

$$\tau_n = n^{-\frac{1}{2}}(T_n - n) \to_{\mathscr{L}} N(0, 1)\,, \qquad S_n = n^{-\frac{1}{2}}(\ln \Delta_n - n) \to_{\mathscr{L}} N(0, 1)\,. \tag{2}$$

For a more detailed treatment of this subject see [2], [3], and papers referenced therein. The rv's τ_n and S_n are dependent and their joint distribution is given by

$$\begin{aligned} P(\Delta_n > \delta, T_n > t) &= \int_t^\infty e^{-y}\, dy \int_0^y \frac{x^{n-1}}{(n-1)!}(1 - e^{-x})^{\delta^*}\, dx \\ &= e^{-t} \int_0^t \frac{x^{n-1}}{(n-1)!}(1 - e^{-x})^{\delta^*}\, dx \\ &\quad + \int_t^\infty \frac{x^{n-1}}{(n-1)!} e^{-x}(1 - e^{-x})^{\delta^*}\, dx\,, \end{aligned} \tag{3}$$

which is easily derivable. By making the substitution $t = n + \tau n^{\frac{1}{2}}$ and $\delta = \exp(n + sn^{\frac{1}{2}})$ in (3) we find that $S_n - \tau_n \to_P 0$ as $n \to \infty$. Thus, the limiting joint distribution of (S_n, τ_n) is degenerate. A transformation which gives non-degenerate limit distribution is given in the following theorem which also shows that $S_n - \tau_n = O_p(n^{-\frac{1}{2}})$.

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THEOREM. *Let $Y_n = \exp[n^{\frac{1}{2}}(S_n - \tau_n)] = \Delta_n e^{-T_n}$. Then Y_n and τ_n are asymptotically independent and*

$$\lim_{n\to\infty} P(Y_n > y) = \int_1^\infty z^{-2}e^{-zy}\,dz\,, \qquad y > 0\,.$$

(The function on the right-hand side is denoted as $E_2(y)$ in [1] and its properties are listed in Chapter 5, therein.)

PROOF. A computational proof with variate transformations in (3) can be given. However, we give a proof which makes use of the Markov and partial sum structures of the sequence $\{\Delta_n, T_n\}$.

We observe that $T_n = T_{n-1} + V$, where V is an exponential (1) rv independent of T_{n-1}, so that $Y_n = \Delta_n e^{-T_{n-1}}U$, where $U = e^{-V}$ is a uniform (0, 1) rv. Now,

$$P(\Delta_n > \delta \mid Z_{n-1}) = F^{\delta^*}(Z_{n-1}) = (1 - e^{-T_{n-1}})^{\delta^*}\,.$$

Substituting $T_{n-1} = (n-1) + \tau_{n-1}(n-1)^{\frac{1}{2}}$, we find that

$$\lim_{n\to\infty} P(\Delta_n e^{-T_{n-1}} > x \mid \tau_{n-1} = \tau) = e^{-x}\,, \qquad \text{independent of} \quad \tau\,.$$

Thus, $\Delta_n e^{-T_{n-1}} \to_{\mathscr{L}} W$, where W is an exponential (1) rv, independent of the limiting distribution of τ_n. Finally, $Y_n \to_{\mathscr{L}} WU$ and $P(WU > y) = E_2(y)$.

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