*The Annals of Probability*
1976, Vol. 4, No. 1, 151–152

# A NOTE ON DOWNCROSSINGS FOR EXTREMAL PROCESSES[1]

BY JUDAH M. FRANKEL

*Brookhaven National Laboratory, Upton, New York*

Asymptotic expressions for the downcrossing probabilities of certain extremal and their related order statistics processes are obtained.

**1. Introduction.** Let $X_1, X_2, \cdots$ be independent random variables uniformly distibuted on (0, 1) and let $X_n{}^k$ denote the $k$th smallest among $X_1, \cdots, X_n$. Then for the Markov process $V^k(t)$, $0 < t < \infty$ defined by $P(V^k(t_i) < a_i\; i = 1, \cdots, r) = \lim_{n\to\infty} P(X^k_{[nt_i]} < a_i/n\; i = 1, \cdots, r)$ the following hold (Frankel 1972):

$$P(\tau(x) > s \mid V^k(t) = x) = e^{-sx} \quad t, s > 0\,, \qquad \text{and} \tag{1}$$

$$P(V^k(t + \tau(x)) \leqq y \mid V^k(t) = x) = (y/x)^k \qquad \text{for} \qquad y \leqq x \tag{2}$$

where $\tau(x)$ is the time until the first jump from $x$. Also for $0 < b \leqq x \leqq a < \infty$

$$\begin{aligned} p(x) &\equiv P(V^k(t) \geqq a/t \text{ before } V^k(t) \leqq b/t \text{ for } t > s \mid V^k(s) = x/t) \\ &= (\textstyle\int_b^x e^u/u^k\,du + e^b/b^k)/(\int_b^a e^u/u^k\,du + e^b/b^k)\,, \end{aligned} \tag{3}$$

$$P(V^k(t) \geqq g(t)/t \text{ i.o. } t \uparrow \infty) = 0 \quad \text{or} \quad 1 \quad \text{according as} \tag{4}$$

$$\textstyle\int_1^\infty e^{-g(t)} g^k(t)/t\,dt < \infty \quad \text{or} \quad = \infty$$

where $g(t) \uparrow \infty$, $g(t)/t \downarrow 0$ ultimately, and

$$P(X_n{}^k \geqq c_n/n \text{ i.o.}) = 0 \quad \text{or} \quad 1 \quad \text{according as} \tag{5}$$

$$\textstyle\sum_1^\infty e^{-c_n} c_n/n < \infty \quad \text{or} \quad = \infty$$

where $c_n \uparrow \infty$ and $c_n/n \downarrow 0$ ultimately.

Wichura (1973) has obtained asymptotic expressions for:

$$P(V^k(t) = g(t)/t \text{ some } t \geqq s) \quad \text{as} \quad s \uparrow \infty \qquad \text{and} \tag{6}$$

$$P(X_n{}^k \geqq g(n)/n \text{ some } n \geqq m) \quad \text{as} \quad m \uparrow \infty\,. \tag{7}$$

We will show that with slight modifications asymptotic expressions may also be obtained for

$$P(V^k(t) \leqq h(t)/t \text{ some } t \geqq s) \quad \text{as} \quad s \uparrow \infty\,, \qquad \text{and for} \tag{8}$$

$$P(X_n{}^k \leqq h(n)/n \text{ some } n \geqq m) \quad \text{as} \quad m \uparrow \infty\,. \tag{9}$$

**2. Results.** Let $s_1(s) = \inf(r > s \mid V^k(r) = 1/r)$, $s_2(s) = \inf(r > s \mid V^k(r) \leqq 1/r)$ and $q(x) = P(V^k(t) \leqq b/t$ before $V^k(t) \geqq 1/t$ for $t > s \mid V^k(s) = x/s)$ for $0 \leqq b \leqq x < 1$. Note that with $a = 1$ in (3) we have $q(x) = 1 - p(x)$.

Received July 10, 1972; revised March 20, 1975.

*AMS* 1970 *subject classifications.* Primary 60F15; Secondary 60J25, 60J50.

*Key words and phrases.* Boundary crossings, extremal processes, order statistics.

[1] Work performed under auspices of U.S. Atomic Energy Commission.

LEMMA 1.

$$P(V^k(t) \leqq b/t \ \ \textit{some} \ \ s < t < s_1(s) \,|\, V^k(s) = 1/s) \sim eb^k \quad \textit{as} \quad b \downarrow 0\,.$$

PROOF.

$$\begin{aligned} &P(V^k(t) \leqq b/t \text{ some } s < t < s_1(s) \,|\, V^k(s) = 1/s) \\ &\quad = P(V^k(s_2(s)) \leqq b/s_2(s) \,|\, V^k(s) = 1/s) \\ &\qquad + \textstyle\int_b^1 q(x)\, dP(V^k(s_2(s)) \leqq x/s_2(s) \,|\, V^k(s) = 1/s) \\ &\quad = b^k + (\textstyle\int_b^1 b^k(\int_x^1 e^u/u^k\, du)kx^{k-1}\, dx)/(e^b + b^k \int_b^1 e^u/u^k\, du) \qquad \text{by (2) and (3)} \\ &\quad = eb^k/(e^b + b^k \textstyle\int_b^1 e^k/u^u\, du) \sim eb^k \quad \text{as} \quad b \downarrow 0\,. \end{aligned}$$

COROLLARY 1. *Let* $Z^k(u) = e^u V^k(e^u)$ *then* $P(Z^k(u) \leqq b$ *before* $Z^k(u) = 1 \,|\, Z^k(0) = 1) = P(V^k(e^u) \leqq b/e^u$ *for some* $1 < e^u < s_1(s) \,|\, V^k(1) = 1) \sim eb^k$ *as* $b \downarrow 0$ *by Lemma* 1.

Now let $h(t) \downarrow 0$ and $H(t) = (\int_t^\infty h^k(s)/s\, ds)/\Gamma(k)$; then if

$$H(\exp(t + t^{1-c})) \sim H(e^t) \qquad \text{as } t \uparrow \infty \text{ for some } c < \tfrac{1}{3} \tag{10}$$

the proofs of Theorem 2.1 and Corollary 2.1 of Wichura (1973) will yield:

THEOREM 1. $P(V^k(t) \leqq h(t)/t$ *some* $t \geqq s) \sim H(s)$ *as* $s \uparrow \infty$.

THEOREM 2. $P(X^k(n) \leqq h(n)/n$ *some* $n \geqq m) \sim H(m)$ *as* $m \uparrow \infty$.

REMARKS. Using martingale techniques, Robbins and Siegmund (1972) have obtained exact although more complicated solutions to (6) when $k = 1$; their method may be extended to get solutions to (6) for all $k$ for the $h(t)$ of greatest interest.

**Acknowledgment.** The author is greatly indebted to Professor David Siegmund for his valuable advice and guidance.

## REFERENCES

[1] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, Mass.

[2] FRANKEL, J. (1972). On the law of the iterated logarithm for order statistics. Ph. D. dissertation, Columbia Univ.

[3] ROBBINS, H. and SIEGMUND, D. (1972). On the law of the iterated logarithm for maxima and minima. *Proc. Sixth Berkeley Symp. Math. Statist.* **3** 51-70. Univ. of California Press.

[4] WICHURA, M. J. (1973). Boundary crossing probabilities associated with Motoo's law of the iterated logarithm. *Ann. Probability* **1** 437-456.

1745 46TH STREET
BROOKLYN, NEW YORK 11204