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CONVERGENCE OF CONDITIONAL p-MEANS GIVEN A σ-LATTICE

BY D. LANDERS AND L. ROGGE

University of Cologne and University of Konstanz

Let $P_n | \mathscr{A}$, $n \in \mathbb{N}$, be a sequence of probability measures converging in total variation to the probability measure $P | \mathscr{A}$ and $\mathscr{C}_n \subset \mathscr{A}$, $n \in \mathbb{N}$, be a sequence of σ-lattices converging increasing or decreasing to the σ-lattice $\mathscr{C}$. Then for every uniformly bounded sequence f_n, $n \in \mathbb{N}$, converging to f in P-measure we show in this paper that the conditional p-mean $P_n^{\mathscr{C}_k} f_j$ converge to $P^{\mathscr{C}} f$ in P-measure if n, k, j tends to infinity.

The methods used in this paper are completely different from those used to prove the corresponding result for σ-fields instead of σ-lattices.

Let P be a probability measure on a σ-field $\mathscr{A}$ of a basic set Ω. In this paper let $1 < p < \infty$. Denote by $\mathscr{L}_p(\Omega, \mathscr{A}, P)$ the system of all $\mathscr{A}$-measurable functions $f: \Omega \to \mathbb{R}$ with $P(|f|^p) < \infty$ and $P(|f|^p)^{1/p}$ by $||f||_p$. Let $\mathscr{C} \subset \mathscr{A}$ be a σ-lattice, i.e. a system of sets which is closed under countable unions and intersections and contains Ω and the empty set $\emptyset$. A function $f: \Omega \to \mathbb{R}$ is $\mathscr{C}$-measurable iff $\{\omega : f(\omega) > a\} \in \mathscr{C}$ for all $a \in \mathbb{R}$.

$\mathscr{L}_p(\Omega, \mathscr{C}, P)$ denotes the system of all $\mathscr{C}$-measurable functions belonging to $\mathscr{L}_p(\Omega, \mathscr{A}, P)$. A conditional p-mean $P^{\mathscr{C}} f$ of a function $f \in \mathscr{L}_p(\Omega, \mathscr{A}, P)$ with respect to the σ-lattice $\mathscr{C}$ is the projection of f on the closed convex cone $\mathscr{L}_p(\Omega, \mathscr{C}, P)$ (see [2], page 315 for the case $p = 2$ and [4] for the general case). The conditional p-mean is P-a.e. uniquely determined and has the following properties:

(i) $a \leqq f \leqq b$ imply $a \leqq P^{\mathscr{C}} f \leqq b$ (see [4]);

(ii) a slight generalization of a result given in [1] yields a martingale theorem for conditional p-means: If $\mathscr{C}_n$ increases or decreases to the σ-lattice $\mathscr{C}$ then

$$||P^{\mathscr{C}_n} f - P^{\mathscr{C}} f||_p \to 0 \qquad \text{as} \quad n \to \infty .$$

The following is shown in [1]:

(iii) Let $\varepsilon > 0$ and $f \in \mathscr{L}_p(\Omega, \mathscr{A}, P)$ with $||f||_p > 0$ be given. There exists $\delta \equiv \delta(\varepsilon, f) > 0$ such that for every convex subset C of $\mathscr{L}_p(\Omega, \mathscr{A}, P)$, $g, h \in C$, $||f - g||_p$, $||f - h||_p \leqq \inf \{||f - c||_p : c \in C\} + \delta$ imply $||g - h||_p \leqq \varepsilon$.

Let $||f||_\infty = |\sup \{|f(\omega)| : \omega \in \Omega\}$ and $||\mu||$ the total variation of a signed measure μ. A sequence of probability measures $P_n | \mathscr{A}$, $n \in \mathbb{N}$, converges in total variation to a probability measure $P | \mathscr{A}$ if and only if $P_n | \mathscr{A}$ converges uniformly to $P | \mathscr{A}$, i.e. if $\sup \{|P_n(A) - P(A)| : A \in \mathscr{A}\} \to 0$.

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LEMMA 1. *Let P, Q be two probability measures on a σ-field $\mathscr{A}$. Then we have for all σ-lattices $\mathscr{C} \subset \mathscr{A}$, and all $\mathscr{A}$-measurable functions f, g with $||f||_\infty, ||g||_\infty \leqq c$,*

$$||Q^{\mathscr{C}}g - f||_p \leqq ||P^{\mathscr{C}}f - f||_p + 2||f - g||_p + 4c||P - Q||^{1/p}$$

for all versions $Q^{\mathscr{C}}g$ with $||Q^{\mathscr{C}}g||_\infty \leqq c$.

PROOF. Using the differential calculus it is easy to see that

$$(1) \qquad |a^x - b^x| \leqq |a - b|^x, \quad \text{if} \quad a, b \geqq 0 \quad \text{and} \quad 0 < x < 1 .$$

For every $\mathscr{A}$-measurable bounded function f we have

$$(2) \qquad |P(f) - Q(f)| \leqq ||P - Q|| \, ||f||_\infty .$$

Using the fact that $||g||_\infty \leqq c$ and $||Q^{\mathscr{C}}g||_\infty \leqq c$, we obtain from (2) applied to $f = |Q^{\mathscr{C}}g - g|^p$ that

$$(3) \qquad |P(|Q^{\mathscr{C}}g - g|^p) - Q(|Q^{\mathscr{C}}g - g|^p)| \leqq (2c)^p||P - Q|| .$$

Hence (1) applied to $x = 1/p$ yields

$$(4) \qquad ||Q^{\mathscr{C}}g - g||_p = P(|Q^{\mathscr{C}}g - g|^p)^{1/p} \leqq Q(|Q^{\mathscr{C}}g - g|^p)^{1/p} + 2c||P - Q||^{1/p} .$$

Let $\mathscr{M}_c \equiv \{h : h \ \mathscr{C}\text{-measurable}, \ ||h||_\infty \leqq c\}$ be the system of all $\mathscr{C}$-measurable functions bounded by c. If $h \in \mathscr{M}_c$, we obtain from (1) and (2) (analogously as in (4)):

$$Q(|h - g|^p)^{1/p} \leqq ||h - g||_p + 2c||P - Q||^{1/p} .$$

Hence using the definition of $Q^{\mathscr{C}}g$ and (i) we have

$$(5) \qquad \begin{aligned} Q(|Q^{\mathscr{C}}g - g|^p)^{1/p} &= \inf\{Q(|h - g|^p)^{1/p} : h \in \mathscr{M}_c\} \\ &\leqq \inf\{||h - g||_p : h \in \mathscr{M}_c\} + 2c||P - Q||^{1/p} \\ &\leqq \inf\{||h - f||_p : h \in \mathscr{M}_c\} + ||f - g||_p + 2c||P - Q||^{1/p} \\ &= ||P^{\mathscr{C}}f - f||_p + ||f - g||_p + 2c||P - Q||^{1/p} . \end{aligned}$$

Using (4) and (5) we obtain:

$$\begin{aligned} ||Q^{\mathscr{C}}g - f||_p &\leqq ||Q^{\mathscr{C}}g - g||_p + ||f - g||_p \\ &\leqq ||P^{\mathscr{C}}f - f||_p + 2||f - g||_p + 4c||P - Q||^{1/p} . \end{aligned}$$

The following theorem yields a generalization of Theorem 1 of [6] to the case of σ-lattices instead of σ-fields.

We remark that the method applied in [6] does not carry over to σ-lattices since the basic equation (see equation (1) in the proof of Theorem 1 of [6]) is not true any more for σ-lattices instead of σ-fields. It seems to the authors that the new method used in this paper is more natural and transparent even in the case of σ-fields.

THEOREM 1. *Let $P_n | \mathscr{A}$, $n \in \mathbb{N}$, be a sequence of probability measures converging in total variation to the probability measure $P | \mathscr{A}$ and $\mathscr{C}_n \subset \mathscr{A}$, $n \in \mathbb{N}$, be a sequence of σ-lattices converging increasing or decreasing to the σ-lattices $\mathscr{C}$. Then* *for every uniformly bounded sequence f_n, $n \in \mathbb{N}$, converging to f in P-measure we have that for every p with $1 < p < \infty$ the conditional p-mean $P_n^{\mathscr{C}_k} f_j$ converges to $P^{\mathscr{C}} f$ in P-measure if n, k, j tends to infinity.*

PROOF. Let $\|f_n\|_\infty \leqq c$ for all $n \in \mathbb{N}$. We prove that there exists a version of the conditional p-mean $P_n^{\mathscr{C}_k} f_j$ (namely every version of $P_n^{\mathscr{C}_k} f_j$ with $\|P_n^{\mathscr{C}_k} f_j\|_\infty \leqq c$) which converges to $P^{\mathscr{C}} f$ in the pth mean with respect to P. This implies that these versions converge in P-measure too. Using the fact that two versions of $P_n^{\mathscr{C}_k} f_j$ differ only on a P_n-null set, and that P_n converges to P in total variation, it is easy to see that each version of $P_n^{\mathscr{C}_k} f_j$ converges to $P^{\mathscr{C}} f$ in P-measure. Let, therefore,

$$\|f_j\|_\infty, \|P_n^{\mathscr{C}_k} f_j\|_\infty \leqq c \qquad \text{for} \quad n, k, j \in \mathbb{N} .$$

We prove $\|P_n^{\mathscr{C}_k} f_j - P^{\mathscr{C}} f\|_p \to 0$ if n, k, j tends to infinity.

Apply now Lemma 1 to $Q^{\mathscr{C}} g \equiv P_n^{\mathscr{C}_k} f_j$. We obtain

$$\|P_n^{\mathscr{C}_k} f_j - f\|_p \leqq \|P^{\mathscr{C}_k} f - f\|_p + 2\|f - f_j\|_p + 4c\|P_n - P\|^{1/p} \tag{6}$$

If $f = 0$ P-a.e. the inequality (6) yields the assertion. Let, therefore, $\|f\|_p > 0$ and choose $\varepsilon > 0$. Define

$$C \equiv \{f : f \ \mathscr{C}_k\text{-measurable}, \ \|f\|_\infty \leqq c\} .$$

From (iii) we obtain that there exists $\delta = \delta(\varepsilon, f) > 0$ such that

$$g, h \in C; \|f - g\|_p, \|f - h\|_p \leqq \min\{\|f - c\|_p : c \in C\} + \delta \tag{7}$$

imply

$$\|g - h\|_p \leqq \varepsilon .$$

According to assumption there exists $n(\varepsilon)$ such that

$$2\|f - f_j\|_p + 4c\|P_n - P\| \leqq \delta \qquad \text{for all} \quad n, j \geqq n(\varepsilon) . \tag{8}$$

(6), (7) and (8) applied to $g = P_n^{\mathscr{C}_k} f_j$, $h = P^{\mathscr{C}_k} f$ yield

$$\|P_n^{\mathscr{C}_k} f_j - P^{\mathscr{C}_k} f\|_p \leqq \varepsilon \qquad \text{for all} \quad n, j \geqq n(\varepsilon) .$$

Whence (ii) implies the assertion. That Theorem 1 cannot be improved may be seen from the examples given in [6].

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MATHEMATISCHES INSTITUT
UNIVERSITÄT KÖLN
5 KÖLN 41
WEYERTAL 86-90
WEST GERMANY

FACHBEREICH STATISTIK
UNIVERSITÄT KONSTANZ
775 KONSTANZ
POSTFACH 7733
WEST GERMANY