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SUPERCRITICAL AGE DEPENDENT BRANCHING PROCESSES WITH GENERATION DEPENDENCE[1]

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This paper examines the size, $Z(t)$, of a population as a function of time. $Z(t)$ is just like the ordinary Bellman–Harris age dependent branching process except that the number of daughters born to an individual in the nth generation is allowed to depend on n. The renewal theory of William Feller and Laplace transform theory are used to obtain the behavior of $EZ(t)$ as t approaches infinity, and the convergence of $Z(t)/E(Z(t))$ in quadratic mean.

1. Introduction. In this paper the following branching process is treated: The process starts with one cell in the nth generation. This cell lives for a random length of time $T_{n,1}$ with distribution function G, and then splits into a random number, $\zeta_{n,1}$, of baby cells in the $n+1$st generation. It is assumed throughout that G is nonlattice, and $G(0) = 0$. Each of these cells respectively live random lengths of time $T_{n+1,1}, \cdots, T_{n+1,\zeta_{n,1}}$ and split respectively into $\zeta_{n+1,1}, \cdots, \zeta_{n+1,\zeta_{n,1}}$ cells in the $n+2$nd generation. This process continues; $Z_n(t)$ is the number of cells alive at time t having started with one cell in the nth generation. It is assumed throughout that the random variables $T_{m,j}$, $\zeta_{n,k}$, where m and n run through the nonnegative integers and j and k run through the positive integers, are mutually independent; that the random variables $T_{n,k}$ for $n \geqq 0$ and $k \geqq 1$ have the same distribution; and that for each fixed n, the random variables $\zeta_{n,k}$, $k = 1, 2, \cdots$ have a probability distribution on the nonnegative integers depending only on n. Let G_0 be the distribution function which puts mass one on $t = 0$. Also the notation

$$(1.1) \qquad (A * B)(t) = \int_0^t A(t-u)\,dB(u) = \lim_{\tau\to 0+} \int_{-\tau}^{t+\tau} A(t-u)\,dB(u)$$

will be adopted whenever the right side of (1.1) exists. All integrals in this paper are Lebesgue–Stieltjes integrals. Also for any sequence a_n

$$(1.2) \qquad \sum_{n=k}^{k-1} a_n = 0 \quad \text{and} \quad \prod_{n=k}^{k-1} a_n = 1\,, \quad \text{for all integers} \quad k\,.$$

The Laplace transform of a function A is

$$\mathscr{L}A(\lambda) = \int_0^\infty e^{-\lambda t}\,dA(t)$$

whenever this exists. If $T_1, \cdots, T_k$ are k independent random variables with distribution function G, let

$$(1.3) \qquad G_k(t) = P[\sum_{j=1}^k T_j \leqq t] = (G_{k-1} * G)(t)\,.$$

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Now suppose throughout for some finite number B that

$$(1.4) \qquad m_n = E(\zeta_{n,k}) \leqq B\,, \qquad n = 0, 1, \cdots; k = 1, 2, \cdots.$$

Let $\bar{Z}_n(t)$ be the total number of cells ever alive by time t, having started with one cell in the nth generation.

It is shown in Theorem 3.1 that, when G has a finite first moment μ, $m_n \geqq 1$, and $(\prod_{j=0}^{n-1} m_j)/(n^\rho L(n)) \to 1$ as $n \to \infty$ with L slowly varying and $\rho \geqq 0$, one has $E(Z_0(t))/[(t/\mu)^\rho L(t/\mu)] \to 1$ as $t \to \infty$. Roughly speaking, the conclusion of this theorem means that $E(Z_0(t))$ behaves like $n^\rho L(n)$ where n is the number of average lifespans in the time interval $(0, t)$, provided $\mu =$ one unit of time. But $n^\rho L(n)$ behaves like $\prod_{j=0}^{n-1} m_j$, so $E(Z_0(t))$ recapitulates $\prod_{j=0}^{n-1} m_j$, the mean of an imbedded Galton–Watson process, Z_n, which is the number of cells in the nth generation. Using the renewal theory contained in Lemmas 2.3, 2.4, 2.5 and 3.2 of this paper, this result is extended (see Theorem 3.2) to the case where $m_n \geqq m > 1$ for all n, and $(\prod_{j=0}^{n-1} (m_j/m))/n^\rho L(n) \to 1$, as $n \to \infty$, with $\rho \geqq 0$ and with L slowly varying and nondecreasing. Finally in Section 4, Theorems 4.2 and 4.3 give sufficient conditions for $Z_0(t)/E(Z_0(t))$ to converge in quadratic mean to a nondegenerate random variable W as $t \to \infty$, thus giving information about the sample path behavior of $Z_0(t)$ for large t.

2. Basic renewal theory.

LEMMA 2.1. *For any finite interval I, there is a finite number B_I such that*

$$(2.1) \qquad E(\bar{Z}_n(t)) = \sum_{k=n}^{\infty} \prod_{j=n}^{k-1} m_j G_{k-n}(t) \leqq B_I$$

for $n = 0, 1, 2 \cdots$, and for all $t \in I$.

PROOF. In Harris (1963, page 139), "$\sum_{i_j} P[\nu_{i_1\cdots i_{j-1}} \geqq i_j]$" becomes m_{j-1}. The equation in (2.1) follows. The boundedness in (2.1) follows upon evaluating the Laplace transform of $E(\bar{Z}_n(t))$ for a suitably large value of λ.

This yields the following basic lemma concerning $Z_n(t)$:

LEMMA 2.2. *For any finite interval I, there is a finite number B_I such that*

$$(2.2) \qquad M_n(t) \equiv E(Z_n(t)) \leqq B_I$$

for $n = 0, 1, \cdots$, and all $t \in I$. Hence, $Z_n(t)$ is almost surely finite.

PROOF. This is an immediate consequence of the fact that $Z_n(t) \leqq \bar{Z}_n(t)$.

Some important lemmas will now be stated whose proofs are obtained by suitably modifying arguments used in Feller (1966), pages 181–183 and pages 346–353.

LEMMA 2.3. *Let Y_n be a sequence of functions uniformly bounded on finite intervals, such that $(Y_n * G)(t)$ exists for all t and all n. Suppose X_n is a sequence of functions, uniformly bounded on finite intervals, and for all t satisfying the renewal equations*

$$(2.3) \qquad X_n(t) = Y_n(t) + m_n(X_{n+1} * G)(t)\,, \qquad n = 0, 1, 2, \cdots.$$

Then, for all t

(2.4) $$X_n(t) = \sum_{k=n}^{\infty} (\prod_{j=n}^{k-1} m_j)(Y_k * G_{k-n})(t), \qquad n = 0, 1, 2, \cdots.$$

PROOF. (This proof uses, with some notational changes, the argument in Feller (1966, pages 181–183).) Call the right hand side of (2.4) $\bar{X}_n(t)$. Using (1.4)

$$\bar{X}_n(t) \leqq \sum_{k=n}^{\infty} B^{k-n}[\max_{0 \leqq u \leqq t} |Y_k(u)|]G_{k-n}(t).$$

The argument showing the uniform boundedness of $E(\bar{Z}_n(t))$ on finite intervals shows that $\bar{X}_n$ has this property since Y_k is uniformly bounded on finite intervals. A direct calculation shows $\bar{X}_n$ satisfies (2.3).

Now let $V_n(t) = \bar{X}_n(t) - X_n(t)$. Then V_n is a sequence of functions, uniformly bounded on finite intervals. Using (2.3) recursively one finds that

$$V_n(t) = (\prod_{j=n}^{n+k} m_j)(V_{n+k+1} * G_{k+1})(t).$$

The uniform boundedness of $V_n(u)$ for $n = 0, 1, 2, \cdots$ and u in $[0, t]$, and the fact that, as has been seen, $\sum_{k=0}^{\infty} (\prod_{j=n}^{n+k} m_j)G_{k+1}(t) < \infty$ show that $V_n(t) = 0$ for all t and all n. So $X_n(t) = \bar{X}_n(t)$ and (2.4) is true for all t and all n.

LEMMA 2.4. *Let h be any positive continuous, nondecreasing function such that for every number a, $h(t + a)/h(t)$ approaches 1 as t increases to ∞. Let H be a right continuous function, nondecreasing on $[0, \infty)$. Assume that $h(t) = h(0)$ if $t < 0$ and $H(t) = 0$ if $t \leqq 0$. Then for every $\varepsilon > 0$,*

(2.5) $$\frac{H(t) - H(t - \varepsilon)}{h(t)} \to c\varepsilon \qquad as \quad t \to \infty,$$

if and only if X is directly Riemann integrable implies

(2.6) $$\frac{(X * H)(t)}{h(t)} \to c \int_0^{\infty} X(u)\, du \qquad as \quad t \to \infty.$$

PROOF. (This proof uses, with notational changes, the methods found in Feller (1966, pages 348–350).) To see that (2.6) implies (2.5), let $X(t) = 1$ if $0 \leqq t \leqq \varepsilon$ and $X(t) = 0$ otherwise. (2.5) follows from (2.6) by a direct calculation. It is now shown that (2.5) implies (2.6).

Let $\varepsilon > 0$ be given. Let $X_n(t) = 1$ if $(n - 1)\varepsilon \leqq t \leqq n\varepsilon$ and $X_n(t) = 0$ otherwise. Let $X_n^*(t) = \int_0^{\infty} X_n(t - u)\, dH(u)$. Then $X_n^*(t) = H(t - (n - 1)\varepsilon) - H(t - n\varepsilon)$. By (2.5) one has

$$\frac{H(t - (n - 1)\varepsilon) - H(t - (n - 1)\varepsilon - \varepsilon)}{h(t - (n - 1)\varepsilon)} \to c\varepsilon \qquad \text{as } t \to \infty,$$

and also $h(t - (n - 1)\varepsilon)/h(t)$ approaches one from below as $t \to \infty$; hence $X_n^*(t)/h(t) \to c\varepsilon$ as $t \to \infty$ for fixed n. Thus $X_1^*(t)/h(t)$ is bounded in t due to the positive, continuous, and nondecreasing nature of h. Also because of these properties of h, $X_1^*(t)/h(t) \leqq X_n^*(t - (n - 1)\varepsilon)/h(t - (n - 1)\varepsilon)$, hence $X_n^*(t)/h(t)$ is bounded for all n and t by (say) $M_\varepsilon < \infty$.

Suppose $a_k \geqq 0$ and $\sum_{k=1}^{\infty} a_k < \infty$. Define $X(t) = \sum_{k=1}^{\infty} a_k X_k(t)$ and $X^*(t) = (X * H)(t)$. Then plainly,

$$(2.7)\qquad \sum_{k=1}^{n} \frac{a_k X_k^*(t)}{h(t)} \leqq \frac{X^*(t)}{h(t)} \leqq \sum_{k=1}^{n} \frac{a_k X_k^*(t)}{h(t)} + M_\varepsilon \sum_{k=n+1}^{\infty} a_k .$$

Let $t \to \infty$ in (2.7), and then in the resulting series of inequalities let $n \to \infty$. Using the Lebesgue dominated convergence theorem to pass limits under summation and integral signs and the fact that $\sum_{k=1}^{\infty} a_k < \infty$ one obtains

$$(2.8)\qquad X^*(t)/h(t) = (X * H)(t)/h(t) \to c \int_0^\infty X(u)\,du \qquad \text{as } t \to \infty .$$

The arbitrary nature of ε and a_k allow (2.8) to hold whenever X is directly Riemann integrable, so (2.5) implies (2.6) and Lemma 2.4 is true.

The next lemma is a slight generalization of the key renewal theorem. The proof is obtained by letting H_n^* and f in the lemma below correspond to U and Z respectively in the proof on page 350 of Feller (1966) and considering $(f * H_n^*)(x)/h(x)$ in the same way as $(Z * U)(x)$ is considered in Feller's proof.

LEMMA 2.5. *If m_j is a sequence of positive numbers approaching* 1 *as j approaches* ∞; *if*

$$H_n^*(t) = \sum_{k=n+1}^{\infty} \left(\prod_{j=n}^{k-1} m_j\right) G_{k-n}(t) , \qquad n = 0, 1, 2 \cdots ;$$

if $h(t)$ is a continuous positive nondecreasing function such that for every number q,

$$\frac{h(t+q)}{h(t)} \to 1 \qquad \text{as } t \to \infty ;$$

then, whenever f is any right continuous, nonnegative, directly Riemann integrable function, with $f(t) = f(0) > 0$ for $t < 0$ and with f satisfying, for some number $c > 0$,

$$(2.9)\qquad \frac{X_f(t)}{h(t)} \equiv \frac{(f * H_0^*)(t)}{h(t)} \to c \qquad \text{as } t \to \infty ,$$

one has

$$(2.10)\qquad \frac{H_0^*(t) - H_0^*(t-\varepsilon)}{h(t)} \to \frac{c\varepsilon}{\int_0^\infty f(u)\,du} \qquad \text{as } t \to \infty .$$

PROOF. The limit statement in (2.9) of the hypothesis of this lemma, the fact that f is directly Riemann integrable, and the positive nondecreasing nature of h together imply that $X_f(t)/h(t)$ is bounded in t by (say) $M_f < \infty$. Also, the right continuity of f allows the choice of a number $A > 0$ sufficiently small so that $f(u) > f(0)/2 = \delta > 0$, whenever $0 \leqq u \leqq A$. Hence for $0 \leqq v \leqq A$, and all $t \geqq 0$,

$$(2.11)\qquad \frac{M_f}{\delta} \geqq \frac{X_f(t)}{\delta h(t)} \geqq \frac{\int_{t-v}^{t} f(t-u)\,dH_0^*(u)}{\delta h(t)} \geqq \frac{H_0^*(t) - H_0^*(t-v)}{h(t)} .$$

Now define for any function E, and any finite interval $I = [a, b]$, $E(t + I) = E(t + b) - E(t + a)$. The inequalities in (2.11), the nondecreasing nature of

h, and the fact that any finite interval I can be broken up into subintervals of length less than or equal to A, allow $H_0^*(t+I)/h(t)$ to be bounded in t for all finite intervals I.

The same argument was used in Feller (1966, page 350), to show that his $U(t+I)$ was bounded in t for finite intervals I, except that there was no function h in the denominator. Following the proof in Feller (1966, page 350), the selection theorem ([3], page 263) gives a measure V_0, and a sequence $t_k \to \infty$, such that for each closed interval I

$$H_0^*(t_k+I)/h(t_k) \to V_0(I) \qquad \text{as } k \to \infty . \tag{2.12}$$

The limit statement (2.12) and the fact that by hypothesis $m_n \to 1$ as $n \to \infty$ allow the conclusion that

$$m_0 H_1^*(t_k+I)/h(t_k) \to V_0(I) \qquad \text{as } k \to \infty . \tag{2.13}$$

A detailed argument to this effect is in [1].

Now suppose a is a positive number and that A is a continuous function vanishing outside of $[0, a]$. Then one has for all t and all t_k,

$$\begin{aligned} \textstyle\int_{-\infty}^{\infty} A(t-u)\, dH_0^*(u+t_k) &= m_0 \textstyle\int_{-\infty}^{\infty} A(t-u)\, dG(u+t_k) \\ &\quad + \textstyle\int_0^{\infty} \left(m_0 \int_{-\infty}^{\infty} A(t-u-v)\, dH_1^*(v+t_k)\right) dG(u) . \end{aligned} \tag{2.14}$$

Then dividing both sides of this equation by $h(t_k)$ and letting $k \to \infty$ yields the all important functional equation

$$\begin{aligned} \zeta(t) &= \textstyle\int_0^{\infty} \zeta(t-u)\, dG(u) \quad \text{where} \\ \zeta(t) &\equiv \textstyle\int_{-\infty}^{\infty} A(t-u)\, dV_0(u) . \end{aligned} \tag{2.15}$$

Equation (2.14), when divided by $h(t_k)$, corresponds to (1.14) in Feller (1966, page 350). Equation (2.15) is equation (1.15) in Feller (1966, page 350), with "G" instead of "F". The rest of the proof of this lemma can now be obtained by plugging $(H_0^*(t_k) - H_0^*(t-h))/h(t_k)$ and V_0 (respectively) in for Feller's $U(t_k) - U(t_k - h)$ and V (respectively) on page 351 of Feller (1966), and using the argument found there. The details of this can be found in [1].

3. Limiting behavior of $M_n(t)$ as $t \to \infty$. Lemmas 2.3, 2.4, and 2.5 constitute a body of renewal theory which will be sufficient to determine the asymptotic behavior of $M_0(t)$ as $t \to \infty$, under certain assumptions on the m_n's. To this end the following lemma is proven.

LEMMA 3.1. *For* $n = 0, 1, 2, \cdots$

$$M_n(t) = 1 - G(t) + m_n(M_{n+1} * G)(t) . \tag{3.1}$$

PROOF. $M_n(t)$ is the expected number of cells alive at time t, starting with one nth generation cell. But this is the expected number of nth generation cells alive at time t (i.e., $1 - G(t)$), plus the expected number, $m_n(M_{n+1} * G)(t)$, of cells descending from an average of m_n cells in the $n + 1$st generation produced by the nth generation cell dying at or before time t.

LEMMA 3.2. *For* $n = 0, 1, 2, \cdots$

$$M_n(t) = \int_0^t 1 - G(t-u)\, dH_n(u) \tag{3.2}$$ *where*

$$H_n(t) = \sum_{k=n}^{\infty} \left(\prod_{j=n}^{k-1} m_j\right) G_{k-n}(t) . \tag{3.3}$$

PROOF. $1 - G(t)$ is bounded, $\int_0^t 1 - G(t-u)\, dG(u)$ exists, so by Lemma 2.3 and Fubini's theorem, (3.2) is true for $n = 0, 1, 2, \cdots$. (Here $1 - G(t)$ plays the same role as $Y_n(t)$ does in Lemma 2.3.)

From now on (3.3) will remain in force, as a definition of H_n.

THEOREM 3.1. *If* $m_n \geqq 1$ *for all* n;

$$\prod_{j=0}^{n-1} m_j/(n^\rho L(n)) \to 1 \qquad \text{as } n \to \infty , \tag{3.4}$$

where $L(n)$ *is slowly varying* (i.e., $L(an)/L(n) \to 1$ *as* $n \to \infty$ *for any* $a > 0$) *and* $\rho \geqq 0$; *and* $\mu = \int_0^\infty t\, dG(t) < \infty$; *then*

$$\frac{M_0(t)}{t^\rho L(t)} \to \frac{1}{\mu^\rho} \qquad \text{as } t \to \infty . \tag{3.5}$$

PROOF. First take the Laplace transform of each side of (3.2). Next divide the resulting equation by

$$(1 - \mathscr{L}G(\lambda))^{-\rho} L\left(\frac{1}{1 - \mathscr{L}G(\lambda)}\right) \Gamma(\rho + 1) .$$

Letting λ decrease to zero one obtains

$$\frac{\mathscr{L}M_0(\lambda)}{\Gamma(\rho + 1)\lambda^{-\rho}\mu^{-\rho}L(1/\lambda)} \to 1 ,$$

using Theorem 5, page 423 of Feller (1966). This is true, since $\mathscr{L}G(\lambda)$ increases to one as λ decreases to zero, $(1 - \mathscr{L}G(\lambda))/\lambda \to \mu$ as λ decreases to zero, and L is slowly varying. Now if it were true that $M_0(t)$ is a monotone function of t, Theorem 2, page 421 of Feller (1966) could be applied to obtain the desired result. This monotonicity can be established by forcing each nth generation cell to split into at least one $n + 1$st generation cell, while splitting into m_n cells on the average. The resulting branching process has the same expected value, and is nondecreasing in t. So (3.5) is true, since the nondecreasing and right continuous nature of $M_0(t)$ allows $M_0(t)$ to define a measure.

THEOREM 3.2. *If* $m > 1$, *as* $n \to \infty$, $m_n \geqq m$ *for all* n;

$$\left(\prod_{j=0}^{n-1} m_j^*\right)/n^\rho L(n) \to 1$$

as $n \to \infty$; $m_n^* = m_n/m$ *for all* n; $\rho > 0$ *and* $L(n)$ *is as in the hypotheses of Theorem* 3.1, *with* L *nondecreasing*; *and* α *is the* (*positive*) *number satisfying*

$m \int_0^\infty e^{-\alpha t}\, dG(t) = 1$, then

$$\frac{M_0(t)}{e^{\alpha t} t^\rho L(t)} \to \frac{\int_0^\infty (1 - G(u))e^{-\alpha u}\, du}{\bar{\mu}^\rho \int_0^\infty (1 - \bar{G}(u))\, du} \tag{3.6}$$

where $\bar{\mu} = m \int_0^\infty te^{-\alpha t}\, dG(t)$ and $\bar{G}(t) = m \int_0^t e^{-\alpha u}\, dG(u)$.

PROOF. By Lemma 3.2, $M_0(t) = \int_0^t 1 - G(t - u)\, dH_0(u)$. Multiply both sides of this equation by $e^{-\alpha t}$. Then

$$\bar{M}_0(t) = M_0(t)e^{-\alpha t} = \int_0^t (1 - G(t - u))e^{-\alpha(t-u)}\, d\bar{H}_0(u),$$

where $\bar{H}_0(t) = \sum_{n=0}^\infty (\prod_{j=0}^{n-1} m_j^*)\bar{G}_n(t)$, and $\bar{G}_n(t) = m^n \int_0^t e^{-\alpha u}\, dG_n(u)$, for $n = 0, 1, 2, \cdots$. To determine the behavior of $M_0(t)$ we determine the behavior of $B(t) = \int_0^t 1 - \bar{G}(t - u)\, d\bar{H}_0(u)$. Now m_n^* satisfies the hypotheses of Theorem 3.1, hence $B(t)/(t^\rho L(t)) \to 1/\bar{\mu}^\rho$ as $t \to \infty$. Since $1/(t^\rho L(t)) \to 0$ as $t \to \infty$, it is true by Theorem 3.1 that $(1 - \bar{G}(t)) * \bar{H}_0^*(t)/(t^\rho L(t))$ approaches $1/\bar{\mu}^\rho$ as $t \to \infty$, where $\bar{H}_0^*(t) = \bar{H}_0(t) - 1$. Now let $f(t) = 1 - \bar{G}(t)$, $h(t) = t^\rho L(t)$, $X_f(t) = \int_0^t 1 - \bar{G}(t - u)\, d\bar{H}_0^*(u)$, $c = 1/\bar{\mu}^\rho$, and $H_0^*(t) = \bar{H}_0^*(t)$. Then f, h, c, H_0^*, X_f satisfy the hypotheses of Lemma 2.5. So, by (2.10)

$$\frac{\bar{H}_0^*(t) - \bar{H}_0^*(t - \varepsilon)}{t^\rho L(t)} \to \frac{\varepsilon}{\bar{\mu}^\rho \int_0^\infty (1 - \bar{G}(u))\, du}$$

for each $\varepsilon > 0$, as $t \to \infty$. Now, since $(1 - G(t))e^{-\alpha t}$ is directly Riemann integrable, it follows from Lemma 2.4 that

$$\frac{((1 - G(t))e^{-\alpha t}) * \bar{H}_0^*(t)}{t^\rho L(t)} \to \frac{\int_0^\infty (1 - G(u))e^{-\alpha u}\, du}{\bar{\mu}^\rho \int_0^\infty (1 - \bar{G}(u))\, du}$$

as $t \to \infty$. Now $(1 - G(t))e^{-\alpha t}/(t^\rho L(t)) \to 0$ as $t \to 0$, so (3.6) follows.

4. The convergence of $W(t) \equiv Z_0(t)/M_0(t)$ in quadratic mean. In this section it is assumed that m_n and $\sigma_n = E(\zeta_{n,k}^2) - m_n$ are both bounded by a positive finite number B independent of n. Moreover it is supposed that $m \geqq 1$; $m_n \geqq m$ for all n; and $((\prod_{j=0}^{n-1})(m_j/m))/(n^\rho L(n)) \to 1$ as $n \to \infty$, where $\rho > 0$ and L is a positive nondecreasing function satisfying $L(cn)/L(n) \to 1$ as $n \to \infty$, whenever $c > 0$. Suppose α is the number defined in Theorem 3.2 (with $\alpha = 0$ when $m = 1$).

The behavior of $W(t)$ in quadratic mean will be determined by considering the asymptotic behavior of $C_n(t, \tau) = E(Z_n(t)Z_n(t + \tau))$ as $t \to \infty$. First we give the renewal equation for $C_n(t, \tau)$.

LEMMA 4.1. *The following renewal equation holds for $n = 0, 1, 2, \cdots$ and for $t, \tau \geqq 0$:*

$$C_n(t, \tau) = F_n(t, \tau) + m_n \int_0^t C_{n+1}(t - u, \tau)\, dG(u) \tag{4.1}$$

where

$$\begin{aligned} F_n(t, \tau) = {} & \sigma_n \int_0^t M_{n+1}(t - u)M_{n+1}(t - u + \tau)\, dG(u) \\ & + m_n \int_t^{t+\tau} M_{n+1}(t + \tau - u)G(u) + 1 - G(t + \tau). \end{aligned} \tag{4.2}$$

PROOF. This lemma is easily established by, as in the proof of Lemma 3.1, conditioning on what happens when the first cell in the branching process dies.

The following lemma is a consequence of (4.1) and Lemma 2.3.

LEMMA 4.2. *For all* $n, t, \tau \geqq 0$,

$$C_n(t, \tau) = F_n(t, \tau) + \sum_{k=n+1}^{\infty} \left(\prod_{j=n}^{k-1} m_j\right) \int_0^t F_k(t-u, \tau)\, dG_{k-n}(u) . \tag{4.3}$$

PROOF. Equation (4.3) follows from Lemma 4.1 once it is seen that $F_n(t, \tau)$ (for fixed τ) is uniformly bounded in n on finite intervals. Due to the boundedness of σ_n and m_n and the fact that G is a probability distribution, it follows that $F_n(t, \tau)$ is bounded uniformly in n on finite intervals if $M_n(t)$ has this property. But this is true by Lemma 3.2. The asymptotic behavior of $C_0(t, \tau)$ is given by the following theorem.

THEOREM 4.1. *If* $\rho > 1$ *or* $m > 1$, *then*

$$C_0(t, \tau)/h(t, \tau) \to \sum_{k=0}^{\infty} \frac{C^2 \sigma_k \prod_{j=0}^{k-1} m_j}{(m^*)^{k+1} (\prod_{j=0}^{k} (m_j/m))^2} \equiv Q \qquad as \quad t \to \infty , \tag{4.4}$$

uniformly in τ, *where*

$$h(t, \tau) = e^{2\alpha t + \alpha \tau} t^\rho (t+\tau)^\rho L(t) L(t+\tau) , \tag{4.5}$$

$$C = \lim_{t \to \infty} M_0(t)/(t^\rho L(t) e^{\alpha t}) , \tag{4.6}$$

and

$$m^* = \left(\int_0^\infty e^{-2\alpha t}\, dG(t)\right)^{-1} . \tag{4.7}$$

PROOF. By Lemma 4.2, we may define $D(t, \tau)$ by

$$\frac{C_0(t, \tau)}{h(t, \tau)} = D(t, \tau) + \sum_{k=0}^{\infty} \frac{\sigma_k \prod_{j=0}^{k-1} m_j \int_0^t M_{k+1}(t-u) M_{k+1}(t-u+\tau)\, dG_{k+1}(u)}{h(t, \tau)} . \tag{4.8}$$

It follows from Theorem 3.2 that $D(t, \tau) \to 0$ uniformly in t as $t \to \infty$. Now, consider the series in (4.8).

First suppose $m > 1$. Using Lemma 3.2, the series in (4.8) is dominated termwise, for a sufficiently large k, chosen independently of t and τ, by the series

$$\sum_{k=0}^{\infty} (k+1)^\rho \sigma_k \int_0^t \frac{M_0(t-u) M_0(t-u+\tau)}{h(t-u, \tau)} (m^*)^k e^{-2\alpha u}\, dG_{k+1}(u) \left(\frac{m}{m^*}\right)^k .$$

By Theorem 3.2 and the definition of m^*, the convolution in the above series is bounded in all variables. Moreover recall that $m = (\int_0^\infty e^{-\alpha t}\, dG(t))^{-1}$, $m^* = (\int_0^\infty e^{-2\alpha t}\, dG(t))^{-1}$ so $m/m^* < 1$ and the above series is dominated termwise by a convergent geometric series independent of t and τ. Hence in case $m > 1$, (4.4) follows by applying the Lebesgue dominated convergence theorem to (4.8) and using Theorem 3.2.

Next suppose $\rho > 1$ and $m = 1$. Now by our assumptions concerning $\prod_{j=0}^{n-1} m_j$, there are constants $B_1 > 0$ and $B_2 < \infty$ such that $B_1 n^\rho L(n) \leqq \prod_{j=0}^{n-1} m_j \leqq B_2 n^\rho L(n)$

for $n = 1, 2, \cdots$. From (3.2) and (3.3),

$$\begin{aligned} M_n(t) &= \sum_{k=n}^{\infty} \prod_{j=n}^{k-1} m_j(G_{k-n}(t) - G_{k-n+1}(t)) \\ &= \sum_{l=0}^{n} \frac{\prod_{j=0}^{l+n-1} m_j}{\prod_{j=0}^{n-1} m_j} (G_l(t) - G_{l+1}(t)) \\ &\quad + \sum_{l=n+1}^{\infty} \frac{\prod_{j=0}^{l+n-1} m_j}{\prod_{j=0}^{l-1} m_j \prod_{j=0}^{n-1} m_j} \prod_{j=0}^{l-1} m_j(G_l(t)) - G_{l+1}(t) . \end{aligned}$$

Hence, for $n = 1, 2, 3, \cdots$

$$\begin{aligned} (4.9) \qquad M_n(t) \leqq \frac{B_2}{B_1}\left(\sum_{l=0}^{n}\left(\frac{l+n}{n}\right)^{\rho} \frac{L(l+n)}{L(n)}\right)(G_l(t) - G_{l+1}(t)) + \frac{1}{\prod_{j=0}^{n-1} m_j} \\ \times \left(\frac{B_2}{B_1} \sum_{l=n+1}^{\infty}\left(\frac{l+n}{l}\right)^{\rho} \frac{L(l+n)}{L(l)}\right) \prod_{j=0}^{l-1} m_j(G_l(t) - G_{l+1}(t)) . \end{aligned}$$

Since L is slowly varying and nondecreasing, there is a finite number B_3 such that $(t + s/s)^{\rho} L(t + s)/L(s) < B_3$ for $0 \leqq t \leqq s$ and $s \geqq 1$. This entails, after putting the appropriate additional nonnegative term into the second summation in (4.9),

$$M_n(t) \leqq \frac{B_2 B_3}{B_1}(1 - G_{n+1}(t)) + \frac{1}{\prod_{j=0}^{n-1} m_j} \frac{B_2 B_3}{B_1} \sum_{l=0}^{\infty} \prod_{j=0}^{l-1} m_j(G_l(t) - G_{l+1}(t)) .$$

But this means that, for all $n \geqq 1$ and all $t \geqq 0$,

$$(4.10) \qquad M_n(t) \leqq B_4 + \frac{B_5}{n^{\rho} L(n)} M_0(t) ,$$

where $B_4 = B_2 B_3/B_1$ and $B_5 = B_4/B_1$. This follows using $n = 0$ in (3.2) and (3.3). Let $E(t, \tau)$ denote the series in (4.8). Define $E_0(t, \tau)$, $E_1(t, \tau)$, and $E_2(t, \tau)$ by

$$\begin{aligned} E(t, \tau) &= [E_0(t, \tau) + E_1(t, \tau) + E_2(t, \tau)]/h(t, \tau) \\ E_1(t, \tau) &= B_4 \sum_{k=0}^{\infty} \sigma_k \prod_{j=0}^{k-1} m_j(M_{k+1} * G_{k+1})(t) \\ &\quad + B_4 \sum_{k=0}^{\infty} \sigma_k \prod_{j=0}^{k-1} m_j \int_0^t M_{k+1}(t + \tau - u)\, dG_{k+1}(u) \\ E_2(t, \tau) &= \sum_{k=0}^{\infty} \sigma_k \prod_{j=0}^{k-1} m_j \int_0^t (M_{k+1}(t-u) - B_4)(M_{k+1}(t - u + \tau) - B_4)\, dG_{k+1}(u) . \end{aligned}$$

Then, plainly using Theorem 5, page 423 and Theorem 2, page 421 of Feller (1966), one has $E_0(t, \tau)/h(t, \tau) \to 0$ uniformly in τ as $t \to \infty$. Upon plugging the right hand side of (4.10) into the definition of $E_1(t, \tau)$, realizing that the resulting quantity defines a measure, one may check that this resulting quantity when divided by $h(t, \tau)$ approaches zero uniformly in τ as $t \to \infty$, using the same theorems from Feller (1966), and Theorem 3.1. Now, by plugging the right hand side of (4.10) into the definition of $E_2(t, \tau)/h(t, \tau)$, we obtain a series whose terms are, by Theorem 3.1, dominated by terms of the form $B_6/k^{\rho}L(k)$ where B_6 is a positive finite number independent of t and τ. Since $\rho > 1$, the Lebesgue dominated convergence theorem allows the evaluation of $\lim_{t\to\infty} E_2(t, \tau)/h(t, \tau)$ termwise, giving (4.4) with $m^* = m = 1$.

THEOREM 4.2. *Under the hypotheses of Theorem* 4.1, *the quantity* $W(t) = Z_0(t)/M_0(t)$ *converges in quadratic mean to a random variable* W *as* t *approaches infinity.*

PROOF. First we notice that

$$E(W(t+\tau) - W(t))^2 = \frac{C_0(t+\tau, 0)}{(M_0(t+\tau))^2} - \frac{2C_0(t,\tau)}{M_0(t)M_0(t+\tau)} + \frac{C_0(t,0)}{(M_0(t))^2}$$

and the right hand side of this equality approaches zero uniformly in τ as t approaches infinity, using Theorem 4.1. Theorem 4.2 now follows by completeness of L_2.

THEOREM 4.3. *Under the hypotheses of Theorem* 4.1, *if* W *is the random variable in Theorem* 4.2, *and if* m^* *is as in Theorem* 4.1 *then* $EW = 1$, $\operatorname{Var} W = \sum_{k=0}^{\infty} \sigma_k(\prod_{j=0}^{k-1} m_j)/[m^{*k+1}(\prod_{j=0}^{k}(m_j/m))^2] - 1$, *and* $\operatorname{Var} W > 0$ *if* $\operatorname{Var} \zeta_{n,1} > 0$ *for some* n.

PROOF. $EW(t) - EW = E(W(t) - W)$. The right hand side of this inequality approaches zero as t approaches infinity, by Theorem 4.2, since L_2 convergence implies L_1 convergence. This means that $1 = EW(t) = EW$. Also, by Minkowski's inequality,

$$(\operatorname{Var} W(t))^{\frac{1}{2}} - (E(W(t) - W)^2) \leqq (\operatorname{Var} W)^{\frac{1}{2}} \leqq (\operatorname{Var} W(t))^{\frac{1}{2}} + (E(W(t) - W)^2)^{\frac{1}{2}}$$

and so, letting t approach infinity,

$$\operatorname{Var} W = \lim_{t\to\infty} \operatorname{Var} W(t) = \lim_{t\to\infty} \frac{C_0(t,0)}{(M_0(t))^2} - 1 .$$

The latter limit is the desired result for Var W, by Theorem 4.1. Now assume $\operatorname{Var} \zeta_{N,1} > 0$.

$$\begin{aligned} E(W^2) &= \textstyle\sum_{k=0}^{\infty} (\sigma_k \prod_{j=0}^{k-1} m_j)/[m^{*k+1}(\prod_{j=0}^{k}(m_j/m))^2] \\ &= \sigma + \textstyle\sum_{k=0}^{\infty} [(1/\prod_{j=0}^{k-1} m_j) - (1/\prod_{j=0}^{k} m_j)]/(m^*/m^2)^{k+1} \end{aligned}$$

where $\sigma = \sum_{k=0}^{\infty} [\operatorname{Var}(\zeta_{k,1}) \prod_{j=0}^{k-1} m_j]/[m^{*k+1} \prod_{j=0}^{k}(m_j/m)^2]$. Note that σ is positive since $\operatorname{Var}(\zeta_{N,1}) > 0$. By Jensen's inequality (or Holder's inequality), $m^*/m^2 = (\int_0^\infty e^{-\alpha t}\, dG(t))^2/\int_0^\infty e^{-2\alpha t}\, dG(t) \leqq 1$. Thus,

$$1 + \operatorname{Var} W = EW^2 \geqq \sigma + \textstyle\sum_{k=0}^{\infty} (1/\prod_{j=0}^{k-1} m_j) - (1/\prod_{j=0}^{k} m_j) = \sigma + 1 ,$$

proving that Var W is positive.

Notice that if $m = m^* = 1$ in the expression for $E(W^2)$, then $\operatorname{Var}(W)$ is $\sum_{k=0}^{\infty} \operatorname{Var}(\zeta_{k,1})/(m_k^2 \prod_{j=0}^{k-1} m_j)$. This recapitulates the corresponding result in Theorem 1 of [2] (provided (48) of [2] is corrected to read "$\operatorname{Var} W = \sum_{k=0}^{\infty} \sigma_k^2/(m_k^2 \prod_{j=0}^{k-1} m_j)$" as in the left hand side of (49) in [2]). This expression after notational changes was also obtained in Theorem 4 of [6]. It is anticipated that by relating the branching process $Z_0(t)$ to Z_n, the size of the nth generation of cells in the branching process $Z_0(t)$, one may show that if $0 \leqq \rho \leqq 1$ and

$m = 1$, $P[Z_0(t) \neq 0]$ Var $W(t) \to \gamma > 0$ as $t \to \infty$. (This is the so-called critical case result.) This approach was successful in Goldstein (1971) and is feasible since it has been shown in [2] that under certain conditions with generation dependence, $P[Z_n \neq 0]$ Var $(Z_n/EZ_n) \to 2$ as n approaches infinity. Weiner's (1972) article contains a survey of results and methods for obtaining critical case results.

Finally, the following corollary to Theorems 4.1 and 4.3 establishes that the convergence in quadratic mean of $W(t)$ to a random variable W is not vacuous under the hypotheses of Theorem 4.1.

COROLLARY 4.1. *Under the hypotheses of Theorem* 4.1, $P[Z_0(t) \to \infty$ *as* $t \to \infty] > 0$.

PROOF. Clearly, since $M_0(t) \to \infty$ as $t \to \infty$ under the hypotheses of Theorem 4.1, $P[Z_0(t) \to \infty$ as $t \to \infty] \geqq P[W > 0]$. From Theorem 4.3, $EW = 1$, insuring that $P[W > 0] > 0$.

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