SOME FINITELY ADDITIVE PROBABILITY

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Lester E. Dubins and Leonard J. Savage have shown how to define a large family of finitely additive probability measures on the lattice of open sets of spaces of the form $X \times X \times \cdots$, where X, otherwise arbitrary, is assigned the discrete topology. This lattice does not include many of the sets which occur in the usual treatment of such probabilistic limit laws as the martingale convergence theorem, and in some unpublished notes Dubins and Savage conjectured that there might be a natural way to extend their measures to such sets. We confirm their conjecture here by showing that every set in the Borel sigma-field can be squeezed between an open and a closed set in the usual manner. It is then possible to generalize to this finitely additive setting many of the classical countably additive limit theorems. If assumptions of countable additivity are imposed, the extension studied here, when restricted to the usual product sigma-field, agrees with the conventional extension.

1. Introduction. Let γ be a countably additive probability defined on all subsets of a denumerably infinite set X. (In the main body of the paper, X will be an arbitrary set.) Let N be the set of positive integers. It is well known that there exists a unique countably additive probability which assigns to each set of the form

$$A^1 \times A^2 \times \cdots \times A^j \times X \times X \times \cdots, \quad j \in N, \quad A^i \subseteq X,$$

the probability $\prod_{i=1}^{j} \gamma(A^i)$, and whose domain is the sigma-algebra generated by these sets. Is there a counterpart to this product measure theorem in case γ is not countably additive? In much greater generality this problem has already been considered by Lester Dubins and the late Leonard Savage in their book *How to Gamble If You Must* (1965). In order to surmount the apparent arbitrariness involved in extension, Dubins and Savage require that a certain natural condition be satisfied, which, for the special case being considered here, rests on the following equality:

$$(1) \qquad \pi(D) = \int_X \pi(Dx) \, d\gamma(x) .$$

Here π is the extension-to-be, $D \subseteq X^N$, $Dx = \{z \in X^N \mid (x, z_1, z_2, \cdots) \in D\}$. For reasons given in Dubins and Savage (1965, pages 12–20), but too lengthy to

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present here, it is natural to require that (1) hold for all sets D which are *clopen* (simultaneously closed and open) in the product topology on X^N determined by assigning X the discrete topology. Then, their method shows that there is exactly one finitely additive probability π defined on the clopen subsets of X^N which meets this requirement.

The collection of clopen sets in X^N includes properly the collection of sets which depend on finitely many coordinates, but is a much smaller class than the domain of the conventional product measure. In fact, the latter coincides with the sigma-field generated by the clopen sets. Is it possible to extend π in a natural way to a larger collection of sets? Dubins and Savage posed this question, again in greater generality, in some unpublished notes written in the fall of 1962. (For a relevant quotation from these notes, see Dubins (1974).) In the same notes they began to answer it by assigning to each open set the supremum of the measures of the clopen sets contained within it, and then showing that the resulting extension, which in this particular instance might be called π_*, satisfied $\pi_*(O \cup P) + \pi_*(O \cap P) = \pi_*(O) + \pi_*(P)$ for all open sets O, P.

This is the point of departure of our efforts. We were privileged to see these notes and were immediately tempted by the possibility, suggested in the notes by Dubins and Savage, of further extension. A time honored first step in such a situation is to form the collection $\mathscr{A}$ of all sets which can be approximated from without by an open set and within by a closed set in such a way that the measure of their set-theoretic difference, which is open, can be made arbitrarily small. Then as described in Section 2 below, $\mathscr{A}$ is an algebra of sets which includes the open sets, and there is a unique finitely additive probability on $\mathscr{A}$ which extends π_*. The next question is: how large is this algebra? For example, does it contain the sigma-field generated by the open sets? We found this question difficult, even in such a special case, and the results to follow come from our attempt to answer it.

The answer, given in greater generality in Theorem 5.1 below, is yes. This theorem makes it possible to state in the usual fashion finitely additive generalizations of such classical limit theorems as the strong law of large numbers and the martingale convergence theorem. (Incidentally, a recent paper of Lester Dubins (1974) contains some pointed remarks concerning the merits of the various ways of formulating a limit law). Many of these finitely additive limit theorems (see Theorems 7.2 and 7.3, for example) can then be proved by applying well-known arguments in conjunction with the stop rule methods devised to establish Theorem 5.1. Finally, if γ is countably additive, the extension of π considered here coincides with the familiar countably additive product measure (see Section 6 for a more comprehensive result).

2. Basic framework. Throughout, probability measures are finitely additive unless explicitly stated otherwise. Let N be the positive integers and X an arbitrary nonempty set. Let $H = X^N = X \times X \times \cdots$ and give H the product topology determined by assigning X the discrete topology.

Let μ be a finitely additive probability on the clopen subsets of H. Following Dubins (1974), μ may be extended from the clopen sets to the open sets by assigning to each open set the supremum of the measures of the clopen sets contained within. Call this extension η. It is shown in Dubins (1974) that $\eta(O \cup P) + \eta(O \cap P) = \eta(O) + \eta(P)$ for all open sets O, P. Further, as asserted in the same reference, η possesses one and only one finitely additive extension to the algebra generated by the open sets, and this extension can, in turn, be completed by squeezing in the usual manner to obtain a finitely additive probability λ on an algebra $\mathscr{A}(\mu)$ of subsets of H. Finally, associated with λ are an outer measure λ^* and an inner measure λ_*, where

$$\lambda^*(A) = \inf \{\lambda(E) \mid A \subseteq E, E \in \mathscr{A}(\mu)\} \quad \text{and}$$

$$\lambda_*(A) = \sup \{\lambda(E) \mid E \subseteq A, E \in \mathscr{A}(\mu)\}, \qquad A \text{ an arbitrary subset of } H.$$

A slightly different description of $\mathscr{A}(\mu)$, λ^*, λ_* is given in the following proposition.

PROPOSITION 1. (i) *The algebra $\mathscr{A}(\mu)$ is the collection of all $E \subseteq H$ such that, for every $\varepsilon > 0$, there are O open, C closed with $C \subseteq E \subseteq O$ and $\eta(O - C) < \varepsilon$.*

(ii) *For all $A \subseteq H$, $\lambda^*(A) = \inf \{\lambda(O) \mid A \subseteq O, O \text{ open}\}$ and $\lambda_*(A) = \sup \{\lambda(C) \mid C \subseteq A, C \text{ closed}\}$.*

PROOF. In (i) the only step which is not immediate is the fact that each set in the algebra generated by the open sets can be appropriately squeezed between an open and a closed set. This algebra is the collection of all finite unions of sets of the form $P - Q$, where P, Q are open and $P \supseteq Q$. Thus, invoking sub-additivity of η, it is only necessary to check that such a $P - Q$ can be so squeezed. To do this, set $O = P - L$, $C = K - Q$, where K, L are clopen sets chosen to satisfy $K \subseteq P$, $L \subseteq Q$ and $\eta(P - K)$, $\eta(Q - L)$ small. It is easy to verify that $C \subseteq P - Q \subseteq O$ and $\eta(O - C)$ is small. This completes the proof of (i). Clause (ii) follows directly. □

At this point it is reasonable to inquire whether the algebra $\mathscr{A}(\mu)$ contains the Borel sets (sigma-field generated by the open sets). This is not so for all μ; an example is given in the next section. The next paragraph introduces a class of μ, the "probabilities determined by strategies," for which it is so. This class, first considered by Dubins and Savage, is essential to our proof, which typically involves working with many of its members simultaneously.

Let X^* be the set of all finite sequences of members of X, including the empty one. A *strategy* σ is a function which assigns to each $p \in X^*$ a probability measure $\sigma(p)$, defined on all subsets of X. The probability measure assigned by σ to the empty sequence will be denoted σ_0. Each strategy determines, in the manner described by Dubins and Savage (1965, pages 7–21), a positive linear functional on the class of all bounded real-valued functions on H which are continuous when the real line is endowed with the discrete topology. If g is such a function, and σ is a strategy, the value of this linear functional will be denoted in what

follows by either $\int g \, d\sigma$, $\int g(h) \, d\sigma(h)$, or σg. The *probability determined by a strategy* σ is the set function $\mu_\sigma : K \to \int 1_K \, d\sigma$, K clopen. (Here, and elsewhere, 1_K is the function which is 1 on K and 0 off K.) As μ_σ is a probability, it may be extended, as described above, to the probability λ on the algebra $\mathscr{A}(\mu_\sigma)$. To keep the notation simple, let $\mathscr{A}(\sigma)$ be an abbreviation for $\mathscr{A}(\mu_\sigma)$ and, with some harmless ambiguity, let σ be the extension λ. Also, let σ^*, σ_* be λ^*, λ_* respectively.

The following notation will appear throughout. Let $p, q \in X^*$ and $h \in H$. Then pq is the member of X^* whose terms consist of the terms of p followed by the terms of q, and ph is the member of H whose terms consist of the terms of p followed by the terms of h. If $A \subseteq H$, $Ap = \{h \in H \,|\, ph \in A\}$ and $pA = \{h \in H \,|\, h = ph'$ for some $h' \in A\}$. If g is a function defined on H, gp is defined by $gp : h \to g(ph)$, $h \in H$. If p consists of a single term $x \in X$, the entities Ap, pA, gp will be written Ax, xA, gx, respectively. For arbitrary $p \in X^*$, $|p|$ is the number of terms of p; if p is empty, $|p| = 0$.

3. An example. Suppose for this paragraph that X is an infinite set. Then, as shown in Corollary 2 of this section, there is a finitely additive probability μ defined on the clopen subsets of H such that μ takes only the values 0 and 1, and if D is any closed nowhere dense set in H, there is a clopen K with $K \supseteq D$ and $\mu(K) = 0$. Such a μ is of interest for two reasons. The first is that the extension λ, of Section 2, does not include all G_δ's in its domain; the second is that μ cannot be approximated (in a sense to be specified later) by any probability which is determined by a strategy. L. Dubins (1975) has already given an example of this latter phenomenon. The reasoning will be given following the proof of existence of μ.

(Nothing in this section is required in the remainder of this paper, and it may be skipped at first reading.)

LEMMA 1. *Let $p \in X^*$ and $Q \subseteq H$.*

 (i) *If $A \subseteq Qp$, then $pA \subseteq Q$.*
 (ii) *If Q is dense in H, Qp is dense in H.*
 (iii) *If Q is open in H, Qp is open in H.*
 (iv) *If Q is clopen in H, pQ is clopen in H.*

LEMMA 2. *Let Z be a topological space. Suppose $(Z_i, i \in I)$ is a family of pairwise disjoint clopen subsets of Z whose union is a clopen subset of Z. Then if $(C_i, i \in I)$ is a family of closed subsets of Z and $C_i \subseteq Z_i$, all $i \in I$, the set $C = \bigcup_{i \in I} C_i$ is a closed subset of Z.*

The next two definitions and lemma establish the existence of a function which inserts a clopen set in each open dense set in such a way that finite intersections of the inserted sets are never empty. For general X, the next definition relies on the axiom of choice, but, for well-ordered X, its use can be avoided.

DEFINITION. Let O be a nonempty open set in H. Let q be any member of X^* such that $qH \subseteq O$ and if $pH \subseteq O$, $p \in X^*$, then $|p| \geq |q|$. Set $\beta(O) = qH$ and $d(O) = |q|$.

There is more freedom in the choice of the function β than is apparent from the above definition. Many other β's will work just as well in what follows.

DEFINITION. The equations below define inductively, for each positive integer n, a function β_n, which has as its domain the collection of all open dense subsets of H and which takes as values subsets of H. The inductive step requires Lemma 1 (ii).

$$\beta_1 = \beta$$
$$\beta_{n+1}(O) = \beta(O) \cup \left(\bigcup_{|p| \leq k} p\beta_n(Op) \right),$$

where O is open dense, $k = d(O)$, $p \in X^*$.

LEMMA 3. *Let n be a positive integer.*
Then if $O, O^1, \cdots, O^n$ are any open dense sets in H,

(a) $\beta_n(O)$ *is a clopen set in H,*
(b) $\beta_n(O) \subseteq 0$,
(c) $\beta_n(O^1) \cap \cdots \cap \beta_n(O^n)$ *is nonempty.*

PROOF. This is done by induction on n. If $n = 1$, the lemma is immediate from the definition of β. Now suppose it holds for a positive integer n. To verify (a) for β_{n+1}, let O be open dense and set

$$Q_j = \bigcup_{|p| = j} p\beta_n(Op)$$

for each $j = 0, \cdots, d(O)$. Now $\beta_n(Op)$ is clopen by Lemma 1 (ii) and the inductive hypothesis. Next $p\beta_n(Op)$ is clopen by Lemma 1 (iv) and since $p\beta_n(Op) \subseteq pH$ for all $p \in X^*$, Lemma 2 applies with $I = \{p \in X^* \mid |p| = j\}$ and $Z_p = pH$ all $p \in I$, to show that Q_j is closed. It is also open, so Q_j is clopen, $j = 0, \cdots, d(O)$. Since $\beta_{n+1}(O)$ is the union of $\beta(O)$ together with the Q_j's, $\beta_{n+1}(O)$ is clopen.

To verify (b), note that the inductive hypothesis and Lemma 1 (ii) together imply that $\beta_n(Op) \subseteq Op$, so that, by Lemma 1 (i), $p\beta_n(Op) \subseteq 0$. This is so for any $p \in X^*$, so $\beta_{n+1}(O) \subseteq 0$.

To verify (c), let $O^1, \cdots, O^{n+1}$ be open dense in H. Assume that $d(O^1) \leq d(O^i)$ for $i = 2, \cdots, n + 1$, and set $\beta(O^1) = qH$, where $q \in X^*$. Since $|q| = d(O^1) \leq d(O^i)$, the set $\beta_{n+1}(O^i)$ includes $q\beta_n(O^iq)$ as a subset, $i = 2, \cdots, n + 1$. By the induction hypothesis, $\beta_n(O^2q), \cdots, \beta_n(O^{n+1}q)$ have a nonempty intersection. Therefore the sets $q\beta_n(O^2q), \cdots, q\beta_n(O^{n+1}q)$ have a nonempty intersection, which is also a subset of qH. As the intersection of $\beta_{n+1}(O^2), \cdots, \beta_{n+1}(O^{n+1})$ includes this subset, and $\beta(O^1) = qH$, the proof of the inductive step is complete. $\square$.

COROLLARY 1. *If X is an infinite set, there is a function $\bar{\beta}$ which assigns to each open dense set O in H a clopen set $\bar{\beta}(O)$ in H such that $\bar{\beta}(O) \subseteq O$; and if n is any positive integer and $O^1, \cdots, O^n$ are any open dense sets, the sets $\bar{\beta}(O^1), \cdots, \bar{\beta}(O^n)$ have a nonempty intersection.*

PROOF. Let $w_1, w_2, \cdots$ be an infinite sequence of the distinct members of X^* with $|w_i| = 1$ all $i \in N$. If O is open dense, set

$$\bar{\beta}(O) = \bigcup_{i \in N} w_i \beta_i(Ow_i)$$

where β_i is defined in Lemma 3. The set Ow_i is in the domain of β_i by Lemma 1 (ii).

It is now easy to check that $\bar{\beta}$ has the desired properties. For example, if $O^1, \cdots, O^n$ are open dense, the set $\bar{\beta}(O^1) \cap \cdots \cap \bar{\beta}(O^n)$ is nonempty essentially because $\beta_n(O^1 w_n) \cap \cdots \cap \beta_n(O^n w_n)$ is. For if h is a member of the latter, $w_n h$ is a member of $w_n \beta_n(O^1 w_n) \cap \cdots \cap w_n \beta_n(O^n w_n)$, which in turn is a subset of $\bar{\beta}(O^1) \cap \cdots \cap \bar{\beta}(O^n)$. The distinctness of the w_i's has not been used. It is required to guarantee that $\bar{\beta}(O)$ be closed (as well as open). In Lemma 2 let $I = N$ and $Z_i = w_i H$, $i \in N$. Then the Z_i's are pairwise disjoint by virtue of the distinctness of the w_i's. ☐

COROLLARY 2. *Let X be an infinite set. There is a finitely additive probability μ defined on the clopen subsets of H such that μ only takes on the values 0 and 1, and if C is any closed nowhere dense set in H, there is a clopen set $K \supseteq C$ for which $\mu(K) = 0$.*

PROOF. Let $\mathscr{F}$ be the collection of all clopen sets L such that $L = \bar{\beta}(O)$ for some open dense set O. Then, since every finite intersection of members of $\mathscr{F}$ is nonempty, there is an ultrafilter of subsets of H which includes $\mathscr{F}$ as a subcollection. This ultrafilter determines a finitely additive probability μ on the clopen subsets of H in the usual manner: assign probability one to all clopen sets which belong to the ultrafilter and probability zero to all other clopen sets. Then, from the definition of μ, every open dense set contains a clopen set L with $\mu(L) = 1$. As a set is closed nowhere dense if and only if its complement is open dense, the proof is completed by taking complements. ☐

For the particular μ of Corollary 2, let λ and $\mathscr{A}(\mu)$ be as described in Section 2. Fix a member x of X and let $S = \{h \in H \mid h_j = x, \text{ for all } j \in N \text{ sufficiently large}\}$. Then S cannot belong to $\mathscr{A}(\mu)$. First, S is dense in H so that $O \supseteq S$ and O open imply O dense and $\lambda(O) = 1$. Second S has an empty interior so that $C \subseteq S$ and C closed imply C nowhere dense and $\lambda(C) = 0$. Thus $S \notin \mathscr{A}(\mu)$, and consequently the complement $H - S$ does not belong to $\mathscr{A}(\mu)$. The set S is a countable union of closed sets, so $H - S$ is a countable intersection of open sets which does not belong to $\mathscr{A}(\mu)$.

The probability μ cannot be approximated by a strategy: for $\varepsilon = \frac{1}{2}$, say, there is no strategy σ such that $|\mu(K) - \sigma(K)| < \varepsilon$ for all clopen K. For, as will be shown in the next paragraph, given any strategy σ, there is a closed nowhere dense set D such that $K \supseteq D$ and K clopen implies $\sigma(K) \geq \frac{3}{4}$. But there is a clopen set K such that $K \supseteq D$ and $\mu(K) = 0$. For that K, $|\mu(K) - \sigma(K)| \geq \frac{3}{4}$.

Let σ be a strategy, $0 < \delta < 1$, and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots$ be positive numbers such that $\sum \varepsilon_i \leq \delta$. Assume X is an infinite set. For each $p \in X^*$, let $X(p)$ be a proper

subset of X such that $\sigma(p)(X(p)) \geqq 1 - \varepsilon_i$, where i is the number of terms of p. Finally, set $K_1 = H$, $K_n = \{h \in H \mid h_n \in X(p_{n-1}(h))\}$, $n \geqq 2$, $n \in N$. Then using Lemma 1 of Section 7 and the elementary inequality $\prod_{i=1}^{\infty}(1 - \varepsilon_i) \geqq 1 - \sum_{i=1}^{\infty} \varepsilon_i$, it can be shown that the set $D = \bigcap_1^{\infty} K_i$ has σ-measure at least $1 - \delta$. From the definition of the sets K_i, D must be closed nowhere dense.

4. The basic integration formula. If σ is a strategy and A a clopen set, then according to the definition given in Dubins and Savage (1965), $\sigma(A)$ can be calculated by first conditioning on the first coordinate of H and then integrating with respect to σ_0. This section gives a straightforward extension of the resulting formula to other sets.

If $p \in X^*$, $\sigma[p]$, the *conditional strategy*, is defined by $\sigma[p](q) = \sigma(pq)$, all $q \in X^*$. If $p = (x)$, $x \in X$, $\sigma[p]$ will be written $\sigma[x]$. The notation $\sigma(A \mid p)$ will often be used for the quantity $\sigma[p](Ap)$, which it is natural to regard as the conditional σ-probability of A given the past p. A similar notation will be used for $\sigma[p]^*(Ap)$ and $\sigma[p]_*(Ap)$.

THEOREM 1. *For every* $A \subseteq H$, $\sigma^*(A) = \int \sigma^*(A \mid x) \, d\sigma_0(x)$ *and* $\sigma_*(A) = \int \sigma_*(A \mid x) \, d\sigma_0(x)$.

PROOF. By Theorem 2.8.1 of Dubins and Savage (1965), the formulae hold if A is clopen. They are next established for open sets.

If $K \subseteq O$, then $Kx \subseteq Ox$ for all x and, hence, $\sigma(O) = \sup \sigma(K) = \sup \int \sigma(K \mid x) \, d\sigma_0(x) \leqq \int \sigma(O \mid x) \, d\sigma_0(x)$ where the supremum is taken over all clopen sets K contained in the open set O.

For the opposite inequality, let $\varepsilon > 0$. If O is open, then, for each $x \in X$, Ox is open and there is a clopen set $K^x \subseteq Ox$ satisfying $\sigma[x](K^x) \geqq \sigma(O \mid x) - \varepsilon$. Set $K = \bigcup_{x \in X} xK^x$. Check that K is clopen and $Kx = K^x$ for every x. Then $K \subseteq 0$ and $\sigma(O) \geqq \sigma(K) = \int \sigma(K \mid x) \, d\sigma_0(x) \geqq \int \sigma(O \mid x) \, d\sigma_0(x) - \varepsilon$.

This establishes the formulae for open sets A. The argument can be easily adapted to give the first formula for all A.

The second follows from the first since $\sigma_*(A) = 1 - \sigma^*(A^c)$. □

A *stop rule* is a function $s: H \to N$ such that if h, h' belong to H and $h_i = h_i'$, $i = 1, \cdots, s(h)$, then $s(h) = s(h')$. For $h \in H$, $n \in N$, set $p_n(h) = (h_1, \cdots, h_n)$ and, if s is a stop rule, set $p_s(h) = p_n(h)$ where $n = s(h)$.

COROLLARY 1. *Let s be a stop rule. Then, for every* $A \subseteq H$, $\sigma^*(A) = \int \sigma^*(A \mid p_s) \, d\sigma$ *and* $\sigma_*(A) = \int \sigma_*(A \mid p_s) \, d\sigma$.

This corollary extends Theorem 1 and is proved from it by induction on the structure of p_s. A similar result is formula 3.7.1 in Dubins and Savage (1965).

Let $A \in \mathscr{N}(\sigma)$. Then, by Theorem 1, $\int \sigma^*(A \mid x) \, d\sigma_0(x) = \int \sigma_*(A \mid x) \, d\sigma_0(x)$. However, it can happen, as the following example shows, that $\sigma^*(A \mid x) > \sigma_*(A \mid x)$ for all x so that for no x is Ax in $\mathscr{N}(\sigma[x])$.

EXAMPLE. Let $X = N$ and γ be a probability on N which gives measure zero

to all finite sets. Let β be the probability which assigns measure $\frac{1}{2}$ to each of the points 1, 2. Set $\sigma_0 = \gamma$ and $\sigma(p) = \beta$ if p is a nonempty element of X^*. The next step will not be executed precisely here, but it can be justified by the contents of Section 6. Namely, $\sigma[x]$ is just the coin tossing measure for every x. For that reason, there is a set B^x such that $\sigma^*(B^x | x) = 1/x$ and $\sigma_*(B^x | x) = 0$. Let $A = \bigcup_{x \in X} xB^x$ so that $Ax = B^x$ for all x.

5. $\mathscr{A}(\sigma)$ contains the Borel sets. A critical ingredient in the theorems of this section is a very weak Heine–Borel property of H, presented in Lemma 1 below. To state it, the following notation will be required. Let $A^1, A^2, \cdots$ be subsets of H. If s is a stop rule, $A^s = \{h \in H \,|\, h \in A^{s(h)}\}$. It is easy to check that if the A^i are all open (closed), then A^s is open (closed).

LEMMA 1. *Let $O^1, O^2, \cdots$ be open sets in H. If $O^1 \subseteq O^2 \subseteq \cdots$ and $H = \bigcup_i O^i$ then there is a stop rule s such that $H = O^s$.*

PROOF. For each $h \in H$, there is a least $n \geq 1$ such that the basic neighborhood $\{h' \in H \,|\, h_i' = h_i, i = 1, \cdots, n\}$ is a subset of at least one of the O^i; and for that n there is a least k such that $O^k \supseteq \{h' \in H \,|\, h_i' = h_i, i = 1, \cdots, n\}$. If $k \leq n$, set $s(h) = n$; if $k > n$, set $s(h) = k$. Then s is a stop rule and $H = O^s$. (This argument, which was suggested to us by David Blackwell, is more perspicuous than our original proof.) □

COROLLARY 1. *Let $O^1, O^2, \cdots$ be open sets in H. If $O^1 \subseteq O^2 \subseteq \cdots$, C is closed, and $C \subseteq \bigcup_i O^i$, there is a stop rule s such that $C \subseteq O^s$.*

PROOF. Set $Q^i = O^i \cup (H - C)$, $i \in N$, and apply the preceding lemma to the sets $Q^1, Q^2, \cdots$. □

COROLLARY 2. *Let $C^1, C^2, \cdots$ be closed sets in H. If $C^1 \supseteq C^2 \supseteq \cdots$, O is open, and $O \supseteq \bigcap_i C^i$, there is a stop rule s such that $O \supseteq C^s$.*

PROOF. Take complements in the preceding corollary. □

COROLLARY 3. *Let μ be a finitely additive probability on the algebra of clopen sets and λ be the extension of μ described in Section 2.*

(i) *If $C^1 \supseteq C^2 \supseteq \cdots$ are closed, and $C = \bigcap_i C^i$, then $\lambda(C) = \inf_s \lambda(C^s)$.*
(ii) *If $O^1 \subseteq O^2 \subseteq \cdots$ are open, and $O = \bigcup_i O^i$ then $\lambda(O) = \sup_s \lambda(O^s)$.*

(The infimum and supremum are taken over all stop rules s.)

PROOF. Use Corollaries 1 and 2.

The next lemma gives a crude sufficient condition for a countable union of open sets to have small probability under a strategy σ. The notation $R^1 \cup \cdots \cup R^s$, where s is a stop rule, refers to the set $\{h \in H \,|\, h \in R^i, \text{ for some } i \leq s(h)\}$.

LEMMA 2. *Let $R^1, R^2, \cdots$ be a sequence of subsets of H. If $\delta \geq 0$ and*

$$\sigma^*(R^n | q) \leq \delta 2^{-n} \quad \text{for all } n \in N \text{ and } q \in X^* \text{ with } |q| = n,$$

then

$$\sigma^*(R^1 \cup \cdots \cup R^s) \leqq \delta \,, \qquad \textit{for all stop rules} \quad s \,.$$

If, in addition, R^1, R^2, $\cdots$ are open, then $\sigma(R^1 \cup R^2 \cup \cdots) \leqq \delta$.

PROOF. The second assertion follows from the first by using Corollary 3.

The first assertion is a consequence of showing the following statement holds for every ordinal β:

Let s be of structure β. If σ is any strategy and R^1, R^2, $\cdots$ is any sequence of subsets of H, and δ any nonnegative quantity such that

$$\sigma^*(R^{|q|} \,|\, q) \leqq \delta/2^{|q|}$$

for all nonempty $q \in X^*$, then $\sigma^*(R^1 \cup \cdots \cup R^s) \leqq \delta$.

This assertion holds if $\beta = 0$. For then there is a positive integer m such that $s(h) = m$, all $h \in H$. Hence,

$$\begin{aligned}
\sigma^*(R^1 \cup \cdots \cup R^s) &= \sigma^*(R^1 \cup \cdots \cup R^m) \\
&\leqq \sum_{i=1}^{m} \sigma^*(R^i) \\
&= \sum_{i=1}^{m} \int \sigma^*(R^i \,|\, p_i(h)) \, d\sigma(h) \qquad \text{(by Corollary 4.1)} \\
&\leqq \sum_{i=1}^{m} \delta/2^i \leqq \delta \,.
\end{aligned}$$

For the inductive step, assume the assertion holds for all $\beta < \alpha$, where $\alpha > 0$ is an ordinal. Let s have structure exactly α. Let R^1, R^2, $\cdots$, σ, and δ satisfy the hypothesis of the assertion. Set $M^i = R^{i+1}$, $i = 1, 2, \cdots$. Then

$$R^1 \cup \cdots \cup R^s \subseteq R^1 \cup (M^1 \cup \cdots \cup M^s) \,.$$

Now $\sigma(R^1) \leqq \delta/2$, so it suffices to show $\sigma^*(M^1 \cup \cdots \cup M^s) \leqq \delta/2$. First,

$$(1) \qquad \sigma^*(M^1 \cup \cdots \cup M^s) = \int \sigma[x]^*(M^1 \cup \cdots \cup M^s)x \, d\sigma_0(x) \,.$$

Fix an $x \in X$ temporarily. Set

$$\begin{aligned}
L^i &= M^i x \qquad i = 1, 2, \cdots, \\
r(h) &= s(xh) \,, \qquad h \in H \,, \\
\rho &= \sigma[x] \,.
\end{aligned}$$

Then r is a stop rule with structure less than α. Further, the sets L^1, L^2, $\cdots$, the strategy ρ, and the quantity $\delta/2$ all satisfy the hypothesis of the assertion. Therefore, the inductive hypothesis implies that $\rho^*(L^1 \cup \cdots \cup L^r) \leqq \delta/2$. But, as is easily checked,

$$L^1 \cup \cdots \cup L^r = (M^1 \cup \cdots \cup M^s)x \,.$$

It now follows that the integrand in (1) cannot exceed $\delta/2$. $\square$

As will be shown shortly, $\mathscr{A}(\sigma)$ need not be a sigma-field. The following subcollection of sets is a sigma-field however, this being the content of Theorem 1.

DEFINITION. $\mathscr{F}(\sigma) = \{A \subseteq H \,|\, Ap \in \mathscr{A}(\sigma[p]) \text{ for all } p \in X^*\}$.

THEOREM 1. $\mathscr{F}(\sigma)$ *is a sigma-field which contains the Borel sigma-field and is contained in* $\mathscr{A}(\sigma)$.

PROOF. First, $\mathscr{F}(\sigma)$ contains the open sets, as A open implies Ap open, and $\mathscr{A}(\sigma[p])$ contains all the open sets. Next, $\mathscr{F}(\sigma)$ is closed under complementation, as $A^c p = (Ap)^c$ and $\mathscr{A}(\sigma[p])$ is closed under complementation. The remainder of the proof is devoted to showing that $\mathscr{F}(\sigma)$ is closed under countable intersection.

Step 1. Let $A^1, A^2, \cdots$ be members of $\mathscr{F}(\sigma)$ and $A = \bigcap_i A^i$. Fix $p \in X^*$ and set

$$B = Ap \, ,$$
$$B^i = A^i p \, , \qquad i = 1, 2, \cdots ,$$
$$\tau = \sigma[p] \, .$$

The aim is to show $B \in \mathscr{A}(\tau)$. In other words, given $\varepsilon > 0$, to show there exists D closed, P open such that $D \subseteq B \subseteq P$ and $\tau(P - D) < \varepsilon$. Let δ be a small positive quantity, to be chosen later.

Step 2. For each $q \neq 0$ in X^*, there is a closed set C^q and an open set O^q such that

$$C^q \subseteq (B^{|q|})q \subseteq O^q \, ,$$

and

$$\tau[q](O^q - C^q) \leq \delta/2^{|q|} \, .$$

To see this, set $i = |q|$ and observe that since $A^i \in \mathscr{F}(\sigma)$, $A^i pq \in \mathscr{A}(\sigma[pq])$. But $A^i pq = B^i q$ and $\sigma[pq] = \tau[q]$.

Step 3. Choose (axiom of choice) for each $q \neq 0$ in X^* a C^q, O^q satisfying the conditions of Step 2. Set, for each $n \in N$,

$$C^n = \bigcup_{|q|=n} qC^q \, , \qquad O^n = \bigcup_{|q|=n} qO^q \, .$$

Then C^n is closed (see Lemma 3.2 for example), O^n is open, $C^n \subseteq B^n \subseteq O^n$, and $C^n q = C^q$, $O^n q = O^q$ for all $q \in X^*$ with $|q| = n$.

Step 4. Set, for $n \in N$,

$$P^n = O^1 \cap \cdots \cap O^n \, ,$$
$$D^n = C^1 \cap \cdots \cap C^n \, .$$

Then, for all stop rules s, $D^s \subseteq P^s$ and

$$\tau(P^s - D^s) \leq \delta \, .$$

The inclusion follows from the fact (Step 3) that $C^i \subseteq O^i$, all $i \in N$, and the inequality is established as follows. Let $R^i = O^i - C^i$, $i \in N$. First,

$$P^s - D^s \subseteq R^1 \cup \cdots \cup R^s \, ,$$

which follows from

$$P^m - D^m = (P^m - C^1) \cup \cdots \cup (P^m - C^m) \, ,$$
$$\subseteq (O^1 - C^1) \cup \cdots \cup (O^m - C^m) \, ,$$

where $m \in N$, and the ith term of the second union is $O^i - C^i$. Now, if $q \neq 0$ in X^*,

$$\tau[q]R^{|q|}q \leq \delta/2^{|q|} \,,$$

using Steps 2 and 3. By Lemma 2, this suffices to show $\tau(R^1 \cup \cdots \cup R^s) \leq \delta$.

Step 5. Let $D = \bigcap_i D^i$. Then

$$D \subseteq \bigcap_i B^i \subseteq P^s$$

for each stop rule s.

Step 6. There is a stop rule r such that

$$\tau(P^r - D) < 2\delta \,.$$

To see this, note that $P^s - D \subseteq (P^s - D^s) \cup (D^s - D)$ for every stop rule s. By Corollary 3, there is a stop rule r such that $\tau(D^r - D) < \delta$. Then, using Step 4, $\tau(P^r - D) < 2\delta$.

To complete the proof of the theorem, set $\delta = \varepsilon/2$, $P = P^r$, and observe that $B = \bigcap_i B^i$. $\square$

It is pleasing that $\mathscr{A}(\sigma)$ is the completion, in the usual sense, of σ restricted to the Borel sets. That is, $\mathscr{A}(\sigma)$ is exactly equal to the collection of all sets A for which there exist Borel E, F with $E \subseteq A \subseteq F$ and $\sigma(F - E) = 0$. For suppose $A \in \mathscr{A}(\sigma)$. For each $n \in N$, there is a closed set C_n, an open set O_n such that $C_n \subseteq A \subseteq O_n$ and $\sigma(O_n - C_n) < 1/n$, set $E = \bigcup_n C_n$, $F = \bigcap_n O_n$. Then E, F are Borel and $F - E \subseteq O_n - C_n$ for every $n \in N$. It follows that $\sigma(F - E) = 0$. The other direction is a consequence of Theorem 1.

Perhaps suprisingly, $\mathscr{A}(\sigma)$ need not be a sigma-field. Take X and σ as in the example of Section 4. Then there is a set B of infinite sequences of 1's and 2's such that $\sigma[x]^*B = 1$, $\sigma[x]_*B = 0$. Set $A^n = nB$, $n \in N$. Then $A^n \in \mathscr{A}(\sigma)$, all $n \in N$, but $\bigcup_n A^n$ is not in $\mathscr{A}(\sigma)$. The set A^n is in $\mathscr{A}(\sigma)$ because

$$\sigma^*(A^n) = \int \sigma[x]^*A^n x \, d\sigma_0(x) = 0$$

as $A^n x = \emptyset$ unless $x = n$. However, since $(\bigcup A^n)x = B$, $\sigma^*(\bigcup_n A^n) = 1$ and $\sigma_*(\bigcup_n A^n) = 0$.

THEOREM 2. *Let $A^1, A^2, \cdots$ be sets in $\mathscr{F}(\sigma)$.*

(i) *If s is a stop rule, $A^s \in \mathscr{F}(\sigma)$.*

(ii) *If $A^1 \supseteq A^2 \supseteq \cdots$, and $A = \bigcap_i A^i$, then $\sigma(A) = \inf_s \sigma(A^s)$.*

(iii) *If $A^1 \subseteq A^2 \subseteq \cdots$, and $A = \bigcup_i A^i$, then $\sigma(A) = \sup_s \sigma(A^s)$.*

(The infimum and supremum are taken over all stop rules s.)

PROOF. (i) $A^s \in \mathscr{F}(\sigma)$, since $\mathscr{F}(\sigma)$ is a sigma-field and

$$A^s = \bigcup_i A^i \cap \{h \in H \mid s(h) = i\} \,.$$

(ii) Let $\varepsilon > 0$ and $\delta = \varepsilon/2$. Take p to be the empty sequence in Steps 1–6 of the proof of Theorem 1. Then there is a closed set D and open sets $P^1, P^2, \cdots$ satisfying $D \subseteq A$, $A^1 \cap \cdots \cap A^n \subseteq P^n$, $n \in N$. Further, there is a stop rule r

such that $\sigma(P^r - D) < 2\delta$. Since $A^1 \supseteq A^2 \supseteq \cdots, A^n = A^1 \cap \cdots \cap A^n, n \in N$. Thus $D \subseteq A \subseteq A^s \subseteq P^s$ for all stop rules s. Use the stop rule r to conclude that

$$\sigma(A^r) - \sigma(A) \leq 2\delta = \varepsilon.$$

(iii) This follows from (ii) by taking complements. $\square$

In the light of Theorems 1 and 2, the final assertion of Lemma 2 holds also for sets $R^1, R^2, \cdots$ in $\mathscr{F}(\sigma)$.

THEOREM 3. $\mathscr{F}(\sigma)$ *is closed under the Souslin operation.*

PROOF. According to the criterion of Szpilrajn–Marczewski (Kuratowski, 1966, page 95) it suffices to show that $\mathscr{F}(\sigma)$ is a sigma-field and that it satisfies the following property: For any $A \subseteq H$ there is a $G \in \mathscr{F}(\sigma)$ with $G \supseteq A$ such that if $\hat{G} \in \mathscr{F}(\sigma)$ with $\hat{G} \supseteq A$, every subset of $G - \hat{G}$ belongs to $\mathscr{F}(\sigma)$. As $\mathscr{F}(\sigma)$ is a sigma-field it remains to check this property. Let $A \subseteq H$. Choose, for each $p \in X^*$ a set G^p such that $G^p \supseteq Ap$, $\sigma[p]G^p = (\sigma[p])^*Ap$, and G^p is a G_δ. Set, for each nonnegative integer n,

$$G^n = \bigcup_{|p|=n} pG^p$$

and $G = \bigcap_{n=0}^{\infty} G^n$. Then the set G is a G_δ which fulfills the requirements of the property. In particular, the set $B = G - \hat{G}$ satisfies $(\sigma[p])^*Bp = 0$, for all $p \in X^*$. $\square$

6. Relation to countably additive theory. If a strategy σ satisfies conventional measurability and countable additivity assumptions, then the present finitely additive extension is consistent with the conventional one and assigns measure to as many sets.

Let $\mathscr{B}$ be a sigma-field of subsets of X, $\mathscr{B}^n = \mathscr{B} \times \cdots \times \mathscr{B}$ (n factors), and $\mathscr{B}^\infty = \mathscr{B} \times \mathscr{B} \times \cdots$ be the product sigma-field of subsets of H. It is assumed in this section that σ is measurable with respect to $\mathscr{B}$. That is, σ is assumed to satisfy

(i) for every $p \in X^*$, $\sigma(p)$ is countably additive when restricted to $\mathscr{B}$,

(ii) for every n and every $B \in \mathscr{B}$, the map $p \to \sigma(p)(B)$ is measurable from $(X^n, \mathscr{B}^n)$ to the real line equipped with its Borel sets.

Then, by the theorem of Ionescu Tulcea (Proposition V.1.1. in Neveu (1965)), there is a unique countably additive probability σ' on $\mathscr{B}^\infty$ such that $\sigma'(A) = \sigma(A)$ for every set A of the form $B_1 \times B_2 \times \cdots$ where each $B_i \in \mathscr{B}$ and $B_i = X$ for all but finitely many i.

THEOREM 1. *If σ is measurable with respect to $\mathscr{B}$ then $\mathscr{A}(\sigma)$ contains $\mathscr{B}^\infty$ and σ' agrees with σ on $\mathscr{B}^\infty$. In particular, σ is countably additive on $\mathscr{B}^\infty$.*

Since $\mathscr{A}(\sigma)$ contains the Borel sigma-field (Theorem 5.1), it certainly contains the smaller sigma-field $\mathscr{B}^\infty$.

The last assertion of the theorem can be proved without relying on the usual

countably additive theory for the existence of σ'. However, we follow a shorter route here.

The proof that $\sigma = \sigma'$ on $\mathscr{B}^\infty$ begins with two rather technical lemmas. The heart of the argument is Lemma 2.

LEMMA 1. *Let K be a clopen set and let* $K \in \mathscr{B}^\infty$. *Then* $\sigma(K) = \sigma'(K)$.

PROOF. The proof is by induction on the structure of K and is presented in detail in Section 2 of Sudderth (1971). ☐

An *incomplete stop rule* is a function $t : H \to N \cup \{\infty\}$ such that if $t(h) < \infty$ and $h_i = h_i'$, $i = 1, \cdots, t(h)$, then $t(h) = t(h')$. If t is an incomplete stop rule, the set $[t < \infty](= \{h \in H \mid t(h) < \infty\})$ is open. Conversely, if O is open there is an incomplete stop rule t such that $O = [t < \infty]$.

LEMMA 2. *Let t be a* $\mathscr{B}^\infty$-*measurable incomplete stop rule. Then* $\sigma[t < \infty] = \sigma'[t < \infty]$.

PROOF. Notice that

$$\sigma[t < \infty] = \sup \{\sigma[t \leq s] : s \text{ a stop rule}\} \quad \text{(by Corollary 5.3)}$$
$$\geq \sup \{\sigma[t \leq n] : n \text{ a positive integer}\}$$
$$= \sup \{\sigma'[t \leq n] : n \text{ a positive integer}\} \quad \text{(by the previous lemma)}$$
$$= \sigma'[t < \infty] \,.$$

The final equation above uses the countable additivity of σ' on $\mathscr{B}^\infty$.

To complete the proof it suffices to show that, for every stop rule s,

(1) $$\sigma[t \leq s] \leq \sup_n \sigma[t \leq n] \,.$$

The proof of (1) is by induction on the structure of s. If s is constant, (1) is clear. It remains to check the inductive step.

Recall that
$$s[x](h) = s(xh) - 1,$$
and set
$$t[x](h) = t(xh) - 1 \,.$$

Notice that, for each x, $s[x]$ is either a stop rule or identically equal to zero. Also, $s[x]$ has smaller structure than that of s if the structure of s is larger than zero. Similarly, $t[x]$ is either a $\mathscr{B}^\infty$-measurable incomplete stop rule or identically zero. Finally, the conditional strategy $\sigma[x]$ is measurable, for each x, because σ is. Now compute

(2)
$$\sigma[t \leq s] = \int \sigma[x]([t \leq s]x) \, d\sigma_0(x)$$
$$= \int \sigma[x][t[x] \leq s[x]] \, d\sigma_0(x)$$
$$\leq \int \sup_n \sigma[x][t[x] \leq n] \, d\sigma_0(x) \,.$$

The inequality follows from the inductive assumption.

Let $\varepsilon > 0$. For $x \in X$, define

$$N(x) = \min \{k : (\sigma[x][t[x] \leq k]) \geq (\sup_n \sigma[x][t[x] \leq n]) - \varepsilon\} \,,$$

and let $M(h) = N(h_1) + 1$ for $h \in H$, where h_1 is the first coordinate of h. Then, by (2),

$$(3) \qquad\qquad \sigma[t \leq s] \leq \int \sigma[x][t[x] \leq N(x)] \, d\sigma_0(x) + \varepsilon \, .$$

As $[t[x] \leq N(x)] = [t \leq M]x$ the right-hand side of (3) is equal to

$$\int \sigma[x]([t \leq M]x) \, d\sigma_0(x) + \varepsilon = \sigma[t \leq M] + \varepsilon$$
$$= \sigma'[t \leq M] + \varepsilon \, .$$

The last step, which follows from Lemma 1, requires that M be $\mathscr{B}^\infty$-measurable. This will follow easily from the $\mathscr{B}$-measurability of the functions

$$x \to \sigma[x][t[x] \leq n] \, , \quad x \in X \, .$$

For each n, this has the form $x \to \sigma[x]Ax$, where A is $\mathscr{B}^\infty$-measurable and depends on only finitely many coordinates. By Lemma 1, $\sigma[x]Ax = (\sigma[x])'Ax$. The $\mathscr{B}^\infty$-measurability of $x \to (\sigma[x])'Ax$ then follows by the standard arguments. $\square$

PROOF OF THEOREM 1. Let $\mathscr{D}$ be the collection of all $A \subseteq H$ such that, for every $\varepsilon > 0$, there are $\mathscr{B}^\infty$-measurable incomplete stop rules t_1, t_2 such that $[t_2 = \infty] \subseteq A \subseteq [t_1 < \infty]$ and $\sigma'[t_1 < \infty] - \sigma'[t_2 = \infty] < \varepsilon$. Then $\mathscr{D}$ is a sigma-field, which can be verified by checking in order that $\mathscr{D}$ is closed under the taking of complements, finite unions, and countable increasing unions.

Now let A be a cylinder set in $\mathscr{B}^\infty$. Then there is an $n \in N$ and a set $B \subseteq X^n$ such that $A = \{h \in H \mid (h_1, \cdots, h_n) \in B\}$. Define $t_1(h) = n$ or ∞ as $h \in A$ or $h \notin A$; and $t_2(h) = n$ or ∞ as $h \notin A$ or $h \in A$. Then t_1, t_2 are $\mathscr{B}^\infty$-measurable incomplete stop rules and $[t_2 = \infty] = A = [t_1 < \infty]$. Thus $A \in \mathscr{D}$ and $\mathscr{D} \supseteq \mathscr{B}^\infty$.

For the final step of the proof, write O for sets of the form $[t_1 < \infty]$ and C for sets of the form $[t_2 = \infty]$ where t_1, t_2 are any $\mathscr{B}^\infty$-measurable incomplete stop rules. Let $A \in \mathscr{B}^\infty$. Since $A \in \mathscr{D}$, there exist sets O_n and C_n for $n \in N$ such that the O_n's are decreasing, the C_n's are increasing, $C_n \subseteq A \subseteq O_n$, and $\sigma'(O_n - C_n) \to 0$. Then $\sigma'(\bigcup C_n) = \sigma'(\bigcap O_n) = \sigma'(A)$. Also, $\sigma'(\bigcup C_n) = \lim \sigma'(C_n) = \lim \sigma(C_n) \leq \sigma(A)$, where Lemma 2 is used for the second equality. Similarly, $\sigma(A) \leq \sigma'(\bigcap O_n)$. It follows that $\sigma(A) = \sigma'(A)$. $\square$

A few brief remarks conclude this section. Let $\mathscr{C}$ be the usual completion of $\mathscr{B}^\infty$ under σ'. Let $\mathscr{D}$ be as in the proof of Theorem 1. It can be shown that $\mathscr{D} = \mathscr{C} \subseteq \mathscr{A}(\sigma)$. Suppose X is finite or countable and $\mathscr{B}$ is the set of all subsets of X. Then every incomplete stop rule is $\mathscr{B}^\infty$-measurable and $\mathscr{A}(\sigma)$ coincides with $\mathscr{C}$. In particular, the usual examples of nonmeasurable sets give examples of sets not in $\mathscr{A}(\sigma)$.

The field $\mathscr{A}(\sigma)$ is sometimes strictly larger than $\mathscr{C}$, since $\mathscr{A}(\sigma)$ always contains all clopen sets and it can easily happen that some clopen sets are not $\mathscr{B}^\infty$-measurable.

The sigma-field $\mathscr{F}(\sigma)$ certainly contains $\mathscr{B}^\infty$ since it contains the collection of cylinder sets. However, one can construct examples to see that $\mathscr{F}(\sigma)$ need not contain $\mathscr{C}$.

7. Remarks on some classical convergence theorems. Many of the almost sure convergence theorems for countably additive probabilities also hold for probabilities determined by strategies. Often the usual proofs hold up when suitably buttressed by the measure theory of Section 5. The purpose of this section is to give a few instances of how this is accomplished. There is no attempt at maximum generality, the idea being to show the finitely additive arguments in the simplest setting possible. Details and further results can be found in Purves and Sudderth (1973) and in Chen (1974).

Before the statement of the first result, a definition and some notation are needed. Let r be a stop rule. A set $K \subseteq H$ is said to be *determined by time* r provided that $h \in K$, $h' \in H$ and $h_i = h_i'$ for $i = 1, \cdots, r(h)$ imply $h' \in K$. It is shown in Dubins and Savage (1965) that the clopen sets are exactly those sets which are determined by time r for some stop rule r. Let $\{K_n\}$ be a sequence of clopen sets and let $\{r_n\}$ be a sequence of stop rules which is pointwise strictly increasing and such that K_n is determined by time r_n for every n. Define $q_n(h) = p_{r_n}(h)$ for every $n \in N$ and $h \in H$. Assume $0 \leq \alpha_n \leq 1$ for all $n \in N$ and let σ be a strategy.

LEMMA 1. *If* $\sigma(K_1) \geq (\leq) \alpha_1$ *and if, for all* $n \in N$ *and* $h \in \bigcap_1^n K_i$, $\sigma(K_{n+1} \mid q_n(h)) \geq (\leq) \alpha_{n+1}$, *then* $\sigma(\bigcap_1^\infty K_i) \geq (\leq) \prod_1^\infty \alpha_i$.

PROOF. The set $\bigcap_1^\infty K_i$ is closed. Let K be clopen and $K \supseteq \bigcap_1^\infty K_i$. It suffices for the first inequality to show $\sigma(K) \geq \prod_1^\infty \alpha_i$.

The argument is by induction on the structure of K. We can and do assume $\alpha_i > 0$ for all i.

Suppose K has structure O. Then either $K = H$ or $K = \varnothing$. If $K = H$, then $\sigma(K) = 1 \geq \prod_1^\infty \alpha_i$. We show K cannot be empty by constructing a history $h \in \bigcap_1^\infty K_i$. Since $\sigma(K_1) \geq \alpha_1 > 0$, there exists $h^1 \in K_1$. Since $\sigma(K_2 \mid q_1(h^1)) \geq \alpha_2 > 0$, there exists $h^2 \in K_2$ such that h^2 agrees with h^1 up to time $r_1(h^1)$. Continue in this fashion to define $h^n \in K_n$ such that h^n agrees with h^{n-1} up to time $r_{n-1}(h^{n-1})$. Then let h be that history which agrees with h^n up to time $r_n(h^n)$ for all n. Since K_n is determined by time r_n and $h^n \in K_n$, we have $h \in K_n$ for all n.

For the inductive step, assume the desired result for sets of structure less than the ordinal α and suppose K has structure $\alpha > 0$. Then, for all h, $Kq_1(h) \supseteq (\bigcap_1^\infty K_n)q_1(h) = \bigcap_1^\infty (K_n q_1(h))$ and $Kq_1(h)$ has structure less than α.

Fix $h \in K_1$. Set $q = q_1(h)$ and define $\sigma' = \sigma[q]$, $K_n' = K_{n+1}q$, and $r_n'(h') = r_{n+1}(qh') - r_1(h)$ for $h' \in H$. Then $\sigma'(K_1') = \sigma(K_2 \mid q_1(h)) \geq \alpha_2$. Also, if $h' \in \bigcap_1^n K_i'$, then $qh' \in \bigcap_1^{n+1} K_i$ and $\sigma'(K_{n+1}' \mid q_n'(h')) = \sigma(K_{n+2} \mid q_{n+1}(qh')) \geq \alpha_{n+2}$.

By the inductive assumption, if $h \in K_1$, then $\sigma(K \mid q_1(h)) \geq \prod_2^\infty \alpha_n$. Hence, by Corollary 4.1,

$$\sigma(K) = \int \sigma(K \mid q_1(h)) \, d\sigma(h)$$

$$\geq \int_{K_1} \sigma(K \mid q_1(h)) \, d\sigma(h)$$

$$\geq \sigma(K_1) \prod_2^\infty \alpha_n$$

$$\geq \prod_1^\infty \alpha_n \, .$$

The first inequality is now proved. The other is simpler since $\sigma(\bigcap_1^\infty K_i) \leqq \sigma(\bigcap_1^n K_i) \leqq \prod_1^n \alpha_i$ for every n as can be proved by ordinary induction on n. $\square$

Let $\{\gamma_n\}$ be a sequence of probabilities defined on all subsets of X. Define the strategy $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ by $\sigma_0 = \gamma_1$ and, for all partial histories p of length n, $\sigma(p) = \gamma_{n+1}$. Such a strategy is said to be *independent*. Notice that $\sigma[p] = \gamma_{n+1} \times \gamma_{n+2} \times \cdots$ for every p of length n.

LEMMA 2. *Let* $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ *be an independent strategy and let* $A_n \subseteq X$ *for* $n \in N$. *Then*

$$\sigma(A_1 \times A_2 \times \cdots) = \gamma_1(A_1)\gamma_2(A_2) \cdots .$$

PROOF. Apply Lemma 1 with $K_n = \{h \mid h_n \in A_n\}$, $r_n = n$ identically, and $\alpha_n = \gamma_n(A_n)$ for each n. $\square$

THEOREM 1 (Borel–Cantelli). *Let* $\sigma = \gamma_1 \times \gamma_2 \times \cdots$ *be an independent strategy. Let* $A_n \subseteq X$, $B_n = \{h \mid h_n \in A_n\}$ *and* $a_n = \sigma(B_n) = \gamma_n(A_n)$ *for each* $n \in N$. *If* $\sum a_n < (=) +\infty$, *then* $\sigma[B_n$ i.o.$] = 0$ $(= 1)$. (*Here "i.o." is short for "infinitely often."*)

PROOF. Let $O_n = \bigcup_{k \geqq n} B_k$. Then $[B_n$ i.o.$] = \bigcap_n O_n$. By Lemma 2, $\sigma(O_n) = 1 - \sigma(O_n^c) = 1 - \prod_{k \geqq n} (1 - a_k)$.

If $\sum a_n < \infty$, then $\sigma[B_n$ i.o.$] \leqq \sigma(O_n) \leqq \sum_{k \geqq n} a_k \to 0$ as $n \to \infty$.

Suppose that $\sum a_n = \infty$. By Theorem 5.2 $\sigma(O_r) \downarrow \sigma[B_n$ i.o.$]$ as $r \to \infty$ through the stop rules. So it suffices to show $\sigma(O_r) = 1$ for every stop rule r. The proof is by induction on the structure of r. If r has structure O, the argument is the familiar one. For the inductive step, use the equality $\sigma(O_r) = \int \sigma(O_r \mid x) \, d\sigma_0(x)$ and then use the inductive hypothesis to establish that the integrand has the value 1 for all $x \in X$. $\square$

There is an alternative proof of Theorem 1 which uses the results of Section 6 to reduce it to the countably additive case. To illustrate the technique assume that $A_n = A$ for all n. Let $\mathscr{B} = \{A, A^c, \phi, X\}$. Then the strategy σ of Theorem 1 is easily seen to be measurable with respect to $\mathscr{B}$. By Theorem 6.1, $\sigma(h \mid h_n \in A$ i.o.$\} = \sigma'\{h \mid h_n \in A$ i.o.$\}$. Since the conclusion of Theorem 1 is true for the countably additive probability σ', it is also true for σ. This technique of reducing to the countably additive case often works, but sometimes does not and we prefer the direct finitely additive proofs.

In the countably additive case, independence is only required in one direction of the Borel–Cantelli lemma. The same is not true for probabilities determined by strategies, as the following familiar example shows.

Let γ be a probability on N which assigns zero to every finite subset of N. Then, if $L_n = \{i \in N \mid i \leqq n\}$, $\gamma(L_n) = 0$; but the set $[L_n$ i.o.$]$ is N. To restate this example in the formal framework, let $X = N$, σ be any strategy with $\sigma_0 = \gamma$, and put $A_n = \{h \in H \mid h_1 \in L_n\}$.

Here is a slightly different example, based on the same γ. Imagine a particle

moving at random amongst the points of N in the following manner: at 1, it moves to an integer chosen according to γ; at any $k > 1$ it moves directly to $k - 1$. Then, regardless of initial position, the particle will return to 1 infinitely often with probability one (use Lemma 1, for example). However, for each sufficiently large n, the chance the particle will be at 1 at time n is zero.

Despite the double reliance on independence in Theorem 1, it can still be used in conventional ways (as David Freedman pointed out to us) to prove convergence theorems. In particular, the proof of Theorem 5.1.2 in Chung (1968) can be modified with the use of Theorem 1 so as to give a "strong law of large numbers for independent, uniformly bounded variables."

THEOREM 2. *Let σ be an independent strategy on H and let $\{Y^n\}$ be a uniformly bounded sequence of real-valued functions on H such that, for each $n \in N$, Y^n depends only on the nth coordinate and $\sigma Y^n = 0$. Then $\sigma\{h \mid 1/n \sum_1^n Y^i(h) \to 0\} = 1$.*

PROOF. Omitted.

Robert Chen (1974) has shown that the assumption in Theorem 2 that the Y^n's are uniformly bounded can be replaced, as in the conventional theory, by the weaker condition that $\sum n^{-2} \int (Y^n)^2 d\sigma < \infty$.

For $n \in N$ and $h = (h_1, \cdots, h_n, \cdots) \in H$, let $Z^n(h) = h_n$. The sequence $\{Z^n\}$ is the *coordinate process on H*. Notice that the functions Z^n are nonnegative if X is a subset of the set of nonnegative real numbers. Let σ be a strategy and suppose that, for $n \in N$ and $h \in H$, $\sigma(Z^{n+1} \mid p_n(h)) \leq Z^n(h)$. Then $\{Z^n\}$ is a *supermartingale under σ*. (The notation $\sigma(g \mid p)$ is short for $\sigma[p](gp)$ and $p_n(h) = (h_1, \cdots, h_n)$.)

THEOREM 3. *If X is contained in the set of nonnegative real numbers and the coordinate process $\{Z^n\}$ is a supermartingale under σ, then the sequence $\{Z^n(h)\}$ converges for h in a set of σ-probability one.*

PROOF. Let I be a finite interval of real numbers and let $\beta(h)$ be the number of upcrossings of I by the sequence $\{Z^n(h)\}$. Then $\sigma[\beta = +\infty] \leq \sigma[\beta \geq n]$ for each $n \in N$ and $\sigma[\beta \geq n] \to 0$ as $n \to \infty$ by Theorem 13.1 of Dubins (1962). (Alternatively the argument in Doob (1953) can be adapted to show $\sigma[\beta = \infty] = 0$.)

Let $\{I_n\}$ be an enumeration of all intervals with rational endpoints and, for each n, let β_n be the number of upcrossings of I_n by $\{Z^n\}$. Set $A^n = [\beta_n = +\infty]$ and $A = \bigcup A^n$.

Claim. $\sigma(A) = 0$.

To verify this, let $p \in X^*$ and $n \in N$. If p has length n, then the coordinate process $\{Z^1(h), Z^2(h), \cdots\}$ is a supermartingale under $\sigma[p]$ and $A^n p$ is the event that $\{Z^n\}$ upcrosses I_n infinitely often. By the first paragraph of the proof, $\sigma(A^n \mid p) = \sigma[p](A^n p) = 0$. The claim now follows from the remark at the end of Theorem 5.2.

Since $A^c \subseteq [\{Z^n\}$ converges], the proof of Theorem 3 is complete. $\square$

It can also be shown that $\lim Z^n$ is finite σ-almost surely and even that $\sigma[\lim Z^n > a] \to 0$ as $a \to \infty$. Other martingale convergence results and also some finitely additive $0 - 1$ laws are in Purves and Sudderth (1973).

Finally, Theorems 2 and 3 imply their classical counterparts. Suppose X_1, X_2, $\cdots$ is a uniformly bounded sequence of independent random variables on a countably additive probability space. For present purposes we can assume the X_n's to be coordinate maps on an infinite product space and the probability measure to be a product measure. Such a product measure can be extended (using some transfinite principle such as the axiom of choice) to be an independent strategy. The strong law for the X_n's would then follow from Theorem 2 and Theorem 6.1. For similar reasons, the almost sure convergence of conventional nonnegative supermartingales follows from Theorem 3.

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