The Annals of Probability
1976, Vol. 4, No. 2, 195-228

A MARTINGALE APPROACH TO INFINITE SYSTEMS OF INTERACTING PROCESSES[1]

BY R. A. HOLLEY AND D. W. STROOCK

University of Colorado

Martingale problems associated with the generators of infinite spin flip systems are considered. The stochastic calculus of spin flip systems is developed and applied to the existence and uniqueness questions. Existence of solutions is proved under the assumption that the flip rates are continuous functions of the configurations. Uniqueness theorems are proved under two different conditions and a counterexample to uniqueness in complete generality is given. The techniques also yield ergodic theorems, including rates of convergence, and results concerning mutual absolute continuity of different processes.

0. Introduction. In this paper we adopt the martingale point of view in a study of some of the infinite systems of interacting stochastic processes which have their origin in statistical mechanics. The approach used here was developed and used extensively by Stroock and Varadhan in their studies of diffusion processes [15], [16], [17].

Let S be any countable set and let $E = \{-1, 1\}^S$. We topologize E by giving $\{-1, 1\}$ the discrete topology and E the resulting product topology. We denote elements of E by Greek letters such as η, σ and ξ, and think of such elements as functions from S into $\{-1, 1\}$. The value of η at k is denoted by η_k. We refer to the elements of E as configurations—thinking of them as representing the configurations of spins in a piece of iron, each η_k representing an individual spin.

Now the configuration is allowed to evolve with time. The individual spins of the configuration interact with each other and flip over at random times, the flip rate at k being governed by a function, c_k, of the entire configuration. The resulting random path of configurations is denoted by $\eta(t)$.

In order to identify these stochastic processes more precisely we let $\mathscr{C}(E)$ be the space of continuous complex valued functions on E, and let $\mathscr{D}$ be the elements of $\mathscr{C}(E)$ which depend on only finitely many coordinates. Suppose that each $c_k \in \mathscr{C}(E)$ and that each $c_k \geqq 0$. We define a linear operator $\mathscr{L}$ from $\mathscr{D}$ into $\mathscr{C}(E)$ by the formula

$$\mathscr{L}f(\eta) = \sum_{k \in S} c_k(\eta)\Delta_k f(\eta)\,, \tag{0.1}$$

where $\Delta_k f(\eta) = f({}_k\eta) - f(\eta)$, and ${}_k\eta$ is the configuration obtained from η by

Received April 4, 1975; revised August 7, 1975.

[1] Research supported in part by N.S.F. Grant MPS74-18926.

AMS 1970 *subject classifications.* Primary 60K35; Secondary 60G45.

Key words and phrases. Martingale problem, infinite particle system, random time change. convergence to equilibrium.

flipping the spin at k. Let

$$\Omega = D([0, \infty), E)\,, \tag{0.2}$$

the space of right continuous functions with left limits which map $[0, \infty)$ into E. We think of Ω as a Polish space endowed with the Skorohod metric. Given $\omega \in \Omega$ and $t \geqq 0$ let $\eta(t, \omega) = \omega(t)$ (i.e., the configuration at time t.) For $0 \leqq s \leqq t$ set $\mathscr{M}_t^s = \mathscr{B}[\eta(u, \cdot): s \leqq u \leqq t]$, the σ-algebra generated by $\eta(u, \cdot)$ for $s \leqq u \leqq t$, and set $\mathscr{M}^s = \mathscr{B}[\eta(u, \cdot): u \geqq s]$. Note that $\mathscr{M} \equiv \mathscr{M}^0$ coincides with the Borel field Ω. We say that a probability measure P_η on $\mathscr{M}$ solves the *martingale problem* for $\mathscr{L}$ (or c) with initial configuration η if

$$P_\eta(\eta(0) = \eta) = 1\,, \tag{0.3}$$

and for all $f \in \mathscr{D}$

$$f(\eta(t)) - \textstyle\int_0^t \mathscr{L}f(\eta(s))\,ds \tag{0.4}$$

is a $\langle\{\mathscr{M}_t^0\}, P_\eta\rangle$ martingale.

To see that this corresponds to our intuitive notion of each spin η_k flipping at the infinitesimal rate $c_k(\eta)$ we must show that for all $s \geqq 0$

$$P_\eta(\eta_k(u) \neq \eta_k(s) \quad \text{for some} \quad u \in [s, s+h] \mid \mathscr{M}_s^0) = hc_k(\eta(s)) + o(h)\,. \tag{0.5}$$

Let $f_k(\eta) = \frac{1}{2}\eta_k$. Then $f_k \in \mathscr{D}$, and $\mathscr{L}f_k(\eta) = -c_k(\eta)\eta_k$. Thus $f_k(\eta(t)) + \int_0^t c_k(\eta(u))\eta_k(u)\,du$ is a P_η martingale. Given $s \geqq 0$ let $\tau = \inf\{t > s: \eta_k(t) \neq \eta_k(s)\}$. Then

$$\begin{aligned}
&P_\eta(\eta_k(u) \neq \eta_k(s) \quad \text{for some} \quad u \in [s, s+h] \mid \mathscr{M}_s^0) \\
&\quad = |E^{P_\eta}\{f_k(\eta(\tau \wedge (s+h))) - f_k(\eta(s)) \mid \mathscr{M}_s^0\}| \\
&\quad = |E^{P_\eta}\{\textstyle\int_s^{\tau\wedge(s+h)} c_k(\eta(u))\eta_k(u)\,du \mid \mathscr{M}_s^0\}| \\
&\quad = E^{P_\eta}\{\textstyle\int_s^{\tau\wedge(s+h)} c_k(\eta(u))\,du \mid \mathscr{M}_s^0\} \\
&\quad = hc_k(\eta(s)) + E^{P_\eta}\{\textstyle\int_s^{\tau\wedge(s+h)} [c_k(\eta(u)) - c_k(\eta(s))]\,du \mid \mathscr{M}_s^0\} \\
&\qquad - c_k(\eta(s))E^{P_\eta}\{(s+h) - [\tau \wedge (s+h)] \mid \mathscr{M}_s^0\}\,.
\end{aligned} \tag{0.6}$$

Since c_k is continuous and the paths are right continuous the dominated convergence theorem shows that the last two terms on the right side of (0.6) are $o(h)$.

The principle purpose of this paper is to investigate the questions of existence and uniqueness of solutions to the martingale problem. After developing some useful machinery in Section 1 we prove in Section 2 that there is always at least one solution to the martingale problem provided the flip rates are continuous functions of the configuration. The proof of existence is surprisingly easy in view of the difficulties encountered in previous existence proofs (see [1], [6], [8], [9], [14], [20]). The reason for this is that in previous existence proofs uniqueness was also proved simultaneously. The uniqueness question is both considerably more difficult and interesting. Among other things, uniqueness implies that the process is a strong Markov process and is stable under various approximation procedures. Section 3 contains an example of nonuniqueness. The most useful conditions which imply uniqueness are those of Liggett [14]. In Section 4 we give an analytic proof of uniqueness under Liggett's conditions, and later, in Section 5, we give a second proof involving simultaneous random time changes. This latter approach has also been used by Helms [7]. Section 6 contains another criterion for uniqueness. The method of Section 6 is to view the operator $\mathscr{L}$ as a perturbation of an operator $\mathscr{L}^0$ for which all of the flip rates are constants. This method has the pleasant feature that it yields ergodic theorems for the processes. These ergodic theorems, together with some results on the stochastic Ising model, are contained in Section 7. Finally Section 8 contains an investigation into when the solutions of the martingale problem for the different operators are mutually absolutely continuous on $\mathscr{M}_t^0$ for each t. This leads to existence and uniqueness theorems in some special cases when the flip rates are only measurable rather than continuous functions of the configuration.

1. Stochastic calculus. Unless the countable set S is given a particular structure it will always be assumed to be the integers. This is for notational convenience only.

Ω and $\mathscr{M}_t^s$ are as in (0.2) and the paragraph which follows it, and the operators Δ_k are as in (0.1). A function φ on $[s, \infty) \times \Omega$ is said to be *s-nonanticipating* (simply *nonanticipating* if $s = 0$), if φ is $\mathscr{B}_{[s,\infty)} \times \mathscr{M}^s$ measurable and $\varphi(t) = \varphi(t, \cdot)$ is $\mathscr{M}_t^s$ measurable for $t \geqq s$. If P is a probability measure on $\langle \Omega, \mathscr{M}^s \rangle$ and α is a complex valued s-nonanticipating function we say that α is a *P-martingale* if $\langle \alpha(t), \mathscr{M}_t^s, P \rangle$, $t \geqq s$, is a martingale.

Given an s-nonanticipating function $c : [s, \infty) \times \Omega \to [0, \infty)^S$ such that c_k is bounded for each $k \in S$, define the random operator

$$\mathscr{L}_t = \textstyle\sum_{k \in S} c_k(t) \Delta_k , \quad t \geqq s ,$$

on $\mathscr{D}$. Let

$$\begin{aligned} \hat{\mathscr{D}} = \{f \in \mathscr{C}([0, \infty) \times E) : f(\cdot, \eta) \in \mathscr{C}^1([0, \infty)) \quad &\text{for all} \quad \eta \\ \text{and there is a finite} \quad F \subset S \quad \text{such that} \quad &\Delta_k f(t, \cdot) \equiv 0 \\ \text{for all} \quad t \geqq 0 \quad \text{and} \quad k \notin F\} . \end{aligned}$$

The following theorem is proved in much the same way as Theorem (2.1) of [16].

(1.1) THEOREM. *Given a probability measure P on $\langle \Omega, \mathscr{M}^s \rangle$ and a right continuous s-nonanticipating function $\alpha : [s, \infty) \times \Omega \to E$ having left limits, the following are equivalent*:

(i) $f(\alpha(t \vee s)) - \int_s^{t \vee s} \mathscr{L}_u f(\alpha(u))\, du$ *is a P martingale for all* $f \in \mathscr{D}$,

(ii) $f(t \vee s, \alpha(t \vee s)) - \int_s^{t \vee s} (\partial f/\partial u + \mathscr{L}_u f)(u, \alpha(u))\, du$ *is a P martingale for all* $f \in \hat{\mathscr{D}}$,

(iii) $f(\alpha(t \vee s)) \exp[-\int_s^{t \vee s} \mathscr{L}_u f(\alpha(u))/f(\alpha(u))\, du]$ *is a P martingale for all uniformly positive* $f \in \mathscr{D}$,

(iv) $\exp[\sum_k (\theta_k \alpha_k(t \vee s) - \int_s^{t \vee s} (e^{-2\theta_k \alpha_k(u)} - 1)c_k(u)\, du)]$ *is a P martingale for all* $\theta \in R^S$ *such that* $\theta_k = 0$ *for all but finitely many k's,*

(v) $\exp[\sum_k (i\theta_k \alpha_k(t \vee s) - \int_s^{t \vee s} (e^{-2i\theta_k \alpha_k(u)} - 1)c_k(u)\, du)]$ *is a P martingale for all* θ *as in* (iv).

Given P on $\langle \Omega, \mathscr{M}^s \rangle$ and a countably generated σ-algebra $\mathscr{A} \subseteq \mathscr{M}^s$, there is a function $\omega \to P^{(\omega)}$ having the following properties:

(i) $\omega \to P^{(\omega)}(C)$ is $\mathscr{A}$ measurable for all $C \in \mathscr{M}^s$,

(ii) $P^{(\omega)}(\cdot)$ is a probability measure on $\langle \Omega, \mathscr{M}^s \rangle$ for all $\omega \in \Omega$,

(iii) $P^{(\omega)}([\omega]_{\mathscr{A}}) = 1$ for all $\omega \in \Omega$, where $[\omega]_{\mathscr{A}}$ is the atom of $\mathscr{A}$ containing ω,

(iv) $P^{(\cdot)}(C) = P(C \mid \mathscr{A})$ (a.s. P) for all $C \in \mathscr{M}^s$.

If $\omega \to P^{(\omega)}$ is such a function, we will call it a *regular conditional probability distribution of P given* $\mathscr{A}$ (r.c.p.d. *of* $P \mid \mathscr{A}$). In particular, if $\tau : \Omega \to [s, \infty)$ is an *s-stopping time* (i.e. $\{\tau \leqq t\} \in \mathscr{M}_t^s$ for all $t \geqq s$) and $\mathscr{M}_\tau^s = \{A : A \cap \{\tau \leqq t\} \in \mathscr{M}_t^s, t \geqq s\}$, then $\mathscr{M}_\tau^s$ is countable generated (in fact $\mathscr{M}_\tau^s = \mathscr{B}[\eta(t \wedge \tau) : t \geqq s]$). Hence a r.c.p.d. of $P \mid \mathscr{M}_\tau^s$, $\omega \to P^{(\tau,\omega)}$, exists and (iii) becomes:

$$P^{(\tau,\omega)}(\eta(t) = \eta(t, \omega), t \leqq \tau(\omega)) = 1 .$$

The next theorem is proved in the same way as Theorem (3.1) in [15].

(1.2) THEOREM. *Let P be a probability measure on* $\langle \Omega, \mathscr{M}^s \rangle$ *and* τ *an s-stopping time. Let* $\alpha : [s, \infty) \times \Omega \to E$ *be a right continuous s-nonanticipating function having left limits. Assume that*

$$f(\alpha(t \vee s)) - \int_s^{t \vee s} \mathscr{L}_u f(\alpha(u))\, du$$

is a P martingale for all $f \in \mathscr{D}$. *If* $\omega \to P^{(\tau,\omega)}$ *is a* r.c.p.d. *of* $P \mid \mathscr{M}_\tau^s$, *then there is an* $N \in \mathscr{M}_\tau^s$ *such that* $P(N) = 0$ *and for* $\omega \notin N$

$$f(\alpha(t \vee \tau(\omega))) - \int_{\tau(\omega)}^{t \vee \tau(\omega)} \mathscr{L}_u f(\alpha(u))\, du$$

is a $P^{(\tau,\omega)}$ *martingale.*

We now want to develop the stochastic integral for these processes. Let $\alpha : [0, \infty) \times \Omega \to E$ be a right continuous nonanticipating function having left limits and suppose P is given on $\langle \Omega, \mathscr{M}^0 \rangle$ so that $f(\alpha(t)) - \int_0^t \mathscr{L}_u f(\alpha(u))\, du$ is a P martingale for all $f \in \mathscr{D}$. Given a nonanticipating function $\theta : [0, \infty) \times \Omega \to R^S$ such that each θ_k is bounded, $\theta_k \equiv 0$ for all but finitely many k's, and there exists n for which θ satisfies $\theta(t) = \theta([nt]/n)$, $t \geqq 0$, we will call θ a *simple* function. For simple functions θ, we define

$$\int_s^t \theta(u)\, d\tilde{\alpha}(u) = \sum_k \left(\int_s^t \theta_k(u-)\, d\alpha_k(u) + 2 \int_s^t c_k(u)\alpha_k(u)\theta_k(u)\, du\right) ,$$

where for each ω

$$\int_s^t \theta_k(u-, \omega)\, d\alpha_k(u, \omega)$$

is the ordinary Lebesgue–Stieltjes integral. It is obvious that this definition is linear in θ and satisfies:

$$\int_{t_0}^{t_1} \theta(u)\, d\tilde{\alpha}(u) = \int_{t_0}^{t_1} \theta(u)\, d\tilde{\alpha}(u) + \int_{t_1}^{t_2} \theta(u)\, d\tilde{\alpha}(u) ,$$

$t_0 \leqq t_1 \leqq t_2$. Moreover, $\int_s^{t\vee s} \theta(u)\, d\tilde{\alpha}(u)$ is right continuous, nonanticipating, and has left limits. Finally, from Theorems (1.1) and (1.2), it is easy to check that:

$$(1.3) \qquad X_\theta^s(t) = \exp\Big[\int_s^{t\vee s} \theta(u)\, d\tilde{\alpha}(u) - \sum_k \int_s^{t\vee s} c_k(u)(e^{-2\theta_k(u)\alpha_k(u)} - 1 + 2\theta_k(u)\alpha_k(u))\, du\Big]$$

is a P martingale. Replacing θ by $\lambda\theta$, differentiating once and then a second time with respect to λ, and setting $\lambda = 0$, we see that

$$(1.4) \qquad \int_s^{t\vee s} \theta(u)\, d\tilde{\alpha}(u)$$

and

$$(1.5) \qquad \Big(\int_s^{t\vee s} \theta(u)\, d\tilde{\alpha}(u)\Big)^2 - 4 \int_s^{t\vee s} |\theta(u)|^2_{c(u)}\, du$$

are P martingales, where

$$|\theta|^2_{c(u)} \equiv \sum_k c_k(u)\theta_k^2\,, \qquad \theta \in R^S\,.$$

Now suppose $\theta\colon [0, \infty) \times \Omega \to R^S$ satisfies

$$(1.6) \qquad E^P\Big[\int_0^T |\theta(u)|^2_{c(u)}\, du\Big] < \infty\,, \qquad T \geqq 0\,.$$

Then we can find a sequence $\{\theta^{(n)}\}$ of simple functions such that

$$E^P\Big[\int_0^T |\theta(u) - \theta^{(n)}(u)|^2_{c(u)}\, du\Big] \to 0 \qquad \text{as } n \to \infty,\ T \geqq 0\,.$$

By (1.5) and Doob's inequality, this means that

$$\int_s^{t\vee s} \theta^{(n)}(u)\, d\tilde{\alpha}(u)$$

converges in $L^2(P)$, uniformly on compact t-intervals, to a function which we define to be $\int_s^{t\vee s} \theta(u)\, d\tilde{\alpha}(u)$. It is easily checked that $\int_s^{t\vee s} \theta(u)\, d\tilde{\alpha}(u)$ is well defined (i.e., independent of the particular choice of $\{\theta^{(n)}\}$), right continuous, nonanticipating and has left limits. Also (1.4) and (1.5) are still P martingales. Finally we want to check that if (1.6) and

$$(1.7) \qquad \Big|\Big|\sum_k c_k(e^{-2\theta_k\alpha_k} - 1)^2\Big|\Big|_T^0 < \infty\,, \qquad T \geqq 0\,,$$

where $||\cdot||_T^0 = \sup_{t \leqq T, \omega \in \Omega} |\cdot|$, then the $X_\theta^0(t)$ in (1.3) is still a P martingale. To see this, first suppose that each θ_k is bounded and that $\theta_k \equiv 0$ for all but a finite number of k's. Then we may take approximating simple functions $\theta^{(n)}$ so that $||\theta_k^{(n)}||_T^0 \leqq ||\theta_k||_T^0$. Each $X^0_{\theta^{(n)}}(\cdot)$ is a P martingale and

$$E^P[(X^0_{\theta^{(n)}}(t))^2] = E^P[X^0_{2\theta^{(n)}}(t)\, Y^0_{\theta^{(n)}}(t)] \leqq ||Y^0_{\theta^{(n)}}||_t^0\,,$$

where

$$Y^0_{\theta^{(n)}}(t) = \exp\Big[\sum_k \int_0^t c_k(u)(e^{-2\theta_k^{(n)}(u)\alpha_k(u)} - 1)^2\, du\Big]\,.$$

Clearly $||Y^0_{\theta^{(n)}}||_t^0$ is bounded, independent of n, by a number depending only on the bounds on the θ_k's and the number of nonvanishing θ_k's. Hence $X^0_{\theta^{(n)}}(t)$ converges to $X_\theta^0(t)$ in $L^1(P)$, and so the latter is a P martingale. In the general case, define $\theta^{(n)}$ so that

$$\theta_k^{(n)} = 0 \quad \text{if} \quad |k| \geqq n \quad \text{or} \quad |\theta_k| \geqq n\,, \qquad = \theta_k \quad \text{otherwise.}$$

Then $X^0_{\theta^{(n)}}(\cdot)$ is a P martingale,

$$E^P[\textstyle\int_0^T |\theta^{(n)}(t) - \theta(t)|^2_{c(t)}\, dt] \to 0\,,$$

and $||\sum_k c_k(e^{-2\theta_k^{(n)}\alpha_k} - 1)^2||_T^0 \leqq ||\sum_k c_k(e^{-2\theta_k\alpha_k} - 1)^2||_T^0$. Hence $X^0_{\theta^{(n)}}(t) \to X_\theta^0(t)$ in P measure. Moreover, we again have

$$E^P[(X^0_{\theta^{(n)}}(t))^2] \leqq ||Y^0_{\theta^{(n)}}||_t^0$$

and the latter is dominated by

$$\exp[t||\textstyle\sum_k c_k(e^{-2\theta_k\alpha_k} - 1)^2||_t^0]\,.$$

Thus $X^0_{\theta^{(n)}}(t) \to X_\theta^0(t)$ in $L^1(P)$, and so X_θ^0 is a P martingale.

Let $\alpha(\cdot)$ in the preceding discussion be equal to $\eta(\cdot)$, and define $\tilde{\gamma}(\cdot)$ by

$$(1.8) \qquad \tilde{\gamma}_k(t) = -\tfrac{1}{2}\textstyle\int_0^t \eta_k(s-)\, d\eta_k(s) - \int_0^t c_k(s)\, ds\,,$$

where the integral $-\frac{1}{2}\int_0^t \eta_k(s-)\, d\eta_k(s)$ is taken in the ordinary Lebesgue–Stieltjes sense and is, of course, equal to the number of times $\eta_k(\cdot)$ changes sign in $[0, t]$. An equivalent description of $\tilde{\gamma}(\cdot)$ is

$$(1.8') \qquad \tilde{\gamma}_k(t) = -\tfrac{1}{2}\textstyle\int_0^t \eta_k(u)\, d\tilde{\eta}_k(u)\,.$$

This latter expression motivates the definition:

$$(1.9) \qquad \textstyle\int_s^t \theta(u)\, d\tilde{\gamma}(u) = -\tfrac{1}{2}\int_s^t (\theta(u)\eta(u))\, d\tilde{\eta}(u)\,, \quad t \geqq s\,,$$

for θ satisfying (1.6), (here $(\theta\eta)_k = \theta_k\eta_k$).

We now summarize the properties of $d\tilde{\gamma}(\cdot)$ integrals inherited from $d\tilde{\eta}(\cdot)$ integrals.

(1.10) THEOREM. *Let P be a measure on $\langle\Omega, \mathscr{M}^0\rangle$ such that $f(\eta(t)) - \int_0^t \mathscr{L}_u f(\eta(u))\, du$ is a P martingale for all $f \in \mathscr{D}$. Given a nonanticipating $\theta: [0, \infty) \times \Omega \to R^S$ satisfying*

$$E^P[\textstyle\int_0^T |\theta(u)|^2_{c(u)}\, du] < \infty\,, \qquad T \geqq 0\,,$$

define $\int_s^{t\vee s} \theta(u)\, d\tilde{\gamma}(u)$ by (1.9). Then $\int_s^{t\vee s} \theta(u)\, d\tilde{\gamma}(u)$ is a right continuous, nonanticipating function having left limits, $\theta \to \int_s^{t\vee s} \theta(u)\, d\tilde{\gamma}(u)$ is linear, and $\int_{t_1}^{t_3} \theta(u)\, d\tilde{\gamma}(u) = \int_{t_1}^{t_2} \theta(u)\, d\tilde{\gamma}(u) + \int_{t_2}^{t_3} \theta(u)\, d\tilde{\gamma}(u)$ for $s \leqq t_1 \leqq t_2 \leqq t_3$. Moreover, if $\theta_k \equiv 0$ for all but finitely many k's and $\{(u, \omega) \in [s, s \vee t] \times \Omega : \theta(u) \neq \theta(u-)\}$ has Lesbesgue $\times$ P measure zero, then

$$(1.11) \qquad \textstyle\int_s^{t\vee s} \theta(u)\, d\tilde{\gamma}(u) = \sum_k \left(\int_s^{t\vee s} \theta_k(u-)\, d\gamma_k(u) - \int_s^{t\vee s} c_k(u)\theta_k(u)\, du\right)\,,$$

where $\gamma_k(t)$ is the number of sign changes $\eta_k(\cdot)$ makes in $[0, t]$, and the integrals on the right hand side are Lebesgue–Stieltjes integrals. Finally

(i) *$\int_s^{t\vee s} \theta(u)\, d\tilde{\gamma}(u)$ is a P martingale,*
(ii) *$(\int_s^{t\vee s} \theta(u)\, d\tilde{\gamma}(u))^2 - \int_s^{t\vee s} |\theta(u)|^2_{c(u)}\, du$ is a P martingale,*

and if $\|\sum_k c_k(e^{\theta_k} - 1)^2\|_T^s < \infty$, $T \geqq s$, *then*

(iii) $M_\theta^s(t)$ *is a P martingale, where*:

$$M_\theta^s(t) \equiv \exp[\textstyle\int_s^{t\vee s} \theta(u)\, d\tilde{\gamma}(u) - \sum_k \int_s^{t\vee s} c_k(u)(e^{\theta_k(u)} - 1 - \theta_k(u))\, du]\,,$$

and

$$E^P[(M_\theta^s(t))^2] \leqq \exp[(t-s)\|\textstyle\sum_k c_k(e^{\theta_k} - 1)^2\|_t^s]\,.$$

(1.12) COROLLARY. *Let* $c, \bar{c}\colon [0, \infty) \times \Omega \to [0, \infty)^S$ *be nonanticipating functions such that* c_k *and* $\bar{c}_k$ *are bounded for each* $k \in S$. *Define* $\mathscr{L}_u$ and $\bar{\mathscr{L}}_u$ *accordingly on* $\mathscr{D}$ *and assume P is a measure on* $\langle\Omega, \mathscr{M}^0\rangle$ *such that* $f(\eta(t)) - \int_0^t \mathscr{L}_u f(\eta(u))\, du$ *is a P martingale for all* $f \in \mathscr{D}$. *If* $c_k = 0$ *if and only if* $\bar{c}_k = 0$ *and*

$$\left\|\sum_k c_k\left(\left(\ln\frac{\bar{c}_k}{c_k}\right)^2 + \left(\frac{\bar{c}_k}{c_k} - 1\right)^2\right)\right\|_T^0 < \infty\,, \qquad T \geqq 0\,,$$

then

$$M(t) = \exp\left[\int_0^t \ln\frac{\bar{c}}{c}(u)\, d\tilde{\gamma}(u) - \sum_k \int_0^t c_k(u)\left(\frac{\bar{c}_k}{c_k}(u) - 1 - \ln\frac{\bar{c}_k}{c_k}(u)\right) du\right]$$

is a P martingale, where $(\ln(\bar{c}/c))_k \equiv \ln(\bar{c}_k/c_k)$. *Moreover, if* $\bar{P}$ *is defined by* $d\bar{P}/dP = M(t)$ *on* $\mathscr{M}_t^0$, $t \geqq 0$, *then* $f(\eta(t)) - \int_0^t \bar{\mathscr{L}}_u f(\eta(u))\, du$ *is a* $\bar{P}$ *martingale for all* $f \in \mathscr{D}$.

PROOF. We need only check the last assertion. By Theorem (1.1), we must show that if $\theta \in R^S$ and $\theta_k = 0$ for all but a finite number of k's, then

$$X_{\theta(t)} = \exp[\textstyle\sum_k \theta_k \eta_k(t) - \int_0^t \bar{c}_k(u)(e^{-2\theta_k\eta_k(u)} - 1)\, du]$$

is a $\bar{P}$ martingale. Equivalently, we must show that

$$X_\theta(t)M(t)$$

is a P martingale. To this end, take $\theta(t)$ so that

$$\theta_k(t) = -2\theta_k\eta_k(t) + \ln\frac{\bar{c}_k}{c_k}(u)\,, \qquad k \in S\,.$$

Then

$$Z(t) \equiv \exp[\textstyle\int_0^t \theta(u)\, d\tilde{\gamma}(u) - \sum_k \int_0^t c_k(u)(e^{\theta_k(u)} - 1 - \theta_k(u))\, du]$$

is a P martingale. Segregating terms in the exponent, one sees that $Z(t) = X_\theta(t)M(t)$.

2. Existence. Again in this section we assume for notational convenience that S is the integers. If f is any complex valued function on E we set

(2.1) $$\|f\| = \sup_{\eta\in\Sigma} |f(\eta)|\,.$$

At the heart of much of our work is the following result about measures on $\langle\Omega, \mathscr{M}^0\rangle$.

(2.2) THEOREM. *Let* $\{A_n\}$ *be a sequence of positive numbers and let* $\mathscr{C}(\{A_n\})$

stand for the set of measurable functions, $c: E \to [0, \infty)^S$, such that

$$\sup_{|k| \leq n} ||c_k|| \leqq A_n, \qquad n \geqq 0.$$

If $\mathscr{P}(\{A_n\})$ is the set of probability measures P on $\langle \Omega, \mathscr{M}^0 \rangle$ such that P solves the martingale problem for some $c \in \mathscr{C}(\{A_n\})$, then $\mathscr{P}(\{A_n\})$ is relatively weakly compact.

An outline of the proof of Theorem (2.2) can be found in the appendix of [18]. The idea is based on criteria for weak compactness of Markov processes on Ω (cf. for example [21]), only here the Markov property is replaced by Theorem (1.2).

The importance of Theorem (2.2) for us is the next theorem.

(2.3) THEOREM. *Let $\{c^{(n)}\}$ be a sequence of continuous functions from E to $[0, \infty)^S$ such that for each k, $||c_k^{(n)} - c_k|| \to 0$. Assume that for each n $P^{(n)}$ is a solution to the martingale problem for $c^{(n)}$ starting from $\eta^{(n)}$, where $\eta^{(n)} \to \eta$. If $\{P^{(n)}\}$ tends weakly to P, then P solves the martingale problem for c starting from η. In particular, if there is exactly one solution, P_η, to the martingale problem for c starting from η, then necessarily $\{P^{(n)}\}$ tends weakly to P_η.*

PROOF. If $P^{(n)}$ converges weakly to P, it is clear that $P(\eta(0) = \eta) = 1$. Given $f \in \mathscr{D}$, we must show that $f(\eta(t)) - \int_0^t \mathscr{L}f(\eta(s))\, ds$ is a P martingale, where $\mathscr{L} = \sum c_k \Delta_k$. Let $0 \leqq t_1 \leqq t_2$ and Φ be an $\mathscr{M}_{t_1}^0$ measurable continuous function on Ω. Then

$$E^{P^{(n)}}[(f(\eta(t_2)) - \int_0^{t_2} \mathscr{L}^{(n)}f(\eta(s))\, ds)\Phi] = E^{P^{(n)}}[(f(\eta(t_1)) - \int_0^{t_1} \mathscr{L}^{(n)}f(\eta(s))\, ds)\Phi]$$

for all n, where $\mathscr{L}^{(n)} = \sum c_k^{(n)}\Delta_k$. Clearly $\mathscr{L}^{(n)}f \to \mathscr{L}f$ uniformly, and $\int_0^t \mathscr{L}f(\eta(s))\, ds$ is continuous on Ω. Also, $f(\eta(t))$ is P almost surely continuous on Ω, since $\eta(t) = \eta(t-)$ a.s. P for each $t \geqq 0$. Thus we can let $n \to \infty$ and get

$$E^P[(f(\eta(t_2)) - \int_0^{t_2} \mathscr{L}f(\eta(s))\, ds)\Phi] = E^P[(f(\eta(t_1)) - \int_0^{t_1} \mathscr{L}f(\eta(s))\, ds)\Phi].$$

This proves the first assertion.

The second assertion is now trivial. Indeed, by Theorem (2.2), $\{P^{(n)}\}$ is necessarily relatively weakly compact. By what has just been said, every weakly convergent subsequence of $\{P^{(n)}\}$ must have P_η as its limit. Hence $\{P^{(n)}\}$ itself tends to P_η.

We now proceed with the proof of existence. Let c be a positive number and define $p_c(t, \pm 1) = 1 \pm e^{-2ct}$. It is easy to see that $p_c(t, \cdot)$ is the transition function for a convolution semi-group over $\{-1, 1\}$ (we are thinking of $\{-1, 1\}$ as a multiplicative group with Haar measure $\mu(-1) = \mu(1) = \frac{1}{2}$). Hence there exist $P_{\pm 1}^c$ on $D([0, \infty), \{-1, 1\})$ such that $P_{\pm 1}^c(\eta(0) = \pm 1) = 1$, $P_{\pm 1}^c(\eta(t) \in \Gamma) = \int_\Gamma p_c(t, \pm\eta)\mu(d\eta)$, and $\eta(\cdot)$ has independent increments. It is trivial to check that $\exp[\theta\eta(t) - c\int_0^t (e^{-2\theta\eta(s)} - 1)\, ds]$ is a $P_{\pm 1}^c$ martingale for all $\theta \in R$. Next let $\{c_k\}_{k \in S} \subseteq [0, \infty)$ and $\eta \in E$ be given. If $c_k > 0$, define $P_{\eta_k}^{c_k}$ on $D([0, \infty), \{-1, 1\})$ as above. If $c_k = 0$, define $P_{\eta_k}^{c_k}$ on $D([0, \infty), \{-1, 1\})$ so that $P_{\eta_k}^{c_k}(\eta(t) = \eta_k, t \geqq 0) = 1$. Then define P_η on Ω to be the measure induced from $\prod_{k \in S} P_{\eta_k}^{c_k}$ by the natural mapping of $(D([0, \infty), \{-1, 1\}))^S$ to Ω. It is again easy to see that if $\theta \in R^S$ and $\theta_k \neq 0$ for only finitely many k's, then $\exp[\sum_k (\theta_k \eta_k(t) - \int_0^t c_k(e^{\theta_k \eta_k(s)} - 1)\, ds)]$ is a P_η martingale. Hence, by Theorem (1.1), we have proved existence in the case when the c_k's are constants.

To get beyond the constant case, we cite the following immediate consequence of Corollary (1.12).

(2.4) THEOREM. *Let $c, \bar{c}\colon E \to [0, \infty)^S$ be measurable functions such that $\|c_k\| < \infty$ for each k, $c_k = 0$ if and only if $\bar{c}_k = 0$, and*

$$\left\| \sum_k c_k \left(\left(\ln \frac{\bar{c}_k}{c_k} \right)^2 + \left(\frac{\bar{c}_k}{c_k} - 1 \right)^2 \right) \right\| < \infty .$$

If P solves the martingale problem for c starting from η and

$$M(t) = \exp\left[\int_0^t \left(\ln \frac{\bar{c}}{c} \right) (\eta(s))\, d\tilde{\gamma}(s) - \sum_k \int_0^t c_k(\eta(s)) \left\{ \frac{\bar{c}_k}{c_k} (\eta(s)) - 1 - \ln \left(\frac{\bar{c}_k}{c_k} (\eta(s)) \right) \right\} ds \right],$$

then $M(t)$ is a P martingale, and the measure $\bar{P}$ defined by $d\bar{P}/dP = M(t)$ on $\mathscr{M}_t^0$, $t \geqq 0$ solves the martingale problem for $\bar{c}$ starting from η.

Now let $c\colon E \to [0, \infty)^S$ be a continuous function such that $c_k > 0$ for $|k| \leqq n$ and $c_k \equiv 0$ for $|k| > n$. Combining Theorem (2.4) with the paragraph preceeding it, we see that for each $\eta \in E$, there is a solution P to the martingale problem for c starting from η. Next suppose $c\colon E \to [0, \infty)^S$ is continuous and put $c_k^{(n)} = c_k + 1/n$ for $|k| \leqq n$ and $c_k^{(n)} \equiv 0$ for $|k| > n$. Given $\eta \in E$, let $P^{(n)}$ solve the martingale problem for $c^{(n)}$ starting from η. Then, by Theorem (2.2), there is a weakly convergent subsequence $\{P^{(n')}\}$ of $\{P^{(n)}\}$, and by Theorem (2.3), the limit P of $\{P^{(n')}\}$ solves the martingale problem for c starting from η. Using this in cojunction with Theorem (2.4) we have the following existence theorem.

(2.5) THEOREM. *Let $c\colon E \to [0, \infty)^S$ be a measurable function. If there exists a continuous function $b\colon E \to [0, \infty)^S$ such that $b_k = 0$ if and only if $c_k = 0$ and*

$$\left\| \sum_k b_k \left(\left(\ln \frac{c_k}{b_k} \right)^2 + \left(\frac{c_k}{b_k} - 1 \right)^2 \right) \right\| < \infty ,$$

then for each η there is a solution to the martingale problem for c starting from η.

(2.6) REMARK. Let $\{c_k\colon k \in S\} \subseteq C(E)$ be given. We have just seen that for each $\eta \in E$ there is a solution to the martingale problem for $\mathscr{L} = \sum_k c_k \Delta_k$ starting from η. By a careful selection procedure, one can show that it is possible to choose for each η a solution P_η such that the family $\{P_\eta\colon \eta \in E\}$ is measurable and enjoys the strong Markov property [13]. However, it is not, in general, possible to make $\{P_\eta\colon \eta \in E\}$ be Feller continuous, and certainly the stability results which follow from uniqueness will fail here. Moreover, if uniqueness for the martingale problem does not hold, then there are at least two distinct choices of such

Markov families $\{P_\eta : \eta \in E\}$; and so there is no way of interpreting $\{P_\eta : \eta \in E\}$ as being canonically associated with $\mathscr{L}$. Finally, when uniqueness fails, there are definitely non-Markovian measurable selections of $\{P_\eta : \eta \in E\}$ (e.g. average two Markovian ones).

3. An example of nonuniqueness. Since the examples of nonuniqueness require a lot of computation and are negative results, we just outline the construction of one example and leave the bulk of the computation to the reader.

In this example the c_k's are uniformly bounded above and uniformly bounded away from zero, and each is an element of $\mathscr{D}$. The c_k's are easiest to define if we give the set S a special structure. Let S be the countable set

$$\{(1)\} \cup \{(1, i_1, i_2, \cdots, i_k) : 1 \leqq k < \infty \text{ and } 1 \leqq i_j \leqq 2^{2^j+100}\} .$$

If $x = (1, i_1, \cdots, i_k) \in S$ set $|x| = k$ ($|(1)| = 0$), and define

$$S(x) = \{(1, i_1, \cdots, i_k, j) : 1 \leqq j \leqq 2^{2^{k+1}+100}\} .$$

The c_x's are given by the formula

$$(3.1) \qquad \begin{aligned} c_x(\eta) &= 1 \qquad \text{if } \eta(x) = 1 \text{ or } \textstyle\sum_{y \in S(x)} \eta_y < 2^{2^{|x|+1}-3|x|+90} \\ &= 10 \qquad \text{otherwise.} \end{aligned}$$

Let $\bar\eta$ be the configuration such that $\bar\eta_{(1)} = -1$ and

$$\begin{aligned} \bar\eta_{(1,i_1,i_2,\cdots,i_k)} &= 1 \qquad \text{if } 1 \leqq i_k \leqq 2^{2^k+99} \\ &= -1 \qquad \text{if } 2^{2^k+99} < i_k \leqq 2^{2^k+100} . \end{aligned}$$

Let

$$\begin{aligned} c_x^{(n)+}(\eta) &= 0 \qquad \text{if } |x| > n , \\ &= 1 \qquad \text{if } |x| = n \text{ and } \eta(x) = 1 , \\ &= 10 \qquad \text{if } |x| = n \text{ and } \eta(x) = -1 , \\ &= c_x(\eta) \qquad \text{if } |x| < n , \end{aligned}$$

and

$$\begin{aligned} c_x^{(n)-}(\eta) &= 0 \qquad \text{if } |x| > n , \\ &= 1 \qquad \text{if } |x| = n , \\ &= c_x(\eta) \qquad \text{if } |x| < n . \end{aligned}$$

Finally let $P^{(n)+}$ and $P^{(n)-}$ be the unique (see Section 4) solution, starting from $\bar\eta$, to the martingale problem for $c^{(n)+}$ and $c^{(n)-}$ respectively. As pointed in Section 4 uniqueness implies that the paths of configurations are Markov processes under $P^{(n)\pm}$.

(3.2) LEMMA. *If* $|x| \leqq n$ *then for all* $t \geqq 4^{-|x|-3}$

$$(3.3) \qquad P^{(n)+}(\eta_x(t) = 1) \geqq e^{-11t}\tfrac{1}{2}(1 + \bar\eta_x) + \tfrac{9}{11}(1 - \exp[-11(t - 4^{-|x|-3})]) .$$

PROOF. If $|x| = n$ we can solve exactly to get

$$P^{(n)+}(\eta_x(t) = 1) = \tfrac{10}{11}(1 - e^{-11t}) + e^{-11t}\tfrac{1}{2}(1 + \bar\eta_x)$$

and the theorem is true in this case.

For the rest of the proof we drop the superscript $(n)+$ and proceed by reverse induction. Assume that the lemma is true if $|x| = k + 1 \leqq n$ and that $|x| = k$. Beginning with the equality

$$\frac{d}{dt} P(\eta_x(t) = 1) = -P(\eta_x(t) = 1) + P(\eta_x(t) = -1) + 9P(\eta_x(t) = -1 \text{ and } \textstyle\sum_{y \in S(x)} \eta_y(t) \geqq 2^{2k+1-3k+90}) ,$$

elementary computations yield

$$P(\eta_x(t) = 1) > \tfrac{10}{11}(1 - e^{-11t}) + e^{-11t}\tfrac{1}{2}(1 + \bar{\eta}_x) - 9e^{-11t} \textstyle\int_0^t e^{11s} P(\sum_{y \in S(x)} \eta_y(s) < 2^{2k+1-3k+90})\, ds .$$

We split the integral into two parts, one from zero to 4^{-k-3} and one from 4^{-k-3} to t. In the first part we bound the probability in the integrand by one and in the second part we bound it by $\sup_{4^{-k-3} \leqq s} P(\sum_{y \in S(x)} \eta_y(s) < 2^{2k+1-3k+90})$. The result is

$$\begin{aligned} P(\eta_x(t) = 1) \geqq{} & \tfrac{9}{11}(1 - \exp[-11(t - 4^{-k-3})]) + e^{-11t}\tfrac{1}{2}(1 + \bar{\eta}_x) \\ & + \tfrac{1}{11}(1 - e^{-11t}) - \tfrac{9}{11}(1 - \exp[-11(t - 4^{-k-3})]) \\ & \times \textstyle\sup_{4^{-k-3} \leqq s} P(\sum_{y \in S(x)} \eta_y(s) < 2^{2k+1-3k+90}) . \end{aligned}$$

Thus to complete the proof it suffices to show that

$$\textstyle\sup_{4^{-k-3} \leqq s} P(\sum_{y \in S(x)} \eta_y(s) < 2^{2k+1-3k+90}) < \tfrac{1}{9} .$$

To do this note that for fixed s the $\eta_y(s)$, $y \in S(x)$ are independent and if $y \in S(x)$ then $|y| = k + 1$. The desired inequality follows from the induction hypothesis and Chebychev's inequality.

(3.4) LEMMA. *If* $|x| \leqq n$ *and* $0 \leqq t < \infty$, then

$$P^{(n)-}(\eta_x(t) = 1) \leqq \tfrac{1}{2}(1 + \bar{\eta}_x e^{-2t}) + 2^{-3|x|-10} . \tag{3.5}$$

PROOF. If $|x| = n$ we can solve exactly to get

$$P^{(n)-}(\eta_x(t) = 1) = \tfrac{1}{2}(1 + \bar{\eta}_x e^{-2t}) .$$

Again we drop the superscript $(n)-$ and proceed by reverse induction. Assume that the lemma is true if $|x| = k + 1$ and that $|x| = k$. By a computation similar to the one in Lemma (3.2) we get

$$\begin{aligned} P(\eta_x(t) = 1) \leqq{} & \tfrac{1}{2}(1 + \bar{\eta}_x e^{-2t}) \\ & + \tfrac{9}{2}(1 - e^{-2t}) \textstyle\sup_{0 \leqq s} P(\sum_{y \in S(x)} \eta_y(s) \geqq 2^{2k+1-3k+90}) . \end{aligned} \tag{3.5}$$

The proof is completed by using the inductive hypothesis, the independence of $\{\eta_y(s)\}$, $y \in S(x)$, and Chebychev's inequality to show that the second term on the right side of (3.5) is bounded by 2^{-3k-10}.

Now $\{P^{(n)+}\}$ and $\{P^{(n)-}\}$ each have subsequences which converge to P^+ and P^- respectively. By Theorem (2.3), both P^+ and P^- solve the martingale problem for the operator $\mathscr{L}$ given by the c_x's in (3.1), and both have initial configuration

$\tilde{\eta}$. However from inequalities (3.3) and (3.5) we see that

$$P^+(\eta_{(1)}(1) = 1) \geqq \tfrac{9}{11}(1 - \exp[-11(1 - 4^{-3})]) > \tfrac{8}{11},$$

and

$$P^-(\eta_{(1)}(1) = 1) \leqq \tfrac{1}{2}(1 - e^{-2}) + \tfrac{1}{1000} < \tfrac{1}{2}.$$

Thus $P^+ \neq P^-$ and the solution of the martingale problem for c given by (3.1) starting from $\tilde{\eta}$ is not unique.

If we are willing to allow the flip rates to take the value zero for some configurations then it is not difficult to make examples of nonuniqueness in which S is a lattice (say the integers) and the c_k's are homogeneous (i.e., if S is the integers and $\tilde{\eta}(k) = \eta(k+1)$ for all k then $c_{j+1}(\eta) = c_j(\tilde{\eta})$ for all j). We will see in the next section that in this case the c_k's can no longer be taken to be in $\mathscr{D}$; however, they can still be taken in $\mathscr{C}(E)$. The reader who is interested in constructing such an example is referred to [11]. Explosion in finite time of the branching process with interference, which appears in [11], is intimately connected with nonuniqueness.

4. Uniqueness, Part I. In this section we again take S to be the integers for notational convenience.

Suppose $c\colon E \to [0, \infty)^S$ is a continuous function and define $\mathscr{L}$ on $\mathscr{D}$ as in (0.1). We extend the definition of $\mathscr{L}$ as follows: Let

$$\mathscr{D}(\mathscr{L}) \equiv \{f \in \mathscr{C}(E)\colon \textstyle\sum_k |c_k \Delta_k f| \text{ is uniformly convergent}\}$$

and define $\mathscr{L}f = \sum_k c_k \Delta_k f$ for $f \in \mathscr{D}(\mathscr{L})$. Define

$$\hat{\mathscr{D}}(\mathscr{L}) \equiv \Big\{f \in \mathscr{C}([0,\infty) \times E)\colon \frac{\partial f}{\partial t} \in \mathscr{C}([0,\infty) \times E),\ f(t, \cdot) \in \mathscr{D}(\mathscr{L}) \text{ for } t \geqq 0, \text{ and } \textstyle\sum_k |c_k \Delta_k f| \text{ is uniformly bounded on compact subsets of } [0,\infty) \times E\Big\}.$$

(4.1) LEMMA. *If P solves the martingale problem for $\mathscr{L}$ and $f \in \mathscr{D}(\mathscr{L})$ ($\hat{\mathscr{D}}(\mathscr{L})$), then*

$$f(\eta(t)) - \int_0^t \mathscr{L}f(\eta(s))\,ds$$

$$\left(f(t, \eta(t)) - \int_0^t \left(\frac{\partial}{\partial s} + \mathscr{L}\right) f(s, \eta(s))\,ds\right)$$

is a P martingale. Moreover, if $f \in \hat{\mathscr{D}}(\mathscr{L})$ and $T > 0$, then

$$f(T - (t \wedge T), \eta(t \wedge T)) - \int_0^{t\wedge T} \left(\frac{\partial}{\partial s} + \mathscr{L}\right) f(T - s, \eta(s))\,ds$$

is a P martingale.

PROOF. Let $f \in \mathscr{D}(\mathscr{L})$. Given $n \geqq 1$, define $\alpha_n\colon E \to E$ by

$$\begin{aligned}(\alpha_n(\eta))_k &= \eta_k \quad &&\text{if } |k| \leqq n\\ &= 1 &&\text{if } |k| > n.\end{aligned}$$

Put $\varphi_n = f \circ \alpha_n$. Clearly $\varphi_n \in \mathscr{D}$ and $||\varphi_n - f|| \to 0$. ($||\cdot||$ is as in (2.1).) Moreover:

$$|\mathscr{L}\varphi_n - \mathscr{L}f| \leqq \sum_{|k|\leqq m} c_k|\Delta_k\varphi_n - \Delta_k f| + \sum_{|k|>m} c_k|\Delta_k\varphi_n| + \sum_{|k|>m} c_k|\Delta_k f|$$

and so

$$||\mathscr{L}\varphi_n - \mathscr{L}f|| \leqq \sum_{|k|\leqq m} ||c_k||\,||\Delta_k\varphi_n - \Delta_k f|| + 2||\sum_{|k|>m} |c_k\Delta_k f|\,||$$

for all $m, n \geqq 0$. Thus

$$\limsup_{n\to\infty} ||\mathscr{L}\varphi_n - \mathscr{L}f|| \leqq 2||\sum_{|k|>m} |c_k\Delta_k f|\,||,$$

and the right hand side goes to zero as m goes to infinity. Now $\varphi_n(\eta(t)) - \int_0^t \mathscr{L}\varphi_n(\eta(s))\,ds$ is a P martingale for all n, and as n goes to infinity it converges to $f(\eta(t)) - \int_0^t \mathscr{L}f(\eta(s))\,ds$ uniformly on finite t intervals. Hence the latter is also a P martingale. The proof when $f \in \hat{\mathscr{D}}(\mathscr{L})$ is similar.

Finally, suppose $f \in \hat{\mathscr{D}}(\mathscr{L})$ and $T > 0$. For $n \geqq 1$, let $\beta_n \in \mathscr{C}_0^\infty(R)$ such that $0 \leqq \beta_n \leqq 1$, $\beta_n \equiv 1$ on $(-\infty, (1 - 1/n)T]$, and $\beta_n \equiv 0$ on $[T, \infty)$. Set

$$g_n(t, \cdot) = \beta_n(t)f(T - t, \cdot)\,.$$

Then $g_n \in \hat{\mathscr{D}}(\mathscr{L})$ and so

$$g_n(t \wedge T, \eta(t \wedge T)) - \int_0^{t\wedge T} \left(\frac{\partial}{\partial s} + \mathscr{L}\right) g_n(s, \eta(s))\,ds$$

is a P martingale. The proof is completed by letting n go to infinity and using the a.s. P left continuity of $\eta(\cdot)$ at T.

(4.2) THEOREM. *Let $\mathscr{H}$ be a dense subset of $\mathscr{C}(E)$. If either of the conditions*:

(a) *for all $f \in \mathscr{H}$ there is a $u \in \hat{\mathscr{D}}(\mathscr{L})$ such that $u(0, \cdot) = f$ and $\partial u/\partial t = \mathscr{L}u$, $t > 0$,*

(b) *for all $f \in \mathscr{H}$ there is a $\lambda_0 \geqq 0$ such that if $\lambda \geqq \lambda_0$ then there is a $u_\lambda \in \mathscr{D}(\mathscr{L})$ satisfying $(\lambda - \mathscr{L})u_\lambda = f$,*

holds, then for each $\eta \in E$ there is exactly one solution, P_η, to the martingale problem starting from η. In particular, $\{P_\eta : \eta \in E\}$ is a homogeneous, strong Markov, Feller continuous family.

PROOF. The details of the last assertion can be found in [15]. To prove the first assertion, define $p(t, \eta, \Gamma) = P_\eta(\eta(t) \in \Gamma)$. Condition (a) or (b) guarantees that $p(t, \eta, \Gamma)$ is a measurable function of (t, η) which is independent of the particular solution P_η of the martingale problem for $\mathscr{L}$ that is used to define $p(\cdot, \cdot, \cdot)$. To see this, assume (a) holds and let $f \in \mathscr{H}$ and take $u \in \hat{\mathscr{D}}(\mathscr{L})$ accordingly. Then by Lemma (4.1), $u(T - (t \wedge T), \eta(t \wedge T))$ is a P_η martingale, and so $\int f(y)p(T, \eta, dy) = E^{P_\eta}[f(\eta(T))] = u(T, \eta)$. Since $\mathscr{H}$ is dense in $\mathscr{C}(E)$, this shows that $p(t, \eta, \cdot)$ is a measurable function of (t, η) and is independent of the particular choice of P_η. The argument in case (b) is similar, only it involves the use of Laplace transforms and the easily derived fact that $e^{-\lambda t}u_\lambda(\eta(t)) + \int_0^t e^{-\lambda s}f(\eta(s))\,ds$ is a P_η martingale.

Once one knows that $p(t, \eta, \Gamma)$ is measurable in (t, η) and independent of the particular solution, one proceeds as follows. Let $s \geqq 0$ and suppose $\omega \to P^{(\omega)}$ is a r.c.p.d. of $P_\eta \mid \mathscr{M}_s^0$. Then there is a P_η null set $N \in \mathscr{M}_s^0$ such that if $\omega \notin N$ and $Q^{(\omega)} = P^{(\omega)} \circ \theta_s^{-1}$ ($\theta_s : \Omega \to \Omega$ is defined by $\eta \circ \theta_s(t) = \eta(t+s)$), then $Q^{(\omega)}$ solves the martingale problem for $\mathscr{L}$ starting from $\eta(s, \omega)$. Hence if $t \geqq s$, then

$$P_\eta(\eta(t) \in \Gamma \mid \mathscr{M}_s^0) = P^{(\omega)}(\eta(t) \in \Gamma) = Q^{(\omega)}(\eta(t-s) \in \Gamma) = p(t-s, \eta(s, \omega), \Gamma)$$

(a.s. P_η). From here the usual induction argument applies and shows that

$$P_\eta(\eta(t_1) \in \Gamma_1, \cdots, \eta(t_n) \in \Gamma_n) = \int_{\Gamma_1} \cdots \int_{\Gamma_n} p(t_1, \eta, d\eta_1) \cdots p(t_n - t_{n-1}, \eta_{n-1}, d\eta_n) .$$

The next lemma is elementary.

(4.3) LEMMA. *If c_k vanishes identically for all but a finite number of k's, then $\mathscr{C}(E) = \mathscr{D}(\mathscr{L})$ and $\{f \in \mathscr{C}([0, \infty) \times E) : \partial f/\partial t \in \mathscr{C}([0, \infty) \times E)\} \subseteqq \hat{\mathscr{D}}(\mathscr{L})$. Moreover, in this case $e^{t\mathscr{L}}$ ($\equiv \sum_{n=0}^{\infty} t^n \mathscr{L}^n/n!$) is a bounded operator from $\mathscr{C}(E)$ to itself; and if $u(t, \cdot) = e^{t\mathscr{L}}f$, then $u \in \hat{\mathscr{D}}(\mathscr{L})$, $u(0, \cdot) = f$, and $\partial u/\partial t = \mathscr{L}u$. In particular, for each η there is exactly one solution P_η to the martingale problem for $\mathscr{L}$ starting from η, and $E^{P_\eta}[f(\eta(t))] = e^{t\mathscr{L}}f(\eta)$, $t \geqq 0$ and $f \in \mathscr{C}(E)$.*

For what follows it will be useful to introduce the norm $|||\cdot|||$ given by

$$|||f||| = \sum_k \|f_{,k}\| ,$$

where

$$f_{,k} \equiv \Delta_k f .$$

We let $\mathscr{C}^1(E)$ stand for the class of $f \in \mathscr{C}(E)$ for which $|||f||| < \infty$, and $\mathscr{C}^1([0, \infty) \times E)$ the class of $f \in \mathscr{C}([0, \infty) \times E)$ for which $\partial f/\partial t \in \mathscr{C}([0, \infty) \times E)$ and $\sup_{0 \leqq t \leqq T} |||f(t, \cdot)||| < \infty$ for $T \geqq 0$. Note that if $f \in \mathscr{C}^1(E)$ and $\eta, \eta' \in E$ satisfy $\eta_k = \eta_k'$ for $|k| \leqq N$, then

$$|f(\eta) - f(\eta')| \leqq \sum_{|k|>N} \|f_{,k}\| . \tag{4.4}$$

The following lemma is obvious.

(4.5) LEMMA. *If $\sup_k \|c_k\| < \infty$, then $\mathscr{C}^1(E) \subseteqq \mathscr{D}(\mathscr{L})$ and $\mathscr{C}^1([0, \infty) \times E) \subseteqq \hat{\mathscr{D}}(\mathscr{L})$.*

We are now going to show that if

$$C = \sup_k (\|c_k\| + |||c_k|||) < \infty , \tag{4.6}$$

then the martingale problem for $\mathscr{L}$ has exactly one solution for each initial point η.

Given c_k's satisfying (4.6), set

$$\begin{aligned} c_k^{(n)} &= c_k && \text{if } |k| \leqq n \\ &= 0 && \text{if } |k| > n , \end{aligned}$$

and

$$\mathscr{L}^{(n)} = \sum c_k^{(n)} \Delta_k .$$

Let $f \in \mathscr{C}^1(E)$ and put $u^{(n)}(t, \cdot) = e^{t\mathscr{L}^{(n)}}f$. Then $u^{(n)} \in \hat{\mathscr{D}}(\mathscr{L}^{(n)})$ and, for each

$j \in S$, $u^{(n)}_{,j} \in \hat{\mathscr{D}}(\mathscr{L}^{(n)})$. Moreover,

$$\frac{\partial u^{(n)}_{,j}}{\partial t}(t, \eta) = \mathscr{L}^{(n)} u^{(n)}_{,j}(t, \eta) + \sum_k c^{(n)}_{k,j}(\eta) u^{(n)}_{,k}(t, {}_j\eta),$$

where ${}_j\eta$ is as in the definition of Δ_j (see (0.1)). Hence, if $T > 0$, then

$$u^{(n)}_{,j}(T - (t \wedge T), \eta(t \wedge T)) + \int_0^{t \wedge T} \sum_k c^{(n)}_{k,j}(\eta(s)) u^{(n)}_{,k}(T - s, {}_j\eta(s))\, ds$$

is a $P_\eta^{(n)}$ martingale, where $P_\eta^{(n)}$ solves the martingale problem for $\mathscr{L}^{(n)}$ starting from η. Therefore,

$$\|u^{(n)}_{,j}(t, \cdot)\| \leqq \|f_{,j}\| + \int_0^t \sum_{|k| \leqq n} \|c_{k,j}\| \|u^{(n)}_{,k}(s, \cdot)\|\, ds. \tag{4.7}$$

Given $N \geqq 1$, we now have:

$$\sum_{|j| \leqq N} \sup_{n \leqq N} \|u^{(n)}_{,j}(t, \cdot)\| \leqq |||f||| + C \int_0^t \sum_{|j| \leqq N} \sup_{n \leqq N} \|u^{(n)}_{,j}(s, \cdot)\|\, ds$$

and so, by Grönwall's inequality,

$$\sum_{|j| \leqq N} \sup_{n \leqq N} \|u^{(n)}_{,j}(t, \cdot)\| \leqq |||f||| e^{Ct}.$$

From this it is immediate that

$$\sum_j \sup_n \|u^{(n)}_{,j}(t, \cdot)\| \leqq |||f||| e^{Ct}. \tag{4.8}$$

We now want to estimate $\sum_{|j| \geqq L} \sup_{0 \leqq t \leqq T} \sup_n \|u^{(n)}_{,j}(t, \cdot)\|$. From (4.7) we have

$$\sum_{|j| \geqq L} \sup_{0 \leqq t \leqq T} \sup_n \|u^{(n)}_{,j}(t, \cdot)\| \leqq \sum_{|j| \geqq L} \|f_{,j}\| + \int_0^T \sum_k \sum_{|j| \geqq L} \|c_{k,j}\| \sup_n \|u^{(n)}_{,k}(t, \cdot)\|\, dt.$$

Hence we will know that

$$\lim_{L \to \infty} \sum_{|j| \geqq L} \sup_{0 \leqq t \leqq T} \sup_n \|u^{(n)}_{,j}(t, \cdot)\| = 0 \tag{4.9}$$

if we show that

$$\lim_{L \to \infty} \int_0^T \sum_k \sum_{|j| \geqq L} \|c_{k,j}\| \sup_n \|u^{(n)}_{,k}(t, \cdot)\|\, dt = 0. \tag{4.10}$$

But, by (4.8),

$$\sum_{|j| \geqq L} \|c_{k,j}\| \sup_n \|u^{(n)}_{,k}(t, \cdot)\| \leqq |||f||| e^{Ct} \sum_{|j| \geqq L} \|c_{k,j}\| \to 0$$

as L goes to infinity, and

$$\sum_k \sum_{|j| \geqq L} \|c_{k,j}\| \sup_n \|u^{(n)}_{,k}(t, \cdot)\| \leqq C \sum_k \sup_n \|u^{(n)}_{,k}(t, \cdot)\| \leqq C |||f||| e^{Ct}.$$

Hence (4.10) follows from the Lebesgue dominated convergence theorem.

Finally,

$$\left\| \frac{\partial u^{(n)}}{\partial t}(t, \cdot) \right\| = \|\mathscr{L}^{(n)} u^{(n)}(t, \cdot)\| \leqq C |||u^{(n)}(t, \cdot)||| \leqq C |||f||| e^{Ct};$$

and so, putting this together with (4.9) and (4.4), we conclude that $\{u^{(n)}\}$ is equicontinuous at each point $(t, \eta) \in [0, \infty) \times E$. Moreover, $\|u^{(n)}(t, \cdot)\| \leqq \|f\|$. Hence there is a subsequence $\{u^{(n')}\}$ and a $u \in C([0, \infty) \times E)$ such that $u^{(n')}$

converges to u uniformly on compact subsets of $[0, \infty) \times E$. From (4.8) it follows that

$$\sup_{0 \leq t \leq T} |||u(t, \cdot)||| \leqq |||f|||e^{Ct} .$$

Also from (4.9) and

$$u^{(n')}(t, \eta) - f(\eta) = \int_0^t \mathscr{L}^{(n')}u^{(n')}(s, \mathscr{L})\, ds ,$$

it is easy to show that $\mathscr{L}^{(n')}u^{(n')}$ converges to $\mathscr{L}u$ uniformly on compacts and

$$u(t, \eta) - f(\eta) = \int_0^t \mathscr{L}u(s, \eta)\, ds .$$

Hence $u \in \mathscr{C}^1([0, \infty) \times E)$, $\partial u/\partial t = \mathscr{L}u$, and $u(0, \cdot) = f$.

We summarize these results in the following lemma.

(4.11) LEMMA. *If the c_k's satisfy* (4.6), *then for each $f \in \mathscr{C}^1(E)$ there is a $u \in \mathscr{C}^1([0, \infty) \times E)$ such that $|||u(t, \cdot)||| \leqq |||f|||e^{Ct}$, $\partial u/\partial t = \mathscr{L}u$, and $u(0, \cdot) = f$.*

(4.12) THEOREM. *If the c_k's satisfy* (4.6), *then for each η there is exactly one solution P_η to the martingale problem for $\mathscr{L}$ starting from η. In particular, $\{P_\eta : \eta \in E\}$ is a homogeneous, strong Markov, Feller continuous family.*

REMARK. Lemma (4.11) is the analogue in the present set-up of the theorem in partial differential equations which says that a parabolic second order equation having smooth coefficients and smooth initial data has a smooth solution. In our case smooth means $\mathscr{C}^1(E)$. In Section 5, we give a second proof of Theorem (4.12) (cf. Remark (5.12)) based on a multiple random time change.

The condition (4.6) coincides with the condition given by Liggett [14] for the existence of a Feller semigroup associated with $\mathscr{L}$. Liggett's techniques are not restricted to the spin flip model considered here. We believe that our method can be modified to include many of his more general results.

5. Random time changes. The contents of this section are of a rather technical nature and are tangential to the rest of the paper. The idea here is to investigate whether a solution to the martingale problem for $\mathscr{L} = \sum_k c_k \Delta_k$ can be represented as a multiple random time change of the solution for $\mathscr{L}^0 = \sum_k \Delta_k$. What we will show is that this is always possible, although it will not be true in general that the $\mathscr{L}$-process is a measurable functional of the $\mathscr{L}^0$-process (cf. Corollary (5.8)). The only case in which we have been able to show that the representation is $\mathscr{L}^0$ measurable is when the c_k's satisfy (4.6) (cf. Theorem (5.11)).

Throughout this section, F will denote a nonempty subset of S and $|F|$ the cardinality of F. When we write $\theta \in R^F$, we actually mean that $\theta \in R^S$ and $\theta_k = 0$ for $k \notin F$.

(5.1) LEMMA. *Let F be a finite subset of S and $k \notin F$. Let $c\colon [0, \infty) \times \Omega \to [0, \infty)^{F \cup \{k\}}$ be a bounded nonanticipating function, and assume that c_k is uniformly positive. Define $\tau_k(t)$ by*

(5.2) $$\int_0^{\tau_k(t)} c_k(u)\, du = t , \quad t \geqq 0 ,$$

and set $\mathscr{B}^k = \mathscr{B}[\eta_k(\tau_k(t)) : t \geqq 0]$. *Let P on* $\langle \Omega, \mathscr{M}^0 \rangle$ *have the property that*

(5.3) $$X_\theta(t) \equiv \exp[\textstyle\sum_j (\theta_j(\eta_j(t) - \eta_j(0)) - \int_0^t c_j(u)(e^{-2\theta_j\eta_j(u)} - 1)\,du]$$

is a P martingale for all $\theta \in R^{F \cup \{k\}}$. *If* $\omega \to P^{(k,\omega)}$ *is a* r.c.p.d. *of* $P \mid \mathscr{B}^k$, *then there is a P-null set* $N \in \mathscr{B}^k$ *such that for* $\omega \notin N$: $\langle X_\theta(\tau_k(t)), \mathscr{M}_{\tau_k(t)}, P^{(k,\omega)} \rangle$ *is a martingale for all* $\theta \in R^F$.

PROOF. First note that $\langle X_\theta(\tau_k(t)), \mathscr{M}_{\tau_k(t)}, P \rangle$ is a martingale for all $\theta \in R^{F \cup \{k\}}$. Thus, if $t_1 \geqq 0$ and $\omega \to P^{(t_1,\omega)}$ is a r.c.p.d. of $P \mid \mathscr{M}_{\tau_k(t_1)}$, then there is a P-null set $A \in \mathscr{M}_{\tau_k(t_1)}$ such that $\langle X_\theta(\tau_k(t \vee t_1)), \mathscr{M}_{\tau_k(t \vee t_1)}, P^{(t_1,\omega)} \rangle$ is a martingale for all $\omega \notin A$ and $\theta \in R^{F \cup \{k\}}$. Hence by Lemma (3.1) in [18], for $\omega \notin A$, $\theta \in R^F$, $\theta_k \in R^{\{k\}}$, $t_2 > t_1$, $t_1 \leqq u < v$, and $C \in \mathscr{M}_{\tau_k(u)}$:

$$E^{P^{(t_1,\omega)}}[X_\theta(\tau_k(t_2))X_{\theta_k}(\tau_k(v))I_C] = E^{P^{(t_1,\omega)}}[X_\theta(\tau_k(t_2))X_{\theta_k}(\tau_k(u))I_C] .$$

Here I_C is the indicator function of C.

Now let $0 \leqq t_1 < t_2$, $\theta \in R^F$, and $\theta_k \in R^{\{k\}}$ be given. Define $\omega \to Q^{(\omega)}$ on $\mathscr{B}^k$ by

$$\frac{dQ^{(\omega)}}{dP^{(k,\omega)}} = E^{P^{(k,\omega)}}[X_\theta^{\tau_k(t_1)}(\tau_k(t_2)) \mid \mathscr{B}^k] ,$$

where $X_\theta^{\tau_k(t_1)}(\tau_k(t_2)) = (X_\theta(\tau_k(t_1)))^{-1}X_\theta(\tau_k(t_2))$. Given $t_1 \leqq u < v$ and $B \in \mathscr{B}_u^k \equiv \mathscr{B}[\eta_k(\tau_k(s)) : 0 \leqq s \leqq u]$, we then have

$$E^{Q^{(\omega)}}[X_{\theta_k}(\tau_k(v))I_B] = E^{Q^{(\omega)}}[X_{\theta_k}(\tau_k(u))I_B]$$

for $\omega \notin A$. In other words, for $\omega \notin A$: $\langle X_{\theta_k}(\tau_k(t \vee t_1)), \mathscr{B}^k_{t \vee t_1}, Q^{(\omega)} \rangle$ is a martingale for all $\theta_k \in R^{\{k\}}$. But

$$X_{\theta_k}(\tau_k(t \vee t_1)) = \exp[\theta_k(\eta_k(\tau_k(t \vee t_1)) - \eta_k(0)) - \textstyle\int_0^{t \vee t_1}(e^{-2\theta_k\eta_k(u)} - 1)\,du] ,$$

and therefore if $\omega \notin A$, the distribution of $\eta_k(\tau_k(\cdot \vee t_1)) - \eta_k(\tau_k(t_1))$ under $P^{(t_1,\omega)}$ and $Q^{(\omega)}$ is the same. Moreover, it is clear that $P^{(t_1,\omega)} = Q^{(\omega)}$ on $\mathscr{B}^k_{t_1}$. Hence $P^{(t_1,\omega)} = Q^{(\omega)}$ on $\mathscr{B}^k$ for $\omega \notin A$.

Finally, let $\theta \in R^F$, $B \in \mathscr{B}^k$ and $C \in \mathscr{M}_{\tau_k(t_1)}$ be given. Then by the preceeding:

$$E^{P^{(t_1,\omega)}}[I_B X_\theta(\tau_k(t_2))] = E^{P^{(t_1,\omega)}}[I_B X_\theta(\tau_k(t_1))]$$

(a.s. P), and so

$$\begin{aligned} E^P[I_B E^{P^{(k,\omega)}}[I_C X_\theta(\tau_k(t_2))]] &= E^P[I_C E^{P^{(t_1,\omega)}}[I_B X_\theta(\tau_k(t_2))]] \\ &= E^P[I_C E^{P^{(t_1,\omega)}}[I_B X_\theta(\tau_k(t_1))]] \\ &= E^P[I_B E^{P^{(k,\omega)}}[I_C X_\theta(\tau_k(t_1))]] . \end{aligned}$$

Hence for each $0 \leqq t_1 \leqq t_2$, $C \in \mathscr{M}_{\tau_k(t_1)}$, and $\theta \in R^F$ there is a P-null set $N(t_1, t_2, C, \theta) \in \mathscr{B}^k$ such that

(5.4) $$E^{P^{(k,\omega)}}[I_C X_\theta(\tau_k(t_2))] = E^{P^{(k,\omega)}}[I_C X_\theta(\tau_k(t_1))]$$

for $\omega \notin N(t_1, t_2, C, \theta)$. Using the facts that $X_\theta(t)$ is right continuous and $\mathscr{M}_{\tau_k(t_1)}$ is countably generated, it is now easy to find a P-null set $N \in \mathscr{B}^k$ so that (5.4) holds for all $\omega \notin N$, $0 \leqq t_1 < t_2$, $C \in \mathscr{M}_{\tau_k(t_1)}$, and $\theta \in R^F$.

(5.5) THEOREM. *Let $c: E \to [0, \infty)^S$ be a measurable function such that each c_k is uniformly positive and bounded. Suppose P and P^0 are solutions to the martingale problem for*

$$\mathscr{L} = \sum c_k \Delta_k \qquad \text{and} \qquad \mathscr{L}^0 = \sum \Delta_k ,$$

respectively, starting from η. Define $\xi(t) = \eta(\tau(t)) \equiv \{\eta_k(\tau_k(t))\}_{k \in S}$, where $\tau_k(t)$ is given by (5.2). *Then the distribution of $\xi(\cdot)$ under P is equal to P^0.*

PROOF. Let F be a finite subset of S. Let $X_\theta(t)$ be as in (5.3). In view of Theorem (1.1), our theorem will be proved if we can show that whenever P on $\langle \Omega, \mathscr{M}^0 \rangle$ has the property that $P(\eta(0) = \eta) = 1$ and $X_\theta(t)$ is a P-martingale for all $\theta \in R^F$, then $\langle X_\theta(\tau^F(t)), \mathscr{B}_t^F, P \rangle$ is a martingale for all $\theta \in R^F$, where

$$X_\theta(\tau^F(t)) = \exp[\textstyle\sum_{j \in F} (\theta_j(\eta_j(\tau_j(t)) - \eta_j(0)) - \int_0^t (e^{-2\theta_j \eta_j(\tau_j(u))} - 1)\, du)]$$

and $\mathscr{B}_t^F = \mathscr{B}[\eta_k(\tau_k(u)) : 0 \leqq u \leqq t \text{ and } k \in F]$. We do this by induction on $|F|$.

This assertion is trivial if $|F| = 1$. Assume it is true when $|F| = n$ and let $k \notin F$. Let $\omega \to P^{(k,\omega)}$ be a r.c.p.d. of $P \,|\, \mathscr{B}^k$, where $\mathscr{B}^k$ is as in Lemma (5.1). Then, by Lemma (5.1), there is a P-null set $N \in \mathscr{B}^k$ such that for all $\omega \notin N$ and all $\theta \in R^F$, $\langle X_\theta(\tau_k(t)), \mathscr{M}_{\tau_k(t)}, P^{(k,\omega)} \rangle$ is a martingale. Let $Q^{(\omega)}$ denote the distribution of $\eta(\tau_k(\cdot))$ under $P^{(k,\omega)}$. Then for $\omega \notin N$, $Q^{(\omega)}$ satisfies the inductive hypothesis with c_j replaced by c_j/c_k, $j \in F$. Thus if

$$\int_0^{\sigma_j(t)} \frac{c_j}{c_k} (\eta(\tau_k(u)))\, du = t , \qquad t \geqq 0 ,$$

then

$$\langle X_\theta(\tau_k \circ \sigma^F(t)), \mathscr{B}[\eta_j(\tau_k \circ \sigma^F(u)) : 0 \leqq u \leqq t, j \in F], P^{(k,\omega)} \rangle$$

is a martingale for all $\omega \notin N$ and $\theta \in R^F$. Since $\tau_j(t) = \tau_k \circ \sigma_j(t)$, $j \in F$, this shows that the distribution of $\{\eta_j(\tau_j(\cdot))\}_{j \in F}$ under $P^{(k,\omega)}$, $\omega \notin N$, is equal to the distribution of $\{\eta_j(\cdot)\}_{j \in F}$ under P^0. In particular we see that $\mathscr{B}^k$ and $\mathscr{B}^F = \mathscr{B}[\eta_j(\tau_j(\cdot)) : j \in F]$ are independent under P, and this completes the induction.

(5.6) COROLLARY. *Let c, P and P^0 be as in Theorem* (5.4). *There exist $\mathscr{B}[0, \infty) \times \mathscr{M}^0$ measurable functions*

$$\xi : [0, \infty) \times \Omega \to E$$

and

$$\sigma : [0, \infty) \times \Omega \to [0, \infty)^S$$

such that

(i) *$\xi(\cdot)$ is right-continuous, has left limits and its distribution under P is equal to P^0,*

(ii) *σ satisfies $\sigma_k(t) = \int_0^t c_k(\xi(\sigma(u)))\, du$, $k \in S$, where $\xi(\sigma(\cdot)) = \{\xi_k(\sigma_k(\cdot))\}_{k \in S}$,*

(iii) *$\eta(\cdot) = \xi(\sigma(\cdot))$.*

PROOF. Define τ_k and $\xi(\cdot)$ as in Theorem (5.4), and take $\sigma_k(\cdot) = \tau_k^{-1}(\cdot)$.

(5.7) REMARK. It should be emphasized that $\sigma(t)$ will not, in general, be $\xi(\cdot)$-measurable. Indeed, if it were, then Corollary (5.6) would provide a uniqueness proof for P, and we know P is not always unique!

(5.8) REMARK. The assumption that the c_k are uniformly positive is not strictly essential. However, if it is dropped,. the $\xi(\cdot)$ and $\sigma(\cdot)$ will live on a larger sample space and in general will not be $\eta(\cdot)$-measurable.

(5.9) LEMMA. *Let* c, $\tau(\cdot)$, *and* P *be as in Theorem* (5.5). *Given* $k \in S$ *let* $\tilde{\mathscr{B}}^k = \mathscr{B}[\eta_j(\tau_j(t)) : t \geqq 0, j \neq k]$ *and* $\omega \to \tilde{P}^{(k,\omega)}$ *be a* r.c.p.d. *of* $P \mid \tilde{\mathscr{B}}^k$. *Then there is a P-null set* $N \in \tilde{\mathscr{B}}^k$ *such that* $\langle X_{\theta_k}(\tau_k(t)), \mathscr{M}_{\tau_k(t)}, \tilde{P}^{(k,\omega)}\rangle$ *is a martingale for all* $\omega \notin N$ *and* $\theta_k \in R^{\{k\}}$, *where* X_{θ_k} *is as in* (5.3).

PROOF. Given $t_1 \geqq 0$, let $\omega \to P^{(t_1,\omega)}$ be a r.c.p.d. of $P \mid \mathscr{M}_{\tau_k(t_1)}$. Then there is a P-null set A such that for $\omega \notin A$ the $P^{(t_1,\omega)}$ distribution of $\eta(\cdot)$ on $[t_1, \infty)$ solves the martingale problem for $\mathscr{L}$ starting from $\eta(\tau_k(t_1, \omega), \omega)$. Hence if $\tau^{t_1}(\cdot)$ is defined by

$$\int_{\tau_k(t_1)}^{\tau_j^{t_1}(t)} c_j(\eta(u))\, du = (t - t_1)\,, \qquad t \geqq t_1\,,$$

then by Theorem (5.5), for $\omega \notin A$ the $P^{(t_1,\omega)}$ distribution of $\eta(\tau^{t_1}(\cdot))$ on $[t_1, \infty)$ solves the martingale problem for $\mathscr{L}^0$ starting from $\eta(\tau_k(t_1, \omega), \omega)$.

For $t_1 < t_2$ and $\theta_k \in R^{\{k\}}$, define

$$\frac{dQ^{(\omega)}}{dP^{(t_1,\omega)}} = E^{P^{(t_1,\omega)}}[X_{\theta_k}^{\tau_k(t_1)}(\tau_k(t_2)) \mid \mathscr{B}[\eta_j(\tau_j^{t_1}(t)) : t \geqq t_1 \text{ and } j \neq k]]\,.$$

Then, by the argument used in Lemma (5.1), $Q^{(\omega)} = P^{(t_1,\omega)}$ on $\mathscr{B}[\eta_j(\tau_j^{t_1}(t)) : t \geqq t_1 \text{ and } j \neq k]$ for $\omega \notin A$. Noting that

$$\tau_j(t) = \tau_j^{t_1}(t - \gamma_j + t_1)\,, \quad t \geqq \gamma_j \equiv \int_0^{\tau_j(t_1)} c_j(\eta(u))\, du\,,$$

and $\tau_j(t) = \tau_j(t, \omega)$ for $t < \gamma_j = \gamma_j(\omega)$ (a.s. $P^{(t_1,\omega)}$), we conclude that

$$E^{P^{(t_1,\omega)}}[X_{\theta_k}(\tau_k(t_2)) \mid \tilde{\mathscr{B}}^k] = X_{\theta_k}(\tau_k(t_1))$$

(a.s. $P^{(t_1,\omega)}$) for $\omega \notin A$.

The rest of the proof is accomplished in exactly the same way as the corresponding part of the proof of Lemma (5.1).

DEFINITION. Let $\xi : [0, \infty) \times \Omega \to E$ be a nonanticipating function and P a probability measure on $\langle \Omega, \mathscr{M}^0\rangle$. For $k \in S$, let $\omega \to \tilde{P}^{(k,\omega)}$ be a r.c.p.d. of $P \mid \mathscr{B}[\xi_j(t) : t \geqq 0 \text{ and } j \neq k]$. We say that $\varphi \cdot \Omega \to [0, \infty)^S$ is a *stopping vector for* $\langle \xi(\cdot), P\rangle$ if

(i) φ is $\xi(\cdot)$-measurable,

(ii) For all $k \in S$ and P-almost all ω, there is a function $\psi_{k,\omega} : \Omega \to [0, \infty)$ such that $\{\psi_{k,\omega} \leqq t\} \in \mathscr{B}[\xi_k(u) : 0 \leqq u \leqq t]$, $t \geqq 0$, and $\tilde{P}^{(k,\omega)}(\varphi_k \neq \psi_{k,\omega}) = 0$.

(5.10) LEMMA. *Let* $\xi : [0, \infty) \times \Omega \to E$ *be a right continuous, nonanticipating function having left limits. Let* $c : E \to (0, \infty)^S$ *be a continuous function satisfying* (4.6). *Suppose P is a probability measure on* $\langle \Omega, \mathscr{M}^0\rangle$ *under which the distribution of* $\xi(\cdot)$ *is the solution to the martingale problem for* $\mathscr{L}^0 = \sum \Delta_k$ *starting from* η. *Then there is a function* $\varphi : [0, \infty) \times \Omega \to [0, \infty)^S$ *such that:*

(a) $\varphi(t) = \int_0^t c(\xi(\varphi(u)))\,du$, $t \geqq 0$ (a.s. P),
(b) $\varphi(t)$ *is a stopping vector for* $\langle \xi(\cdot), P\rangle$ *for each* $t \geqq 0$.

PROOF. For $N \geqq 0$ and measurable function $\alpha: [0, \infty) \to E$, define $F^N(t, \alpha)$ by

$$F^N(t, \alpha) = \int_0^t c(\alpha(F^N([2^N u]/2^N, \alpha)))\,du\,, \qquad t \geqq 0\,.$$

Put $\varphi^N(\cdot, \omega) = F^N(\cdot, \xi(\cdot, \omega))$ and $\psi^N_{k,\omega}(\cdot, \omega') = F_k{}^N(\cdot, \alpha)$, where

$$\begin{aligned} \alpha_j(\cdot) &= \xi_j(\cdot, \omega) \qquad \text{for } j \neq k \\ &= \xi_k(\cdot, \omega') \qquad \text{for } j = k\,. \end{aligned}$$

Clearly $\varphi^N(t, \cdot)$ is $\xi(\cdot)$-measurable, $\{\psi_{k,\omega}(s) \leqq t\} \in \mathscr{B}[\xi_k(u): 0 \leqq u \leqq t]$ for all $s, t \geqq 0$, and $\tilde{P}^{(k,\omega)}(\varphi_k{}^N(\cdot) \neq \psi^N_{k,\omega}(\cdot)) = 0$.

Note that the components of $\xi(\cdot)$ are mutually independent under P. Thus

$$\langle \xi_k(t) + 2\int_0^t \xi_k(u)\,du, \mathscr{B}[\xi_k(u): 0 \leqq u \leqq t], \tilde{P}^{(k,\omega)}\rangle$$

is a martingale for P-almost all ω. Hence if σ and τ are bounded stopping times relative to $\{\mathscr{B}[\xi_k(u): 0 \leqq u \leqq t]: t \geqq 0\}$, then P-almost surely

$$\begin{aligned} E^{\tilde{P}^{(k,\omega)}}[\xi_k(\tau)\xi_k(\sigma)] &= E^{\tilde{P}^{(k,\omega)}}[\xi_k(\sigma \vee \tau)\xi_k(\sigma \wedge \tau)] \\ &= 1 + E^{\tilde{P}^{(k,\omega)}}[\xi_k(\sigma \wedge \tau)(\xi_k(\sigma \vee \tau) - \xi(\sigma \wedge \tau))] \\ &= 1 - 2E^{\tilde{P}^{(k,\omega)}}[\xi_k(\sigma \wedge \tau)\int_{\sigma\wedge\tau}^{\sigma\vee\tau} \xi_k(u)\,du]\,, \end{aligned}$$

and so

$$\begin{aligned} E^{\tilde{P}^{(k,\omega)}}[(\xi_k(\tau) - \xi_k(\sigma))^2] &= 4E^{\tilde{P}^{(k,\omega)}}[\xi_k(\sigma \wedge \tau)\int_{\sigma\wedge\tau}^{\sigma\vee\tau} \xi_k(u)\,du] \\ &\leqq 4E^{\tilde{P}^{(k,\omega)}}[|\tau - \sigma|]\,. \end{aligned}$$

Hence

$$\begin{aligned} E^P[|\xi_k(\varphi_k{}^N(t)) - \xi_k(\varphi_k{}^M(s))|] &= \tfrac{1}{2}E^P[(\xi_k(\varphi_k{}^N(t)) - \xi_k(\varphi_k{}^M(s)))^2] \\ &= \tfrac{1}{2}E^P[E^{\tilde{P}^{(k,\omega)}}[(\xi_k(\psi^N_{k,\omega}(t)) - \xi_k(\psi^M_{k,\omega}(s)))^2]] \\ &\leqq 2E^P[E^{\tilde{P}^{(k,\omega)}}[|\psi^N_{k,\omega}(t) - \psi^M_{k,\omega}(s)|]] \\ &= 2E^P[|\varphi_k{}^N(t) - \varphi_k{}^M(s)|]\,. \end{aligned}$$

We now use the results in the preceding paragraph to prove that $\varphi^N(\cdot)$ converges as N goes to infinity. In fact, if $M < N$, then

$$\begin{aligned} &E^P[\sup_{0\leqq s\leqq t} |\varphi_k{}^N(s) - \varphi_k{}^M(s)|] \\ &\qquad \leqq E^P[\int_0^t |c_k(\xi(\varphi^N([2^N u]/2^N) - c_k(\xi(\varphi^M([2^M u]/2^M)))|\,du] \\ &\qquad \leqq C\int_0^t \sup_j E^P[|\xi_j(\varphi_j{}^N([2^N u]/2^N)) - \xi_j(\varphi_j{}^M([2^M u]/2^M))|\,du] \\ &\qquad \leqq 2C\int_0^t \sup_j E^P[|\varphi_j{}^N([2^N u]/2^N) - \varphi_j{}^M([2^M u]/2^M)|]\,du \\ &\qquad \leqq \frac{C^2}{2^{M-1}} + 2C\int_0^t \sup_j E^P[\sup_{0\leqq u\leqq s} |\varphi_j{}^N(u) - \varphi_j{}^M(u)|\,ds\,, \end{aligned}$$

and so

$$\sup_k E^P[\sup_{0\leqq s\leqq t} |\varphi_k{}^N(s) - \varphi_k{}^M(s)|] \leqq \frac{C^2}{2^{M-1}}\,e^{2Ct}\,.$$

Let $\varphi_k(t, \cdot) = \liminf_{N\to\infty} \varphi_k^N(t, \cdot)$. Then we see that, except on a P-null set, $\varphi_k^N(\cdot)$ converges uniformly on finite time intervals to $\varphi_k(\cdot)$. Hence $\varphi(\cdot)$ fulfills (a). Also, $\varphi(t)$ is clearly $\xi(\cdot)$-measurable, and it is not hard to check that $\varphi(t)$ is a stopping vector for $\langle \xi(\cdot), P\rangle$. Indeed, simply take $\psi_{k,\omega}(\cdot) = \liminf_{N\to\infty} \psi_{k,\omega}^N(\cdot)$.

(5.11) THEOREM. *Let $c : E \to (0, \infty)^S$ be a continuous function satisfying* (4.6) *and let $\mathscr{L} = \sum c_k \Delta_k$. Suppose P is any solution to the martingale problem for $\mathscr{L}$ starting from η, and let P^0 be the solution for $\mathscr{L}^0 = \sum \Delta_k$ starting from η. Define τ_k by*

$$\int_0^{\tau_k(t)} c_k(\eta(u))\, du = t\,, \qquad t \geqq 0\,,$$

and set $\xi(t) = \eta(\tau(t)) \equiv \{\eta_k(\tau_k(t))\}_{k\in S}$. Then the distribution of $\xi(\cdot)$ under P is equal to P^0. Moreover, there is a function $\varphi : [0, \infty) \times \Omega \to [0, \infty)^S$ such that (a) *and* (b) *of Lemma* (5.10) *hold. Finally, if $\varphi(\cdot)$ is any function satisfying* (a) *and* (b), *then $\varphi_k(\cdot) = \tau_k^{-1}(\cdot)$* (a.s. P) *for all k; and in particular, $\eta(\cdot) = \xi(\varphi(\cdot))$* (a.s. P).

PROOF. The only assertion requiring comment is the last. To prove this we use Lemma (5.9). By that lemma, we know that $\langle \xi_k(t) + 2\int_0^t \xi_k(u)\, du, \mathscr{M}_{\tau_k(t)}, \tilde{P}^{(k,\omega)}\rangle$ is a martingale for P-almost all ω. Thus, as in the proof of Lemma (5.10), P-almost surely:

$$E^{\tilde{P}^{(k,\omega)}}[(\xi_k(\psi_{k,\omega}(t)) - \xi_k(\tau_k^{-1}(t)))^2] \leqq 2E^{\tilde{P}^{(k,\omega)}}[|\psi_{k,\omega}(t) - \tau_k^{-1}(t)|]\,,$$

since $\{\tau_k^{-1}(t) \leqq u\} \in \mathscr{M}_{\tau_k(u)}$, $u \geqq 0$, and $\{\psi_{k,\omega}(t) \leqq u\} \in \mathscr{B}[\xi_k(s) : 0 \leqq s \leqq u] \subseteq \mathscr{M}_{\tau_k(u)}$, $u \geqq 0$. One can now proceed as in the proof of Lemma (5.10) to show that

$$\sup_k E^P[\sup_{0\leqq s\leqq t} |\varphi_k(s) - \tau_k^{-1}(s)|] \leqq 2C \int_0^t \sup_k E^P[\sup_{0\leqq u\leqq s} |\varphi_k(u) - \tau_k^{-1}(u)|]\, ds\,,$$

since $\tau^{-1}(t) = \int_0^t c_k(\xi(\tau^{-1}(s)))\, ds$. But this shows that $\varphi(\cdot) = \tau^{-1}(\cdot)$ (a.s. P).

(5.12) REMARK. The situation described in Theorems (5.5) and (5.11) should be compared with the analogous results in diffusion theory. If $x(\cdot)$ is a diffusion associated with a strictly elliptic operator L, then $x(\cdot)$ can be represented as the solution of stochastic integral involving an $x(\cdot)$ measurable Brownian motion $\beta(\cdot)$. However, unless the coefficients of L satisfy smoothness conditions, one cannot show that $x(\cdot)$ is $\beta(\cdot)$-measurable.

(5.13) REMARK. Assume that the c_k's satisfy (4.6) but do not assume they are positive. Given $\eta \in E$ and the solution P^0 to the martingale problem for $\mathscr{L}^0$ starting from η, there is, up to a P^0-null set, exactly one stopping vector $\varphi(t)$ for $\langle \eta(\cdot), P^0\rangle$ such that

$$\varphi(t) = \int_0^t c(\eta(\varphi(u)))\, du\,, \qquad t \geqq 0\,, \quad (\text{a.s.}, P^0)\,.$$

Moreover, if $\xi(\cdot) = \eta(\varphi(\cdot))$ and P is the distribution of $\xi(\cdot)$ under P^0, then P is the unique solution to the martingale problem for $\mathscr{L} = \sum_k c_k \Delta_k$ starting from η. In fact, one does not need to know a priori that the martingale problem for $\mathscr{L}$ is well posed in order to make this last statement. Hence, this technique gives an independent derivation of Theorem (4.12).

6. Uniqueness, Part II. In this section we give a condition on the flip rates, c_k, which neither implies nor is implied by (4.6) and which guarantees uniqueness of the solution of the martingale problem for c.

We begin with some definitions and notation. F, G and H always denote finite subsets of S. We may think of E as the group which is the direct product of the groups $\{-1, 1\}_k$, $k \in S$, and we denote the characters of this group by X_F;

$$X_F(\eta) = \prod_{j \in F} \eta_j .$$

Note that

$$X_F X_G = X_{F \triangle G} ,$$

where $F \triangle G$ is the symmetric difference of F and G.

Let μ be Haar measure on E (i.e., $\mu = \prod_{k \in S} (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1})$). Since $\mathscr{C}(E) \subset L_2(\mu)$, each $f \in \mathscr{C}(E)$ may be written as $f = \sum_F \hat{f}(F) X_F$, where $\hat{f}(F) = \int_E f(\eta) X_F \mu(d\eta)$. Of course we are only sure that the above Fourier expansion of f converges in $L_2(\mu)$, and we need uniform convergence. Therefore we let $\hat{L}$ be the Banach space of functions of the form $f = \sum_F \hat{f}(F) X_F$ with norm

$$|f|_{\hat{L}} = \sum_F |\hat{f}(F)| < \infty .$$

Note that if $f \in \hat{L}$, then $f \in \mathscr{C}(E)$ and

$$\|f\| \leqq |f|_{\hat{L}} . \tag{6.1}$$

The conditions which we want on the flip rates, c_k, are as follows. Each $c_k \in \hat{L}$, and we write

$$c_k = \bar{c}_k + \sum_{G \neq \phi} \gamma(k, G) X_G , \tag{6.2}$$

where $\bar{c}_k = \hat{c}_k(\phi)$. To simplify notation we set $\gamma(k, \phi) = 0$. Since the c_k's are nonnegative it follows that $\bar{c}_k \geqq 0$. The critical assumption which we make is that there is some $\alpha < 1$ such that for all $k \in S$

$$\sum_G |\gamma(k, G)| \leqq \alpha \bar{c}_k . \tag{6.3}$$

Inequality (6.3) clearly implies that

$$\|c_k\| \leqq (1 + \alpha) \bar{c}_k . \tag{6.4}$$

For each finite set $F \subset S$ let

$$c_F = \sum_{j \in F} \bar{c}_j , \quad (c_\phi = 0) ,$$

and define

$$\mathscr{D}_{\{c\}} = \{f \in \hat{L} : \sum_F c_F |\hat{f}(F)| < \infty\} .$$

(6.5) LEMMA. *Let H be any finite subset of S and let $f \in \hat{L}$. Then if* (6.3) *holds,*

$$\Big|\sum_{k \in H} \sum_G \gamma(k, G) X_G \Delta_k f\Big|_{\hat{L}} \leqq 2\alpha \sum_F c_{F \cap H} |\hat{f}(F)| . \tag{6.6}$$

PROOF. We first note that $\Delta_k X_F = -2 I_F(k) X_F$, where I_F is the indicator function of F. Thus

$$\Delta_k f = -2 \sum_F I_F(k) \hat{f}(F) X_F . \tag{6.7}$$

Hence

$$\sum_{k\in H}\sum_G \gamma(k, G)X_G\Delta_k f = -2\sum_{k\in H}\sum_G\sum_F \gamma(k, G)I_F(k)\hat{f}(F)X_{G\triangle F} = -2\sum_{k\in H}\sum_G\sum_F \gamma(k, G\triangle F)I_F(k)\hat{f}(F)X_G\,, \tag{6.8}$$

and therefore

$$\begin{aligned}|\textstyle\sum_{k\in H}\sum_G \gamma(k, G)X_G\Delta_k f|_{\hat{L}} &= 2\textstyle\sum_G|\sum_{k\in H}\sum_F \gamma(k, G\triangle F)I_F(k)\hat{f}(F)| \\ &\leqq 2\textstyle\sum_F\sum_{k\in H}\sum_G|\gamma(k, G\triangle F)||I_F(k)||\hat{f}(F)| \\ &\leqq 2\alpha\textstyle\sum_F\sum_{k\in H}\bar{c}_k I_F(k)|\hat{f}(F)| \\ &= 2\alpha\textstyle\sum_F c_{F\cap H}|\hat{f}(F)|\,.\end{aligned} \tag{6.9}$$

(6.10) LEMMA. *Let $f\in\mathscr{D}_{\{c\}}$ and let* (6.3) *hold. Let $\{H_n\}$ be an increasing sequence of finite sets whose union is S. Then*

$$\sum_{k\in H_n} c_k(\cdot)\Delta_k f(\cdot) \quad \textit{and} \quad \sum_{k\in H_n}\bar{c}_k\Delta_k f(\cdot)$$

are both Cauchy sequences in the $|\cdot|_{\hat{L}}$ norm.

PROOF. Let $m > n$ and set $H(m, n) = H_m\backslash H_n$. Then

$$\begin{aligned}|\textstyle\sum_{k\in H_n}\bar{c}_k\Delta_k f - \sum_{k\in H_m}\bar{c}_k\Delta_k f|_{\hat{L}} \\ &= |\textstyle\sum_{k\in H(m,n)}\bar{c}_k\Delta_k f|_{\hat{L}} \\ &= |-2\textstyle\sum_F\sum_{k\in H(m,n)}\bar{c}_k I_F(k)\hat{f}(F)X_F|_{\hat{L}} \\ &= 2\textstyle\sum_F c_{F\cap H(m,n)}|\hat{f}(F)|\,.\end{aligned} \tag{6.11}$$

Since $f\in\mathscr{D}_{\{c\}}$, $c_{F\cap H(m,n)}\leqq c_F$, and $c_{F\cap H(m,n)}\to 0$ as $m, n\to\infty$, it follows from (6.11) that $\sum_{k\in H_n}\bar{c}_k\Delta_k f$ is a Cauchy sequence. Now

$$\begin{aligned}&|\textstyle\sum_{k\in H_n} c_k\Delta_k f - \sum_{k\in H_m} c_k\Delta_k f|_{\hat{L}} \\ &\quad\leqq |\textstyle\sum_{k\in H(m,n)}\bar{c}_k\Delta_k f|_{\hat{L}} + |\sum_{k\in H(m,n)}\sum_G\gamma(k, G)X_G\Delta_k f|_{\hat{L}}\,.\end{aligned} \tag{6.12}$$

We have already seen that the first term on the right side of (6.12) goes to zero as $m, n\to\infty$. According to Lemma (6.5) the second term is bounded by $2\alpha\sum_F c_{F\cap H(m,n)}|\hat{f}(F)|$, which again goes to zero as $m, n\to\infty$.

(6.13) LEMMA. *Let $f\in\mathscr{D}_{\{c\}}$ and let* (6.3) *hold. Then $f\in\mathscr{D}(\mathscr{L})$. ($\mathscr{D}(\mathscr{L})$ is as in Section* 4.)

PROOF. We saw in the proof of Lemma (6.5) that

$$|c_k\Delta_k f|_{\hat{L}} \leqq 2(1+\alpha)\sum_F\bar{c}_k I_F(k)|\hat{f}(F)|\,.$$

Therefore, by (6.1)

$$\begin{aligned}\sup_\eta\textstyle\sum_{k\in H}|c_k(\eta)\Delta_k f(\eta)| &\leqq \textstyle\sum_{k\in H}||c_k\Delta_k f|| \\ &\leqq \textstyle\sum_{k\in H}|c_k\Delta_k f|_{\hat{L}} \\ &\leqq 2(1+\alpha)\textstyle\sum_{k\in H}\sum_F\bar{c}_k I_k(k)|\hat{f}(F)| \\ &= 2(1+\alpha)\textstyle\sum_F c_{F\cap H}|\hat{f}(F)|\,.\end{aligned}$$

The conclusion follows just as in Lemma (6.10).

Let H_n be an increasing sequence of finite sets whose union is S. We define operators $\hat{\mathscr{L}}$ and $\mathscr{L}^0$ on $\mathscr{D}_{\{c\}}$ as the limits in the $|\cdot|_{\hat{L}}$ norm of $\{\sum_{k\in H_n} c_k\Delta_k f\}$ and $\{\sum_{k\in H_n} \bar{c}_k\Delta_k f\}$. From Lemma (6.10) we see that these limits exist and are independent of the particular sequence of sets H_n used to define them. Since $\mathscr{D}_{\{c\}} \subset \mathscr{D}(\mathscr{L})$, it is easily seen, using (6.1), that we may view $\hat{\mathscr{L}}$ as $\mathscr{L}$ extended to $\mathscr{D}(\mathscr{L})$ as in Section 4 and then restricted to $\mathscr{D}_{\{c\}}$. Hence it $f\in\mathscr{D}_{\{c\}}$

(6.14) $$\hat{\mathscr{L}}f(\eta) = \mathscr{L}f(\eta) \qquad \text{for all} \quad \eta\in E\,.$$

Now the eigenvectors of $\mathscr{L}^0$ are just the characters X_F. In fact

(6.15) $$\mathscr{L}^0 X_F = -2c_F X_F\,.$$

Let $\sigma(c)$ be the closure of $\{-2c_F\colon F\subset S,\ F \text{ finite}\}$. Then if λ is any complex number which is not in $\sigma(c)$, it is easily checked that the operator $R_\lambda{}^0$ on $\hat{L}$ given by the formula

(6.16) $$R_\lambda{}^0 f = \textstyle\sum_F (\lambda + 2c_F)^{-1}\hat{f}(F)X_F$$

is the resolvent of $\mathscr{L}^0$. Moreover

(6.17) $$R_\lambda{}^0\hat{L} = \mathscr{D}_{\{c\}}\,.$$

Our immediate goal is to determine the resolvent of $\hat{\mathscr{L}}$ by thinking of $\hat{\mathscr{L}}$ as differing from $\mathscr{L}^0$ by a perturbation. The next lemma is an immediate consequence of (6.16), (6.17), the definitions of $\hat{\mathscr{L}}$ and $\mathscr{L}^0$ and a calculation identical to (6.8).

(6.18) LEMMA. *If $\lambda\notin\sigma(c)$ and (6.3) holds, then for all $f\in\hat{L}$*

(6.19) $$(\hat{\mathscr{L}} - \mathscr{L}^0)R_\lambda{}^0 f = \textstyle\sum_H\sum_F\sum_K \gamma(k, F\triangle H)\dfrac{-2I_F(k)}{\lambda+2c_F}\hat{f}(F)X_H\,.$$

We denote $(\hat{\mathscr{L}} - \mathscr{L}^0)R_\lambda{}^0$ by A_λ.

(6.20) LEMMA. *Let (6.3) hold and let $\lambda > 0$. Then for all $f\in\hat{L}$*

(6.21) $$|A_\lambda f|_{\hat{L}} \leqq \frac{\alpha+1}{2}|f|_{\hat{L}}\,.$$

For the proof of Lemma (6.20) see Lemma (A.1) in the appendix.

(6.22) COROLLARY. *If (6.3) holds and $f\in\hat{L}$, then for all $\lambda>0$, $B_\lambda f\equiv \sum_{k=0}^\infty (A_\lambda)^k f\in\hat{L}$ and*

(6.23) $$|B_\lambda f|_{\hat{L}} \leqq \frac{2}{1-\alpha}|f|_{\hat{L}}\,.$$

For $\lambda>0$ we define the operator R_λ by the formula

(6.24) $$R_\lambda = R_\lambda{}^0 B_\lambda\,.$$

(6.25) LEMMA. *Let $\lambda>0$ and $f\in\hat{L}$, then*

(6.26) $$(\lambda - \mathscr{L})R_\lambda f = f\,.$$

PROOF. From Corollary (6.22), (6.24) and (6.17) we see that $R_\lambda f \in \mathscr{D}_{\{c\}}$. Therefore, by (6.14)

$$(\lambda - \mathscr{L})R_\lambda f = (\lambda - \hat{\mathscr{L}})R_\lambda f = (\lambda - (\hat{\mathscr{L}} - \mathscr{L}^0) - \mathscr{L}^0)R_\lambda^0 B_\lambda f$$
$$= B_\lambda f - (\hat{\mathscr{L}} - \mathscr{L}^0)\, R_\lambda^0 B_\lambda f = (I - A_\lambda)B_\lambda f \; = f .$$

(6.27) THEOREM. *If the c_k's are given by* (6.2) *and* (6.3) *holds for some $\alpha < 1$ then for each $\eta \in E$ the solution to the martingale problem for $\mathscr{L}$ starting from η is unique.*

PROOF. Since $\mathscr{D} \subset \hat{L}$, and $\mathscr{D}$ is dense in $\mathscr{C}(E)$, the theorem follows from Theorem (4.2b) and Lemma (6.25).

7. An ergodic theorem. In this section we use the same definitions and notation as in Section 6.

We begin by determing some more of the properties of R_λ.

Let α be as in (6.3) and let

$$D_0 = \{\lambda \neq 0 : |\arg \lambda| \leqq \pi - \arcsin(2\alpha/(\alpha + 1))\} .$$

If $a > 0$ let

$$D_1 = \{\lambda : \text{The real part of } \lambda \text{ is greater than } 2a(\alpha - 1)\} .$$

Recall that A_λ is given by (6.19). If all $\bar{c}_k > 0$ then the right side of (6.19) makes sense and is an element of $\hat{L}$ even when $\lambda = 0$. In this case we define $A_0 f$ by the right side of (6.19) with $\lambda = 0$ and define B_0 as in Corollary (6.22).

The proof of the following lemma is given in the appendix.

(7.1) LEMMA. *For all $\lambda \in D_0$ we may define B_λ as in Corollary* (6.22) *and R_λ as in* (6.24). *R_λ so defined has the following properties*:

(a) *For all $f \in \hat{L}$ and all $\lambda \in D_0$*

$$|R_\lambda f(\eta)| \leqq |R_\lambda f|_{\hat{L}} \leqq \frac{\alpha + 1}{1 - \alpha}\,\frac{1}{\alpha}\,\frac{1}{|\lambda|}\,|f|_{\hat{L}} .$$

(b) *If all $\bar{c}_k > 0$, then for all $f \in \hat{L}$*

$$\lim_{\lambda \searrow 0} \lambda R_\lambda f = \widehat{B_0 f}(\phi) .$$

If we denote $\widehat{B_0 f}(\phi)$ by Πf, then for all $f \in \hat{L}$

$$|\Pi f|_{\hat{L}} \leqq \frac{2}{1 - \alpha}\,|f|_{\hat{L}} .$$

(c) *If $\inf_k \bar{c}_k = a > 0$ then the operator B_λ may be defined as in Corollary* (6.22) *on all of $D_0 \cup D_1$ and R_λ may be defined as in* (6.24) *on $(D_0 \cup D_1)\backslash\{0\}$. In this case $R_\lambda f(\eta)$ is an analytic function on $(D_0 \cup D_1)\backslash\{0\}$ for all $f \in \hat{L}$ and all $\eta \in E$, and the singularity at* 0 *of $R_\lambda(f - \Pi f)(\eta)$ is removable.*

(d) *If $\inf_k \bar{c}_k = a > 0$ then for every compact subset D of $(D_0 \cup D_1)\backslash\{0\}$ there is a constant $K(D)$ such that*

$$\sup_{\lambda \in D} |R_\lambda f|_{\hat{L}} \leqq K(D)|f|_{\hat{L}} .$$

(7.2) LEMMA. *Assume that the flip rates, c_k, satisfy* (6.2) *and* (6.3). *Let P_η be the unique solution to the martingale problem for $\mathscr{L}$ starting from η, and let $f \in \hat{L}$. Then for all $\lambda > 0$*

$$\int_0^\infty e^{-\lambda t} E^{P_\eta}[f(\eta(t))]\, dt = R_\lambda f(\eta) . \tag{7.3}$$

PROOF. This follows from Lemma (6.25) as in the proof of Theorem (4.2b).

(7.4) THEOREM. *Let* (6.2) *and* (6.3) *hold and assume that $c_k > 0$ for each k. Then there exists a unique stationary measure, ν, for the family $\{P_\eta : \eta \in E\}$* (i.e., for all Borel sets $E_0 \subset E$ and all $t \geqq 0$

$$\int_E P_\eta[\eta(t) \in E_0]\nu(d\eta) = \nu(E_0)) . \tag{7.5}$$

Here P_η is the unique solution to the martingale problem for $\mathscr{L}$ starting from η.

PROOF. From Theorem (4.1) we know that $\{P_\eta : \eta \in E\}$ is a homogeneous, strong Markov, Feller continuous family. Thus the integrand in (7.5) is measurable and since E is compact there is at least one stationary measure. In order to show there is at most one, it is enough to show that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T E^{P_\eta}[f(\eta(t))]\, dt = \Pi f \tag{7.6}$$

for all f in the dense subset $\hat{L} \subset \mathscr{C}(E)$. First note that (7.6) is immediate it f is constant. Therefore since both sides of (7.6) are linear in f, it suffices to prove (7.6) for $f \geqq 0$. In this case (7.6) is a consequence of a Tauberian theorem (see Theorem 2, page 445 of [3]), (7.3), and Lemma (7.1b).

In order to obtain a rate of convergence to equilibrium we need to invert the Laplace transform in (7.3). To this end we let Γ be the curve consisting of the segments

$$\{re^{i\theta} : r \geqq 1\} \cup \{re^{-i\theta} : r \geqq 1\} \cup \{e^{i\varphi} : -\theta \leqq \varphi \leqq \theta\} ,$$

where θ is any angle between $\pi/2$ and $\pi - \arcsin(2\alpha/(\alpha + 1))$. Integrals over the curve Γ will always be taken from bottom to top.

(7.7) LEMMA. *Let f be an analytic function on D_0 such that $|f(z)| \leqq$* constant$/|z|$. *Define*

$$g(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} f(z)\, dz .$$

Then for all $w > 1$

$$f(w) = \int_0^\infty e^{-wt} g(t)\, dt .$$

PROOF.

$$\int_0^\infty e^{-wt} g(t)\, dt = \frac{1}{2\pi i} \int_0^\infty {}^{-wt} \int_\Gamma e^{tz} f(z)\, dz\, dt$$

$$= \frac{1}{2\pi i} \int_\Gamma \int_0^\infty e^{t(z-w)} f(z)\, dt\, dz = \frac{-1}{2\pi i} \int_\Gamma \frac{f(z)}{z - w}\, dz .$$

We use the assumption that $w > 1$ to interchange the order of integration, since then one can easily establish absolute integrability. To evaluate the last integral let $S(R, w)$ be the circle with radius R and center w. For large enough R $S(R, w)$ intersects Γ in exactly two points. If Γ_R is the curve made up of that part of $S(R, w)$ to the right of Γ and the part of Γ between its intersection with $S(R, w)$ and we orient Γ_R in the clockwise direction, then we have

$$-\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-w}\,dz = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(z)}{z-w}\,dz = \lim_{R\to\infty} f(w) = f(w)\,.$$

(7.8) LEMMA. *Let* (6.2) *and* (6.3) *hold and* $\inf_k \bar{c}_k = a > 0$. *If* P_η *is the unique solution to the martingale problem for* $\mathscr{L}$ *starting from* η, *then for all* $f \in \hat{L}$

$$E^{P_\eta}[f(\eta(t))] = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_\lambda f(\eta)\,d\lambda\,. \tag{7.9}$$

(The formula for the semigroup associated with R_λ given by (7.9) is a familiar one. See for example Kato [12] pages 477–478.)

PROOF. From (7.3), Lemma (7.7), and Lemma (7.1(a), (c)) the Laplace transforms of both sides of (7.9) agree, at least for $\lambda > 1$.

(7.10) THEOREM. *Let* (6.2) *and* (6.3) *hold and* $\inf_k \bar{c}_k = a > 0$. *Then for all* $\gamma < 2a(1-\alpha)$ *there is a constant*, $\bar{K}(\gamma)$, *such that for* $\eta \in E$ *and all* $f \in \hat{L}$

$$|E^{P_\eta}[f(\eta(t))] - \Pi f| \leqq \bar{K}(\gamma)e^{-\gamma t}|f|_{\hat{L}}\,. \tag{7.11}$$

PROOF. Since $A_\lambda X_\phi = 0$, we have $B_\lambda X_\phi = X_\phi$ and hence $R_\lambda X_\phi = (1/\lambda)X_\phi$. Thus for all $t > 0$

$$\Pi f = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_\lambda \Pi f\,d\lambda\,. \tag{7.12}$$

Combining (7.9) and (7.12) we see that

$$E^{P_\eta}[f(\eta(t))] - \Pi f = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}R_\lambda(f - \Pi f)(\eta)\,d\lambda\,, \tag{7.13}$$

and by Lemma (7.1(c)) the integrand in (7.13) is analytic on $D_0 \cup D_1$.

Now let $0 < \gamma < 2a(1-\alpha)$ be fixed, and let z_0 and z_1 be the two points on Γ whose real part is equal to $-\gamma$. Let Γ_0 be the curve obtained from Γ by cutting off the part of Γ whose points have real part larger than $-\gamma$ and replacing it by the line segment between z_0 and z_1. We call this line segment L. Note that L is a compact subset of $D_1\backslash\{0\}$.

Now since $R_\lambda(f - \Pi f)(\eta)$ is analytic on $D_0 \cup D_1$ we have

$$\int_\Gamma e^{\lambda t}R_\lambda(f - \Pi f)(\eta)\,d\lambda = \int_{\Gamma_0} e^{\lambda t}R_\lambda(f - \Pi f)(\eta)\,d\lambda\,. \tag{7.14}$$

Hence from (7.13), (7.14), (6.1) and Lemma (7.1)

$$\begin{aligned} &|E^{P_\eta}[f(\eta(t))] - \Pi f| \\ &\qquad = \left|\int_{\Gamma_0} e^{\lambda t}R_\lambda(f - \Pi f)(\eta)\,d\lambda\right| \\ &\qquad \leqq \int_{\Gamma_0} \exp[t \,\mathrm{Real}\,\lambda]|R_\lambda(f - \Pi f)|_{\hat{L}}\,d\lambda \end{aligned} \tag{7.15}$$

$$\leqq e^{-\gamma t}\left[\frac{\alpha+1}{1-\alpha}\frac{2}{\alpha}\int_{|z_0|}^{\infty}\exp[t(r\cos\theta+\gamma)]\frac{-1}{r\cos\theta}\,dr\right.$$
$$\left.+\,K(L)|z_0-z_1|\right]\frac{3-\alpha}{1-\alpha}|f|_{\hat{L}}\,.$$

since $r\cos\theta < -\gamma$ for all $r > |z_0|$, the theorem is proved.

(7.16) REMARK. If we relax the assumption that α in (6.3) is less than one then Theorems (7.4) and (7.10) are no longer true. For example if we let S be the integers, $\bar{c}_k = 1$ for all k, $\gamma(k, \{k, k+1\}) = -1$ for all k, and all other $\gamma(k, F) = 0$, then (6.3) holds with $\alpha = 1$. However this process has more than one stationary measure since the measures concentrated on the configurations which are identically one or identically minus one are all stationary. This is not to say, however, that $\alpha < 1$ is necessary for the conclusions of these theorems.

We conclude this section with an example in order to compare Theorem (7.10) with previously known results. The example is the stochastic Ising model, which was originally proposed by Glauber [4] as a model for the evolution of the configuration of spins of the atoms in a piece of iron. We let S be the d-dimensional integer lattice and let

$$c_x(\eta) = [1 + \exp(2\beta\eta_x \textstyle\sum_y \eta_y)]^{-1}\,,$$

where $\beta > 0$ and the summation on y over the $2d$ nearest neighbors of x. For the motivation behind this choice of c_x see [4] or [10].

If $d = 1$ (the case considered by Glauber) then we may write

$$c_x(\eta) = \tfrac{1}{2} + \tfrac{1}{4}[(1+e^{4\beta})^{-1} - (1+e^{-4\beta})^{-1}]\eta_x[\eta_{x-1} + \eta_{x+1}]\,. \tag{7.17}$$

Since $\frac{1}{2}[(1+e^{-4\beta})^{-1} - (1+e^{4\beta})^{-1}] < \frac{1}{2}$, the hypotheses of Theorem (7.10) are satisfied and the system converges to equilibrium exponentially fast for all values of β. Special cases of this result were proved by Glauber in his original paper and the full result has been obtained many times since (see, for example, [2] or [19]).

If $d = 2$ the situation is not so simple. In this case we take $\bar{c}_x = \frac{1}{2}$ and if $|x - y| = 1$ let

$$\gamma(x, \{x, y\}) = \frac{-1}{16}\{[1+e^{-8\beta}]^{-1} - [1+e^{8\beta}]^{-1} + 2[1+e^{-4\beta}]^{-1} - 2[1+e^{4\beta}]^{-1}\}\,.$$

If $|x - y| = |x - z| = |x - w| = 1$ and $y \neq z$, $y \neq w$, $z \neq w$, let

$$\gamma(x, \{x, y, z, w\}) = \frac{-1}{16}\{1+e^{-8\beta}]^{-1} - [1+e^{8\beta}]^{-1} - 2[1+e^{-4\beta}]^{-1}$$
$$+\,2[1+e^{4\beta}]^{-1}\}\,.$$

Setting $\gamma(x,F) = 0$ for all other F it is easily checked that $c_x(\eta)$ is given by (6.2). Also

$$\textstyle\sum_F |\gamma(x, F)| = [1+e^{-4\beta}]^{-1} - [1+e^{4\beta}]^{-1}\,,$$

and therefore (6.3) holds for some $\alpha < 1$ if and only if $\beta < (\ln 3)/4 \doteq .275$. Hence if $\beta < (\ln 3)/4$, the system converges to its unique equilibrium state exponentially fast. It is known that for $d = 2$ there is only one stationary measure if and only if $\beta \leqq \operatorname{arcsinh}(1)$, and in this case the system converges to that measure as t goes to infinity (see [10]). However, no rates are known if $\beta \geqq (\ln 3)/4$. The exponential rate of convergence for $\beta < (\ln 3)/4$ can also be obtained by applying the results of [20]. The main advantage of the technique used here is that the integral of functions in $\hat{L}$ with respect to the stationary measure can be obtained from (7.6) and Lemma (7.1b). This allows one to draw conclusions about the Gibbs states. We will give a demonstration of this in a future paper.

8. Absolute continuity, existence, and uniqueness. Let $c, \bar{c} : E \to [0, \infty)^S$ be measurable functions such that $||c_k|| + ||\bar{c}_k|| < \infty$ and $c_k = 0$ if and only if $\bar{c}_k = 0$ for $k \in S$. Define $\mathscr{L}$ and $\bar{\mathscr{L}}$ accordingly.

(8.1) THEOREM. *Suppose* $||\sum_k c_k((\ln(\bar{c}_k/c_k))^2 + (\bar{c}_k/c_k - 1)^2)|| < \infty$. *If there is a solution P to the martingale for $\mathscr{L}$ starting from η, then there is a solution $\bar{P}$ for $\bar{\mathscr{L}}$ starting from η, If there is at most one solution $\bar{P}$ for $\bar{\mathscr{L}}$ starting at η, then there is at most one solution P for $\mathscr{L}$ starting at η. In fact, if P exists and $\bar{P}$ is unique, then P and $\bar{P}$ are equivalent on $\mathscr{M}_t^0$ for all $t \geqq 0$; and $d\bar{P}/dP = M(t)$ on $\mathscr{M}_t^0$, where*

(8.2)
$$M(t) = \exp\left[\int_0^t \ln \frac{\bar{c}}{c}(\eta(u))\, d\tilde{\gamma}(u) - \sum_k \int_0^t c_k(\eta(u))\left(\frac{\bar{c}_k}{c_k}(\eta(u)) - 1 - \ln\frac{\bar{c}_k}{c_k}(\eta(u))\right) du\right].$$

PROOF. The first assertion is contained in Theorem (2.4). To prove the second assertion, let P be solution for $\mathscr{L}$ starting from η and define $M(\cdot)$ by (8.2). By Theorem (2.4) $M(t)$ is a P-martingale, and the measure $\bar{P}$ given by $d\bar{P}/dP = M(t)$ on $\mathscr{M}_t^0$, $t \geqq 0$ solves for $\bar{\mathscr{L}}$ starting from η. Since $M(t) > 0$ (a.s., P) $dP/d\bar{P} = (M(t))^{-1}$ (a.s. $\bar{P}$); and if $\bar{P}$ is the unique solution for $\bar{\mathscr{L}}$, then P is uniquely determined by this relationship. This argument also proves the final assertion.

(8.3) COROLLARY. *Assume that $\bar{c}_k/c_k$ is uniformly positive and bounded for each $k \in S$ and that $||\sum c_k(\bar{c}_k/c_k - 1)^2|| < \infty$. If there exists a solution P to the martingale problem for $\mathscr{L}$ starting from η, then there exists a solution $\bar{P}$ to the martingale problem for $\bar{\mathscr{L}}$ starting from η such that $\bar{P} \ll P$ on $\mathscr{M}_t^0$, $t \geqq 0$; and in fact*

(8.4)
$$E^P\left[\left(\frac{d\bar{P}}{dP}\Big|_{\mathscr{M}_t^0}\right)^2\right] \leqq \exp\left[t\left\|\sum c_k\left(\frac{\bar{c}_k}{c_k} - 1\right)^2\right\|\right].$$

PROOF. For each $n \geqq 0$, define

$$c_k^{(n)} = \bar{c}_k \quad \text{if} \quad |k| \leqq n$$
$$\quad\;\; = c_k \quad \text{if} \quad |k| > n,$$

and let $M^{(n)}(t)$ be given by (8.2) with $c^{(n)}$ in place of $\bar{c}$. Define $P^{(n)}$ by $dP^{(n)}/dP = M^{(n)}(t)$ on $\mathscr{M}_t^0$, $t \geqq 0$. Then $P^{(n)}$ solves the martingale problem for $c^{(n)}$ starting at η and

$$E^P[(M^{(n)}(t))^2] \leqq \exp\left[t\left\|\sum c_k\left(\frac{\bar{c}_k}{c_k} - 1\right)^2\right\|\right].$$

Now let $\{P^{(n')}\}$ be a weakly convergent subsequence of $\{P^{(n)}\}$ and put $\bar{P}$ equal to its weak limit. From the preceding estimate it is easy to see that $P^{(n')}$ converges strongly to $\bar{P}$, $\bar{P} \ll P$ on each $\mathscr{M}_t^0$, and (8.4) is satisfied. Finally, if $f \in \mathscr{D}$ and $\Delta_k f \equiv 0$ for $|k| > m$, then $f(\eta(t)) - \int_0^t \mathscr{L}f(\eta(u))\,du$ is a $P^{(n)}$-martingale for $n \geqq m$. Hence, since $P^{(n')} \to P$ strongly $f(\eta(t)) - \int_0^t \mathscr{L}f(\eta(u))\,du$ is a $\bar{P}$-martingale.

(8.5) COROLLARY. *Suppose P and $\bar{P}$ are the only solutions to the martingale problem for c and $\bar{c}$, respectively, starting at η. If $\bar{c}_k/c_k$ is uniformly positive and bounded for each $k \in S$ and $\|\sum c_k(\bar{c}_k/c_k - 1)^2\| < \infty$, then $\bar{P} \ll P$ on each $\mathscr{M}_t^0$ and (8.4) holds. If $\|\sum c_k((\ln(\bar{c}_k/c_k))^2 + (\bar{c}_k/c_k - 1)^2)\| < \infty$, then P and $\bar{P}$ are equivalent on each $\mathscr{M}_t^0$ and $(d\bar{P}/dP)|_{\mathscr{M}_t^0}$ is given by (8.2).*

(8.6) COROLLARY. *Suppose $\{c^{(n)}\}_1^\infty$ is a sequence of measurable functions on E to $[0, \infty)^S$ such that $\|c_k^{(n)}\| < \infty$ for each $k \in S$ and $n \geqq 1$, and $c_k^{(n)} = 0$ if and only if $c_k = 0$. Assume that for each n there is exactly one solution $P^{(n)}$ to the martingale problem for $c^{(n)}$ starting from η, and let P be a solution for c starting at η. If $c_k^{(n)}/c_k$ is uniformly positive and bounded for each $n \geqq 1$ and $k \in S$ and $\|\sum c_k(c_k^{(n)}/c_k - 1)^2\| \to 0$, then $P^{(n)}$ tends to P in variation on each $\mathscr{M}_t^0$, $t \geqq 0$. In particular, P is unique.*

PROOF. We know that $P^{(n)} \ll P$ on $\mathscr{M}_t^0$ and that

$$E^P\left[\left(\frac{dP^{(n)}}{dP}\Big|_{\mathscr{M}_t^0}\right)^2\right] \leqq \exp\left[t\left\|\sum c_k\left(\frac{c_k^{(n)}}{c_k} - 1\right)^2\right\|\right].$$

Hence

$$\begin{aligned}\|(P - P^{(n)})|_{\mathscr{M}_t^0}\|_{\mathrm{Var}}^2 &= E^P\left[\left|1 - \frac{dP^{(n)}}{dP}\Big|_{\mathscr{M}_t^0}\right|\right]^2 \\ &\leqq E^P\left[\left(1 - \frac{dP^{(n)}}{dP}\Big|_{\mathscr{M}_t^0}\right)^2\right] \\ &\leqq \exp\left[t\left\|\sum c_k\left(\frac{c_k^{(n)}}{c_k} - 1\right)^2\right\|\right] - 1 \to 0 .\end{aligned}$$

(8.9) REMARK. It is interesting to see these results say when the functions c_k and $\bar{c}_k$ are constant. Let $\{a_k\}_{k\in S}$ and $\{b_k\}_{k\in S}$ be sequences of positive constants and let P and Q be, respectively, the solutions to the martingale problem for $\sum a_k \Delta_k$ and $\sum b_k \Delta_k$ starting from η. Since P and Q may be regarded as product measures, it follows that they are either equivalent or mutually singular on each $\mathscr{M}_t^0$. If $\sum a_k(b_k/a_k - 1)^2 < \infty$ then by Corollary (8.3) P and Q are equivalent on each $\mathscr{M}_t^0$. We will now show that $P \perp Q$ on $\mathscr{M}_t^0$ for all $t > 0$ if

$$\sum \left(\frac{a_k}{a_k^{\frac{1}{2}} + b_k^{\frac{1}{2}}}\right)^2 \left(\frac{b_k}{a_k} - 1\right)^2 = \infty .$$

Combining this with the above, we will then know that when there is a $\lambda > 0$ for which $\lambda \leqq b_k/a_k \leqq 1/\lambda$ for all $k \in S$; $\sum a_k(b_k/a_k - 1)^2 < \infty$ implies P is equivalent to Q on each $\mathscr{M}_t^0$, $t \geqq 0$, and $\sum a_k(b_k/a_k - 1)^2 = \infty$ implies $P \perp Q$ on each $\mathscr{M}_t^0$, $t \geqq 0$.

To prove the assertion about singularity of P and Q, let

$$\mathscr{B}_t^n = \mathscr{B}[\eta_k(u) : 0 \leqq u \leqq t \text{ and } |k| \leqq n] .$$

Then it is easy to see that $P \ll Q$ on $\mathscr{B}_t^n$ and $(dQ/dP)|_{\mathscr{B}_t^n} = M^{(n)}(t)$, where

$$M^{(n)}(t) = \exp\left[\sum_{|k| \leqq n} \left(\tilde{\gamma}(t) \ln \frac{b_k}{a_k} - a_k t \left(\frac{b_k}{a_k} - 1 - \ln \frac{b_k}{a_k}\right)\right)\right] .$$

A simple computation shows that:

$$E^P[(M^{(n)}(t))^{\frac{1}{2}}] = \exp\left[-\tfrac{1}{2} \sum_{|k| \leqq n} \left(\frac{a_k}{a_k^{\frac{1}{2}} + b_k^{\frac{1}{2}}}\right)^2 \left(\frac{b_k}{a_k} - 1\right)^2\right] .$$

Since $M^{(n)}(t) \to M(t)$ (a.s., P), where $M(t)$ is the density of the Lebesgue part of Q with respect to P on $\mathscr{M}_t^0$, this shows that $M(t) = 0$ (a.s., P) if and only if

$$\sum \left(\frac{a_k}{a_k^{\frac{1}{2}} + b_k^{\frac{1}{2}}}\right)^2 \left(\frac{b_k}{a_k} - 1\right)^2 = \infty .$$

9. Appendix. We prove here Lemmas (6.20) and (7.1). All of the notation and definitions are the same as in Sections 6 and 7.

If D is a region of the complex plane let $X(D)$ be the Banach space of functions g_λ from D into $\hat{L}$ such that for each F the function $\lambda \to \hat{g}_\lambda(F)$ is an analytic function of λ. The norm on $X(D)$ is given by

$$|||g_{\bullet}|||_D = \sum_F \sup_{\lambda \in D} |\hat{g}_\lambda(F)| .$$

Notice that if $f \in \hat{L}$, then the function $\lambda \to f$, which we again denote by f, is in $X(D)$ and $|||f|||_D = |f|_{\hat{L}}$.

(A.1) LEMMA. *Let* (6.3) *hold and let* $g_\lambda \in X(D)$. *Then* $A_\lambda g_\lambda \in X(D_0)$, *and*

$$|||A_{\bullet} g_{\bullet}|||_{D_0} \leqq \frac{\alpha + 1}{2} |||g_{\bullet}|||_{D_0} . \tag{A.2}$$

In particular if $f \in \hat{L}$ *and* $\lambda > 0$, *then*

$$|A_\lambda f|_{\hat{L}} \leqq |||A_{\bullet} f|||_{D_0} \leqq \frac{\alpha + 1}{2} |||f|||_{D_0} = \frac{\alpha + 1}{2} |f|_{\hat{L}} .$$

PROOF. From the definition of A_λ and (6.19) we see that

$$\widehat{A_\lambda g_\lambda}(H) = \sum_F \sum_k \gamma(k, F \triangle H) \frac{-2I_F(k)}{\lambda + 2c_F} \hat{g}_\lambda(f) .$$

$2I_F(k)/(\lambda + 2c_F)$ and $\hat{g}_\lambda(F)$ are both analytic on D_0. Thus to see that $\widehat{A_\lambda g_\lambda}(H)$ is analytic on D_0 we need only check that the summation is uniformly convergent. This as well as (A.2) will follow if we can show that

$$\text{(A.3)} \qquad \sum_H \sum_F \sum_k |\gamma(k, F \triangle H)| 2I_F(k) \sup_{\lambda \in D_0} \left| \frac{\hat{g}_\lambda(F)}{\lambda + 2c_F} \right| \leqq \frac{\alpha + 1}{2} \sum_F \sup_{\lambda \in D_0} |\hat{g}_\lambda(F)| .$$

But the left hand side of (A.3) is bounded by

$$\text{(A.4)} \qquad \sum_F \sum_k \alpha \bar{c}_k 2I_F(k) \sup_{\lambda \in D_0} \left| \frac{\hat{g}_\lambda(F)}{\lambda + 2c_F} \right| \leqq \sum_F \alpha \sup_{\lambda \in D_0} \left| \frac{2c_F}{\lambda + 2c_F} \right| \sup_{\lambda \in D_0} |\hat{g}_\lambda(F)| .$$

Finally (A.3) follows from (A.4) since for any number $c \geqq 0$, $\sup_{\lambda \in D_0} |2c/(\lambda + 2c)| \leqq (\alpha + 1)/2\alpha$.

(A.5) COROLLARY. *If* (6.3) *holds and* $f \in \hat{L}$, *then* $B_\lambda f \equiv \sum_{k=0}^\infty (A_\lambda)^k f \in X(D_0)$, *and*

$$\text{(A.6)} \qquad |||B_{\boldsymbol{\cdot}} f|||_{D_0} \leqq \frac{2}{1 - \alpha} |f|_{\hat{L}} .$$

(A.7) LEMMA. *Let* (6.3) *hold and suppose that* $\inf_{k \in S} \bar{c}_k = a > 0$. *Let D be any compact subset of* D_1. *Then for* $\lambda \in D$ *and* $f \in \hat{L}$ *we may define* $A_\lambda f$ *by the right side of* (6.19) *and* $B_\lambda f$ *as in Corollary* (A.5). *Moreover there is a constant* $\tilde{K}(D) < \infty$ *such that for all* $f \in \hat{L}$ *we have* $|||B_{\boldsymbol{\cdot}} f|||_D \leqq \tilde{K}(D)|f|_{\hat{L}}$.

The proof is the same as the proof of Corollary (A.5) and Lemma (A.1) except that in (A.4) we use the bound $\alpha \sup_{\lambda \in D} |2c_F/(\lambda + 2c_F)| \equiv a(D) < 1$. Of course $\tilde{K}(D)$ is then $1/[1 - a(D)]$.

PROOF OF LEMMA (7.1). (a) The first inequality follows from (6.1). From (6.16) we see that

$$\text{(A.8)} \qquad R_\lambda f = \sum_F \frac{1}{\lambda + 2c_F} \widehat{B_\lambda f}(F) \mathrm{X}_F .$$

Thus

$$|R_\lambda f|_{\hat{L}} = \sum_F \left| \frac{1}{\lambda + 2c_F} \right| |\widehat{B_\lambda f}(F)| \leqq \sup_F \left| \frac{1}{\lambda + 2c_F} \right| B_\lambda f|_{\hat{L}} .$$

By Corollary (A.5), $|B_\lambda f|_{\hat{L}} \leqq |||B_{\boldsymbol{\cdot}} f|||_{D_0} \leqq (2/(1 - \alpha))|f|_{\hat{L}}$, and it is easily checked that if $\lambda \in D_0$ then $|1/(\lambda + 2c)| \leqq (1/|\lambda|)((\alpha + 1)/2\alpha)$ for all $c \geqq 0$.

(b) A review of the proof of Corollary (A.5) shows that if all $c_k > 0$, then $B_0 f$ exists for all $f \in \hat{L}$, $|B_0 f|_{\hat{L}} \leqq (2/(1 - \alpha))|f|_{\hat{L}}$, and as $\lambda \searrow 0$ $|B_\lambda f - B_0 f|_{\hat{L}} \to 0$. From (A.8) we have, for $\lambda > 0$,

$$\lambda R_\lambda f = \sum_F \frac{\lambda}{\lambda + 2c_F} \widehat{B_\lambda f}(F) \mathrm{X}_F .$$

Thus

$$(A.9)\quad \begin{aligned} &|\lambda R_\lambda f - B_0 f(\varphi)|_{\hat{L}} \\ &= \textstyle\sum_{F\neq\phi} \frac{\lambda}{\lambda + 2c_F} |\widehat{B_\lambda f}(F)| + |\widehat{B_\lambda f}(\phi) - \widehat{B_0 f}(\phi)| \\ &\leqq \textstyle\sum_F \frac{\lambda}{\lambda + 2c_F} |\widehat{B_\lambda f}(F) - \widehat{B_0 f}(F)| + \sum_{F\neq\phi} \frac{\lambda}{\lambda + 2c_F} |\widehat{B_0 f}(F)| \\ &\leqq |B_\lambda f - B_0 f|_{\hat{L}} + \textstyle\sum_{F\neq\phi} \frac{\lambda}{\lambda + 2c_F} |\widehat{B_0 f}(F)| \,. \end{aligned}$$

Since $|B_\lambda f - B_0 f|_{\hat{L}} \to 0$ as $\lambda \searrow 0$ and $B_0 f \in \hat{L}$, both terms on the right side of (A.9) go to zero with λ.

The last inequality in (b) follows from

$$|\Pi f|_{\hat{L}} = |\widehat{B_0 f}(\phi)| \leqq |B_0 f|_{\hat{L}} \leqq \frac{2}{1-\alpha} |f|_{\hat{L}} \,.$$

(c) The existence of B_λ on $D_0 \cup D_1$ and R_λ on $(D_0 \cup D_1)\backslash\{0\}$ follows from Lemma (A.7) and the observation that $\sigma(c) \cap [(D_0 \cup D_1)\backslash\{0\}] = \phi$.

From Corollary (A.5) and Lemma (A.7) we know that if D is a compact subset of $D_0 \cup D_1$ and $f \in \hat{L}$, then $B_\lambda f \in X(D)$. Since $c_F \geqq a$ if $F \neq \phi$ and $c_\phi = 0$, it follows from (A.8) that $R_\lambda f \in X(D)$ for every compact subset of $(D_0 \cup D_1)\backslash\{0\}$. Now it follows immediately from the definition of $X(D)$ that if $g_\lambda \in X(D)$ and $\eta \in E$, then $g_\lambda(\eta)$ is analytic on D.

It remains to show that for all $f \in \hat{L}$ and all $\eta \in E$, $|R_\lambda(f - \Pi f)(\eta)|$ remains bounded as λ goes to zero. As remarked in the proof of Theorem (7.10), $R_\lambda \mathrm{X}_\phi = (1/\lambda)\mathrm{X}_\phi$. Thus from (A.8) and the definition of Πf we have

$$(A.10)\quad \begin{aligned} &R_\lambda(f - \Pi f)(\eta) \\ &= \textstyle\sum_{F\neq\phi} \frac{1}{\lambda + 2c_F} \widehat{B_\lambda f}(F)\mathrm{X}_F(\eta) + \frac{1}{\lambda}(\widehat{B_\lambda f}(\phi) - \widehat{B_0 f}(\phi)) \,. \end{aligned}$$

Using the uniform positivity of the c_F's, $F \neq \phi$, and an argument similar to the one in part (b) we see that the first term on the right of (A.10) remains bounded. The second term remains bounded since $\widehat{B_\lambda f}(\phi)$ is analytic by Lemma (A.7).

(d) This follows immediately from Lemma (A.7) and (A.8).

REFERENCES

[1] DOBRUSHIN, R. L. (1971). Markov processes with a large number of locally interacting components—existence of the limiting process and its ergodicity. *Problemy Peredači Informacii* **7** 70–87.

[2] FELDERHOF, B. U. (1970). Spin relaxation of the Ising chain. *Rep. Math. Phys.* **1** 215–234.

[3] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications* **2**, 2nd ed. Wiley, New York.

[4] GLAUBER, R. (1963). Time dependent statistics of the Ising model. *J. Mathematical Phys.* **4** 294–307.

[5] GRIFFEATH, D. (1975). Ergodic theorems for graph interactions. *Advances in Appl. Probability* **7** 179–194.

[6] HARRIS, T. E. (1972). Nearest neighbor Markov interaction processes on multidimensional lattices. *Advances in Math.* **9** 66–89.

[7] HELMS, L. L. (1974). Ergodic properties of several interacting Poisson particles. *Advances in Math.* **12** 32–57.

[8] HOLLEY, R. (1970). A class of interactions in an infinite particle system. *Advances in Math.* **5** 291–309.

[9] HOLLEY, R. (1972). Markov interaction processes with finite range interactions. *Ann. Math. Statist.* **43** 1961–1967.

[10] HOLLEY, R. (1974). Recent results on the stochastic Ising model. *Rocky Mt. J. of Math.* **4** 479–496.

[11] HOLLEY, R. and LIGGETT, T. M. (1975). Ergodic theorems for weakly interacting systems and the voter model. *Ann. Probability* **3** 643–663.

[12] KATO, T. (1966). *Perturbation Theory for Linear Operations.* Springer-Verlag, New York.

[13] KRYLOV, N. V. (1973). The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes. *Math. USSR-Izv.* **7**, No. 3, 691–709.

[14] LIGGETT, T. M. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.* **165** 471–481.

[15] STROOCK, D. W. and VARADHAN, S. R. S. (1969). Diffusion processes with continuous coefficients, I. *Comm. Pure Appl. Math.* **22** 345–400.

[16] STROOCK, D. W. and VARADHAN, S. R. S. (1969). Diffusion processes with continuous coefficients, II. *Comm. Pure Appl. Math.* **22** 479–530.

[17] STROOCK, D. W. and VARADHAN, S. R. S. (1971). Diffusion processes with boundary conditions. *Comm. Pure Appl. Math.* **24** 147–225.

[18] STROOCK, D. W. (1975). Diffusion processes associated with Levy generators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **32** 209–244.

[19] SULLIVAN, W. G. (1975). Mean square relaxation times for evolution of random fields. *Comm. Math. Phys.* **40** 249–258.

[20] SULLIVAN, W. G. (1974). A unified existence and ergodic theorem for Markov evolution of random fields. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **31** 47–56.

[21] VARADHAN, S. R. S. (1968). *Stochastic Processes.* Courant Inst. Math. Sciences, New York.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO
BOULDER, COLORADO 80302