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ON A LOCAL LIMIT THEOREM CONCERNING VARIABLES IN THE DOMAIN OF NORMAL ATTRACTION OF A STABLE LAW OF INDEX α, $1 < \alpha < 2$

BY SUJIT K. BASU

Indian Institute of Management, Calcutta

Let $\{X_n\}$ be a sequence of independent and identically distributed random variables with $EX_1 = 0$. Suppose that there exists a constant $a > 0$, such that $Z_n = (an^r)^{-1}(X_1 + X_2 + \cdots + X_n)$ converges in law to a stable distribution function (df) $V(x)$ as $n \to \infty$. If, in addition, we assume that the characteristic function of X_1 is absolutely integrable in mth power for some integer $m \geqq 1$, then for all large n, the df F_n of Z_n is absolutely continuous with a probability density function (pdf) f_n such that the relation

$$\lim_{n\to\infty} |x|\,|f_n(x) - v(x)| = 0$$

holds uniformly in x, $-\infty < x < \infty$, where v is the pdf of V.

1. Introduction. Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random variables belonging to the domain of normal attraction of a stable distribution function (df) V of index α, $1 < \alpha < 2$. We assume that $EX_1 = 0$. This means that there exists a constant $a > 0$, such that $Z_n = (an^r)^{-1}(X_1 + X_2 + \cdots + X_n)$ converges in law to V, where $r = \alpha^{-1}$. Moreover, if the characteristic function (ch.f.) of X_1 is absolutely integrable in mth power for some integer $m \geqq 1$, then for all large n, the df F_n of Z_n is absolutely continuous with a probability density function (pdf) f_n such that the relation

$$\lim_{n\to\infty} |f_n(x) - v(x)| = 0 \tag{1.1}$$

holds uniformly in x, $-\infty < x < \infty$, where $v(x) = dV(x)/dx$. This follows from Theorem 2, page 227 in [3]. In the present paper this density convergence result is further investigated and it is found that, in fact, under the same aforesaid conditions, it is possible to go a step further to claim that the relation

$$\lim_{n\to\infty} |x|\,|f_n(x) - v(x)| = 0 \tag{1.2}$$

holds uniformly in x, $-\infty < x < \infty$.

Similar and more sophisticated results of this kind, in cases where the central limit theorem applies, have been given by Petrov [6], Smith [7], Höglund [4], Smith and Basu [8], Basu [1] and [2].

2. Notations and preliminary lemmas. Let Y denote a stable random variable of index α, $1 < \alpha < 2$, and $V(\cdot)$ and $W(\cdot)$ denote the df and ch.f. of Y respectively. Throughout this paper $\{X_n\}$ represents a sequence of i.i.d. random variables each with df $F(\cdot)$, ch.f. $w(\cdot)$ and $EX_1 = 0$. We assume that F belongs

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to the domain of normal attraction of V. With no loss of generality, the constant a in the previous section may be assumed to be 1. We then set $Z_n = n^{-r}(X_1 + X_2 + \cdots + X_n)$ with $r = \alpha^{-1}$, $F_n(x) = \Pr\{Z_n \leqq x\}$ and $W_n(t) = E\exp(itZ_n)$ so that $W_n(t) = \{w(tn^{-r})\}^n$. For any function $g(t)$ and positive integer k, we shall write $g^{(k)}(t)$ to denote $(d/dt)^k g(t)$, whenever such a derivative exists.

Under these notations and assumptions, we then have the following lemmas.

LEMMA 2.1. $\lim_{n\to\infty} E|Z_n| = E|Y|$.

PROOF. See Theorem 2 of [5].

LEMMA 2.2. *$W_n^{(1)}(t)$ and $W^{(1)}(t)$ exist for all t and $W_n^{(1)}(t)$ converges to $W^{(1)}(t)$ for all t.*

PROOF. That the derivatives exist is more or less obvious. Now, note that

$$W_n^{(1)}(t) - W^{(1)}(t) = D_{1n} + D_{2n} + D_{3n}$$

where

$$\begin{aligned} D_{1n} &= \int_{-A}^{A} ix\exp(itx)F_n(dx) - \int_{-A}^{A} ix\exp(itx)V(dx)\,, \\ D_{2n} &= \int_{|x|>A} ix\exp(itx)F_n(dx)\,, \\ D_{3n} &= -\int_{|x|>A} ix\exp(itx)V(dx)\,, \end{aligned}$$

A being a positive number to be suitably chosen later. Since X_n belongs to the domain of attraction of V, D_{1n} converges to zero as $n \to \infty$. Also,

$$|D_{2n}| \leqq \int_{|x|>A} |x| F_n(dx)$$

which, because of Lemma 1, can be made as small as we please by a suitable choice of A. D_{3n} can obviously be made small by choosing a large A. This completes the proof of the lemma.

LEMMA 2.3. *For any fixed $\varepsilon > 0$*

$$\int_{|x|>\varepsilon n^r} |x| F(dx) = O(n^{r-1}) \qquad \textit{as } n \to \infty\,. \tag{2.1}$$

PROOF. Since F belongs to the domain of normal attraction of V, it follows (see e.g., Theorem 5, page 181 of [3]) that there exists some constant $C > 0$ such that $F(x) \leqq C|x|^{-\alpha}$ if $x < 0$ and $1 - F(x) \leqq Cx^{-\alpha}$ if $x > 0$. On carrying out the integration in (2.1) by parts, the lemma follows.

LEMMA 2.4. *There exist positive constants M_1 and M_2 such that the inequality*

$$n^{1-r}|w^{(1)}(tn^{-r})| \leqq M_1|t| + M_2$$

holds for all large n and for all t.

PROOF. Let

$$\begin{aligned} M_n(x) &= nF(n^r x) && \text{for } \ x < 0 \\ &= n\{F(n^r x) - 1\} && \text{for } \ x > 0\,. \end{aligned}$$

Then by the necessary and sufficient conditions for convergence to a stable distribution (see [3], page 116), we know that for all x, $M_n(x)$ converges to $M(x)$

as $n \to \infty$ where

$$M(x) = c_1/|x|^\alpha \quad \text{if} \quad x < 0 ,$$
$$= -c_2/x^\alpha \quad \text{if} \quad x > 0$$

with $c_1, c_2 \geqq 0$ and $c_1 + c_2 > 0$. Now keeping in mind that $EX_1 = 0$, we observe that for $A > 0$,

$$\begin{aligned} n^{1-r}|w^{(1)}(tn^{-r})| &= n^{1-r}|\textstyle\int_{-\infty}^{\infty} ix \exp(itx/n^r)F(dx)| \\ &= |\textstyle\int_{-\infty}^{\infty} y \exp(ity)M_n(dy)| \\ &= |\textstyle\int_{-\infty}^{\infty} y\{\exp(ity) - 1\}M_n(dy)| \\ &\leqq \textstyle\int_{-A}^{A} |y|\,|\exp(ity) - 1|M_n(dy) + 2\int_{|y|>A} |y|M_n(dy) \\ &\leqq |t| \textstyle\int_{-A}^{A} y^2 M_n(dy) + 2\int_{|y|>A} |y|M_n(dy) . \end{aligned}$$

Finally, applying Lemma 2.3 to the second integral and using the fact that

$$\lim_{n\to\infty} \int_{-A}^{A} y^2 M_n(dy) = \int_{-A}^{A} y^2 M(dy) ,$$

we easily obtain the result.

3. The main theorem.

With all these preliminaries we are now ready to prove:

THEOREM 3.1. *Let $\{X_n\}$ be a sequence of independent and identically distributed random variables each with a common* df F *and* ch.f. w. *Assume that $EX_1 = 0$. If*

(i) F belongs to the domain of normal attraction of a stable distribution function v of index α, $1 < \alpha < 2$ with $V^{(1)}(x) = v(x)$; and

(ii) w is absolutely integrable in mth *power for some integer m, $m \geqq 1$, then for all large n, the* df F_n *of $Z_n = n^{-r}(X_1 + \cdots + X_n)$ is absolutely continuous with a* pdf f_n *such that the relation*

$$\lim_{n\to\infty} |x|\,|f_n(x) - v(x)| = 0$$

holds uniformly in x, $-\infty < x < \infty$.

PROOF. From the canonical representation of $W(t)$ (see e.g. page 164 of [3]), it follows that $W(t)$ is absolutely integrable. Also, by (ii), $W_n(t)$ is absolutely integrable for all large n. Therefore, using the inversion formula for Fourier transforms, both $W(t)$ and $W_n(t)$ can be inverted to obtain $v(x)$ and $f_n(x)$ respectively. Moreover, using similar arguments, both $W^{(1)}(t)$ and $W_n^{(1)}(t)$ can also be shown to be absolutely integrable (recall that $\alpha > 1$) and hence may be inverted. Thus

$$\begin{aligned} |xf_n(x) - xv(x)| &= (2\pi)^{-1}|\textstyle\int_{-\infty}^{\infty} \{W_n^{(1)}(t) - W^{(1)}(t)\} \exp(-itx)\, dt| \\ &\leqq I_{1n} + I_{2n} + I_{3n} + I_{4n} , \end{aligned}$$

where

$$\begin{aligned} I_{1n} &= (2\pi)^{-1} \textstyle\int_{-A}^{A} |W_n^{(1)}(t) - W^{(1)}(t)|\, dt \\ I_{2n} &= (2\pi)^{-1} \textstyle\int_{|t|\geqq A} |W^{(1)}(t)|\, dt \\ I_{3n} &= (2\pi)^{-1} \textstyle\int_{A\leqq|t|\leqq\delta n^r} |W_n^{(1)}(t)|\, dt \\ I_{4n} &= (2\pi)^{-1} \textstyle\int_{|t|\geqq\delta n^r} |W_n^{(1)}(t)\, dt \end{aligned}$$

where the positive constants A and δ will be chosen later.

Since by Lemma 2.1, $W_n^{(1)}$ and $W^{(1)}$ are bounded by constants, Lemma 2.2 implies that for any fixed $A > 0$, $I_{1n} \to 0$ as $n \to \infty$. I_{2n} can be made as small as we please by choosing a large A. Also, by the lemma on page 238 in [3] and Lemma 2.4, it follows that given a sufficiently small $\delta > 0$, there exist constants λ, M_1 and M_2 such that the inequality

$$|W_n^{(1)}(t)| \leqq (M_1|t| + M_2) \exp(-\lambda|t|^\alpha)$$

holds for $|t| < \delta n^r$ and all large n. This would imply that I_{3n} can be made as small as we please for all large n by a suitable choice of δ. Further, there exists a $c > 0$ such that $|w(t)| < \exp(-c)$ for $|t| > \delta$. Thus for all $n > m$

$$I_{4n} \leqq n^{1-r}E|X_1| \int_{|t|>\delta n^r} |w(tn^{-r})|^m \, dt \exp\{-c(n - m - 1)\} .$$

Since the integral on the right-hand side converges as $n \to \infty$,

$$I_{4n} \to 0 \qquad \text{as } n \to \infty .$$

This completes the proof of the theorem.

Remark. It may be mentioned here that the assumption that $\alpha < 2$ is not strictly necessary; in fact, similar arguments as above with minor and obvious modifications show that Theorem 3.1 remains true also when $\alpha = 2$. However, stronger results are already in existence in such a case and some of these may be found in [1], [2], [6], [7] and [8].

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Indian Institute of Management
Diamond Harbour Road
P. O. Joka via Calcutta 27
Dist. 24 Parganas
West Bengal, India