

## ON THE STRUCTURE OF SEMIMARKOV PROCESSES COMPARED WITH CHUNG PROCESSES<sup>1</sup>

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Semimarkov processes with discrete state spaces are considered without restrictions on their probability laws. They admit states where every visit lasts a positive time even though there may be infinitely many such visits in a finite interval. These are called unstable holding states as opposed to the stable holding states which are encountered in Markov processes. Further, it is possible to have instantaneous states at which the behavior is that at an ordinary instantaneous state of a Chung process, or that at a sticky boundary point, or that at a nonsticky point. To convert such processes to Chung processes, each unstable state is split into infinitely many stable ones, and then a random time change is effected whereby some sets of constancy are dilated, and sojourn intervals are altered to have exponentially distributed lengths.

**1. Introduction.** While introducing semimarkov<sup>3</sup> processes, Lévy (1954) asserted that any such process can be transformed into a Markov process through a random time change using a suitable strictly increasing time scale. The assertion implies that the qualitative structure of the sample paths of a semimarkov process is the same as that of a Markov process. The only differences, therefore, would be restricted to “the probability laws governing the sojourns” at a holding state: they are exponential for Markov processes and arbitrary for semimarkov processes.

This, however, is not true. The source of error is the implicit assumption that sojourn times at a fixed state are well-defined random variables. It turns out that there are states such that every visit lasts a positive time but that between any two sojourn intervals there is a third. Of course, then, the left end points of those intervals cannot be well-ordered, which implies that we need to talk of random variables such as the length of the first (or second or third . . .) sojourn interval whose length exceeds  $\epsilon$ . And the law of such intervals, of course, depends on  $\epsilon$ . Here is an illustration.

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<sup>3</sup> Spelling is deliberate; the alternative “semi-Markov” looks ugly in addition to conveying a meaning which is not quite right; it is the strong Markov property which will be relaxed here. *AMS 1970 subject classifications.* Primary 60K15; Secondary 60G17, 60J25.

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(1.1) EXAMPLE. Consider a Brownian motion on  $(-\infty, \infty)$  starting from 0. Let  $M$  be the set of all  $t > 0$  at which the process is at the point 0. This is closed in  $(0, \infty)$ . Let  $K_0$  be the set of all  $t$  in  $M$  which are not isolated on the right. Let  $K_1$  be the complement of  $K_0$  in  $(0, \infty)$ . Define a stochastic process  $(X_t)$  by

$$\begin{aligned} X_t &= 0 & \text{if } t \in K_0, \\ &= 1 & \text{if } t \in K_1. \end{aligned}$$

The process  $(X_t)$  is semimarkov in all the senses of which we are aware: Lévy (1954), Yackel<sup>4</sup> (1968), Pyke and Schaufele (1964), page 1740. State 0 is instantaneous, and state 1 is holding. The set  $K_1$  is a countable union of intervals each of which has positive length, but any neighborhood of  $t = 0$  contains infinitely many of those intervals. No two of these sojourn intervals have an end point in common, and between any two intervals there lies a third.

We note that the phrase "length of a sojourn interval at state 1" has no definite meaning, and it is faulty to speak of its probability law. However, there is a measure  $\lambda$  on  $(0, \infty)$ , whose total mass is infinite, and which governs the sojourns at 1 in the following sense: Consider those sojourn intervals whose lengths exceed  $\varepsilon$ ; the lengths of the first, the second, . . . such intervals are independent and identically distributed as

$$\lambda(B)/\lambda((\varepsilon, \infty)), \quad B \subset (\varepsilon, \infty), B \text{ Borel.}$$

It is well known that the set  $K_0$  is the range of a right continuous increasing additive process with zero drift. The measure  $\lambda$  above is the Lévy measure of this additive process.

Our object is to clarify the qualitative structure of semimarkov processes on discrete spaces. We will classify the states of a semimarkov process  $X$  first as holding versus instantaneous just as with Markov processes. A holding state is either stable or unstable depending on whether it is entered only finitely often or infinitely often during a finite interval. An instantaneous state  $i$  is attractive or repellent depending on whether its set of constancy  $\{t: X_t = i\}$  is perfect or discrete (perfect means right closed and has no isolated points, discrete means every point is isolated). An attractive state is light or heavy according as the Lebesgue measure of its set of constancy is 0 or positive.

Every Chung process (see the definition below) is semimarkov. In a Chung process, every holding state is stable, and every instantaneous state is heavy. Supposing that a Chung process has a purely atomic boundary, in the terminology of Chung (1968), a sticky boundary point is like a light attractive state, and a nonsticky boundary point is like a repellent state.

The most important qualitative difference between semimarkov and Chung processes is due to unstable states. While converting a semimarkov process to

<sup>4</sup> Our claim is true in spirit although not actually true: Yackel excludes this possibility by assuming that the transition functions involved are "strong."

a Chung process, each unstable state  $i$  is "split" into infinitely many stable ones in addition to dilating light attractive states and replacing sojourn intervals by exponentially distributed ones. Details of this conversion problem will be reported elsewhere.

Finally, we would like to introduce the following much needed definition.

(1.2) *Chung processes.* Let  $(X_t)$  be a (continuous time-parameter) Markov process defined on a complete probability space, and taking values in a discrete state space  $E$  (a countable set with the discrete topology which is further compactified by adding an extra point if not already compact). Then  $X$  is said to be a *Chung process* provided that

- (a) the transition function of  $X$  be standard, and
- (b) the sample paths be "right lower semicontinuous,"

that is, for any  $t$  and  $\omega$ , the sample path  $X_s(\omega)$  has at most one limiting value in  $E$  as  $s$  decreases to  $t$ ;  $X_t(\omega)$  is equal to that limiting value in  $E$  if it exists, and is equal to the "point at infinity" if there is no such limit in  $E$ .

Chung processes are important because Chung, Doob, Feller, Kolmogorov, and Lévy have worked on them; they are interesting because their sample paths have intricate discontinuity properties not possessed by "standard" Markov processes on more general state spaces; and they need to be singled out because they do not fit neatly into the general potential-theoretic framework. Finally, on sheer volume of publications concerning them alone, Chung processes deserve a name of their own amongst other processes such as Brownian motion, Feller processes, Hunt process. We hope that the wisdom and the justice implicit in this definition will be recognized.

Our terminology for elements of stochastic processes in general follows Dellacherie (1972). In addition, we write  $\mathbb{R}_+ = [0, \infty)$ ,  $\mathbb{R}_t = (t, \infty)$  for any  $t \in \mathbb{R}_+$ ; and  $\mathcal{R}_+$  and  $\mathcal{R}_t$  for the associated  $\sigma$ -algebras of all Borel subsets.

**2. Semimarkov processes.** In this section we describe the processes we are interested in. A more fundamental definition (which seems to have been what Lévy had in mind) will be given elsewhere along with comparisons with other definitions.

Let  $E$  be a discrete set (countable with discrete topology). If  $E$  is finite, let  $\bar{E} = E$ . If  $E$  is not finite, let  $\bar{E} = E \cup \{\varphi\}$  be its one point compactification. Let  $(\Omega, \mathcal{M}, P)$  be a complete probability space, let  $S_0$  be a random variable on it with values in  $(\mathbb{R}_+, \mathcal{R}_+)$ , and let  $X = (X_t)_{t>0}$  be a stochastic process defined on  $(\Omega, \mathcal{M})$  and taking values in  $\bar{E}$ . Let  $\mathcal{H}_t^0 = \sigma(X_s; 0 < s \leq t) \vee \sigma(S_0)$  and  $\mathcal{H}_\infty^0 = \bigvee_t \mathcal{H}_t^0$ . Let  $\mathcal{H}_\infty$  be the completion of  $\mathcal{H}_\infty^0$ , and let  $\mathcal{H}_t$  be the  $\sigma$ -algebra generated by  $\mathcal{H}_{t+}^0 = \bigcap_{s>t} \mathcal{H}_s^0$  and all the negligible sets in  $\mathcal{H}_\infty$ . We will refer to the family  $\mathcal{H} = (\mathcal{H}_t)$  as the history generated by  $(X, S)$ . Stopping times, progressive sets, etc., are all relative to this history unless specified otherwise.

For each  $\omega \in \Omega$  let  $M(\omega)$  be the set of all  $t > 0$  at which the path  $X(\omega) : s \rightarrow X_s(\omega)$

is not continuous plus those  $t > 0$  at which  $X(\omega)$  is continuous and has the value  $\varphi$ . The random set  $M$  will be called the *discontinuity set* of  $X$ .

Fix  $\omega \in \Omega$ . The set  $M(\omega)$  is closed in  $(0, \infty)$ . Its complement (if not empty) is a countable union of open intervals. Each such open interval is said to be contiguous to  $M(\omega)$ . We define  $L(\omega)$  to be the set of all left end points of the intervals contiguous to  $M(\omega)$ .

For every  $t > 0$  and  $\omega \in \Omega$  we define

$$(2.1) \quad S_t(\omega) = S_0(\omega) + t \quad \text{if } (0, t] \cap M(\omega) = \emptyset, \\ = t - \sup \{s \leq t : s \in M(\omega)\} \quad \text{otherwise.}$$

We call  $S_t(\omega)$  the *sojourn value* at  $t$  for the path  $X(\omega)$ . If  $t$  is a time of discontinuity for  $X(\omega)$ , then  $S_t(\omega) = 0$ . Otherwise, if  $t$  belongs to the contiguous interval  $(a(\omega), b(\omega))$  (on which, by the way  $M(\omega)$  is defined, the path  $X(\omega)$  remains constant), the sojourn value at  $t$  is  $S_t(\omega) = t - a(\omega)$ , unless  $a(\omega) = 0$  in which case  $S_t(\omega) = S_0(\omega) + t$ .

Throughout this paper we assume that the following axioms hold.

- (2.2) *Regularity.* For almost every  $\omega$  and every  $t > 0$ ,
- (a) if  $t \in L(\omega)$ , then the path  $X(\omega)$  is right continuous at  $t$ ;
  - (b) if  $t \in M(\omega) \setminus L(\omega)$ , and if there exists a sequence  $(t_n) \subset M(\omega) \setminus L(\omega)$  decreasing to  $t$  and such that  $X_{t_n}(\omega) = i$  for every  $n$  for some  $i \in E$ , then  $X_t(\omega) = i$ .

(2.3) *Point  $\varphi$ .* For every  $t > 0$ ,  $P\{X_t = \varphi\} = 0$ .

(2.4) *Progressive measurability.* The process  $X$  is progressively measurable with respect to the history  $\mathcal{H}$ .

(2.5) *Regeneration.* The process  $(X, S) = (X_t, S_t)_{t \geq 0}$  is a time-homogeneous Markov process which further enjoys the strong Markov property at every stopping time  $T$  of  $\mathcal{H}$  such that  $X_T \in E$  a.s. on  $\{T < \infty\}$ .

Then  $X$  is a semimarkov process in the strict sense.

Regularity axiom (2.2) is close to a separability axiom. By (2.2a) every interval of constancy has the form  $[ \quad )$ . Axiom (2.2b) is in fact an axiom on random sets on which  $X$  is in a fixed instantaneous state. These axioms might require the state space to be closed, and thus the need for compactifying the state space. Hence (2.3) delimits  $\varphi$  to the role of "an extra point added for reasons of smoothness." Axiom (2.4) is technical. The crucial assumption is (2.5): First, note that each  $S_t$  is  $\mathcal{H}_t$  measurable, that  $(S_t)$  is right continuous, and that  $(X, S)$  is progressively measurable relative to  $\mathcal{H}$ . So, it is meaningful to talk of strong Markov property for  $(X, S)$ . This property is of particular interest for stopping times  $T$  whose graphs  $[T]$  are contained in  $M$ , i.e.,  $T(\omega) \in M(\omega)$  for almost every  $\omega \in \{T < \infty\}$ . Then,  $S_T = 0$ , and the strong Markov property at  $T$  may be rephrased as follows. For every state  $i \in E$  there exists a probability measure  $P^i$  on  $(\Omega, \mathcal{H}_\infty^0)$  such that

$$(2.6) \quad E[W \circ \theta_T | \mathcal{H}_T] = E^i[W] \quad \text{a.s. on } \{X_T = i\}$$

for any bounded  $\mathcal{H}_\infty^0$  measurable random variable  $W$  and any stopping time  $T$  such that  $[T] \subset M$ . Here and below we assume the existence of a family  $(\theta_t)$  of shift operators defined on  $\Omega$  such that

$$X_t \circ \theta_u = X_{t+u}, \quad S_0 \circ \theta_u = S_u$$

for all  $t \in \mathbb{R}_0$  and  $u \in \mathbb{R}_+$ . Note that, then,  $S_t \circ \theta_u = S_{t+u}$  also.

(2.7) **EXAMPLE.** Consider the process  $X$  of Example (1.1), and put  $S_0 = 0$ . Then, it is well known that  $S$  is a strong Markov process (see Meyer (1970), for instance). For any stopping time  $T$ , on  $\{S_T > 0\}$  we have  $X_T = 1$ . There is no stopping time  $T$  such that  $[T] \subset L$ , other than  $T = \infty$ . Therefore, for any stopping time  $T$ , on  $\{T < \infty, S_T = 0\}$  we have  $X_T = 0$ . It follows that the strong Markov property of  $S$  implies that of  $(X, S)$ ; that is, Axiom (2.5) is satisfied. It is easy to check the other three axioms. So,  $X$  is semimarkov.

(2.8) **Chung processes.** Suppose  $X$  is a Chung process, put  $S_0 = 0$ . Then right lower semicontinuity implies (2.2). Point  $\varphi$  being “fictitious,” (2.3) is true. Axiom (2.4) follows from right lower semicontinuity again. And it is known that (2.5) holds. So,  $X$  is semimarkov.

(2.9) **Regenerative sets.** Let  $K_0$  be a regenerative set in the sense of Maisonneuve (1971) (Markov random set is another equivalent term; and the set on which a regenerative phenomenon in the sense of Kingman (1972) is equal to 1 is a particular regenerative set). Define  $X$  to be 0 on  $K_0$  and 1 on  $\mathbb{R}_0 \setminus K_0$ . The resulting process  $X$  is semimarkov. Example (1.1) is a special case.

We end this section by giving a regularization of sample paths, due to Lévy (1954), which simplifies certain questions.

For each  $i \in \bar{E}$  and  $\omega \in \Omega$  define

$$(2.10) \quad K_i(\omega) = \{t > 0: X_t(\omega) = i\}.$$

The random set  $K_i$  is called the *constancy set* for  $i$ .

(2.11) **DEFINITION.** A point  $i \in \bar{E}$  is said to be *instantaneous* if for a.e.  $\omega$  the set  $K_i(\omega)$  has an empty interior. It is said to be a *holding point* if for a.e.  $\omega$  the set  $K_i(\omega)$  is a countable union of intervals each of which has a finite positive length. It is said to be *absorbing* if, for a.e.  $\omega$ ,  $K_i(\omega)$  is either empty or consists of one infinite interval.

By Fubini’s theorem, Axiom (2.3) implies that the point  $\varphi$  is instantaneous. If  $X$  is a Chung process then every point  $i$  is either instantaneous or holding or absorbing. However, in general, a semimarkov process may have states which are neither instantaneous nor holding nor absorbing. The following “splitting of states” is to eliminate this unpleasantness.

We define

$$(2.12) \quad R(\omega) = \inf M(\omega), \quad R_t(\omega) = \inf \{s > t: s \in M(\omega)\} - t.$$

Then,  $R_t$  is the remaining time to be spent in the state being occupied at  $t$ , and  $R = R_0$  and  $R_t = R \circ \theta_t$ .

(2.13) PROPOSITION. For every  $\omega \in \Omega$  and  $t > 0$  define

$$\begin{aligned} X'_t(\omega) &= ai && \text{if } R_t(\omega) = +\infty \\ &= hi && \text{if } R_t(\omega) \in (0, \infty) \\ &= ii && \text{if } R_t(\omega) = 0 \end{aligned}$$

when  $X_t(\omega) = i$ . Then, the process  $X'$  is again a semimarkov process, and every state  $i'$  is either instantaneous, or holding and not absorbing, or absorbing.

This proposition can be found in Lévy (1954). Throughout the remainder of this paper we assume that the regularization implicit in this proposition is already carried out, so that we may assume the following:

(2.14) Regularity. Each point  $i \in \bar{E}$  of the process  $X$  is either absorbing or holding or instantaneous. Further, the defining conditions in (2.11) hold for every  $\omega$  instead of almost every  $\omega$ .

**3. Behavior at a holding point.** Throughout this section  $i$  is a fixed holding point. For every  $\omega$ , the set  $K_i(\omega) = \{t > 0 : X_t(\omega) = i\}$  is a countable union of intervals. By Axiom (2.2b), each component interval has the form  $[ \ )$  except possibly the first which may have the form  $(0, t)$ . Since it is possible that  $K_i(\omega) \cap B$  have infinitely many components for even bounded intervals  $B$ , first we consider those component intervals of  $K_i(\omega)$  whose lengths (strictly) exceed  $\varepsilon$ .

For  $\varepsilon > 0$  consider the collection of instants  $t \in K_i(\omega)$  such that  $S_t(\omega) = \varepsilon$ . Let  $Q_{\varepsilon n}(\omega)$  be the  $n$ th such instant if it exists, and set  $Q_{\varepsilon n}(\omega) = +\infty$  if not. Note that each  $Q_{\varepsilon n}$  is a stopping time, and that

$$(3.1) \quad X_{Q_{\varepsilon n}} = i, \quad S_{Q_{\varepsilon n}} = \varepsilon \quad \text{on } \{Q_{\varepsilon n} < \infty\}.$$

Recall the definition (2.12) of  $R$ , note that  $\varepsilon + R \circ \theta_{Q_{\varepsilon n}}$  is the length of that component interval of  $K_i$  which straddles  $Q_{\varepsilon n}$ , and that  $X_R \circ \theta_{Q_{\varepsilon n}}$  is the next state to be visited. By the strong Markov property at  $Q_{\varepsilon n}$ , for  $A \subset \bar{E}$  and  $B \subset \mathbb{R}_\varepsilon = (\varepsilon, \infty)$  Borel, we have

$$(3.2) \quad \begin{aligned} P\{X_R \circ \theta_{Q_{\varepsilon n}} \in A, \varepsilon + R \circ \theta_{Q_{\varepsilon n}} \in B \mid \mathcal{H}_{Q_{\varepsilon n}}\} \\ = L_\varepsilon(i, A, B) \quad \text{a.s. on } \{Q_{\varepsilon n} < \infty\} \end{aligned}$$

for some measure  $L_\varepsilon(i, \cdot)$  on the Borel subsets of  $\bar{E} \times \mathbb{R}_\varepsilon$ , where we write  $L_\varepsilon(i, A, B)$  for  $L_\varepsilon(i, A \times B)$ , and below we will write  $L_\varepsilon(i, j, B)$  for  $L_\varepsilon(i, \{j\} \times B)$ .

We now explore the meaning and shape of  $L_\varepsilon(i, \cdot)$ . Consider the transition semigroup  $(P_t)$  of the Markov process  $(X_t, S_t)_{t>0}$ , and let  $P_{t,s}$  be the probability law of the process corresponding to the entrance law  $P_t((i, s), \cdot)$ ,  $t > 0$ . It is clear that

$$(3.3) \quad L_\varepsilon(i, A, B) = P_{i\varepsilon}\{X_R \in A, R \in B - \varepsilon\}, \quad A \subset \bar{E}, B \in \mathcal{R}_\varepsilon.$$

Concerning the shape of  $L_\varepsilon$  we have the following:

(3.4) PROPOSITION. There is a measure  $L(i, \cdot)$ , on the Borel subsets of  $\bar{E} \times \mathbb{R}_0$ ,

unique up to a multiplicative constant, such that  $L(i, i, \mathbb{R}_0) = 0$ ,

$$(3.5) \quad n(i, t) = L(i, \bar{E}, \mathbb{R}_t) < \infty$$

for every  $t \in \mathbb{R}_0$ , and with  $0/0 = 0$ ,

$$(3.6) \quad L_\varepsilon(i, j, B) = L(i, j, B)/n(i, \varepsilon), \quad j \in \bar{E}, \varepsilon \in \mathbb{R}_0, B \in \mathcal{R}_\varepsilon.$$

PROOF. Define

$$n_s(i, t) = L_s(i, \bar{E}, \mathbb{R}_t) = P_{is}\{R > t - s\}, \quad 0 < s \leq t.$$

Let  $0 < s \leq t$  and  $B \in \mathcal{R}_t$ ; then, the event  $\{R \in B - s\}$  implies  $\{R > t - s\}$ , and on the latter set we have  $X_{t-s} = i$  and  $S_{t-s} = t$  a.s.  $P_{is}$ . Thus, applying the Markov property at  $t - s$  to the right side of the equation (3.3) written for  $\varepsilon = s$ , we obtain

$$(3.7) \quad L_s(i, j, B) = n_s(i, t)L_t(i, j, B), \quad 0 < s \leq t, B \in \mathcal{R}_t,$$

and in particular, summing over  $j \in \bar{E}$  and putting  $B = \mathbb{R}_u$ ,

$$(3.8) \quad n_s(i, u) = n_s(i, t)n_t(i, u) \quad 0 < s \leq t \leq u.$$

Next we show that there exists  $a > 0$  such that  $n_b(i, a) > 0$  for all  $b < a$  unless  $K_i = \emptyset$  almost surely. If  $K_i$  is not a.s. empty, then there exists  $a > 0$  such that  $P\{Q_a < \infty\} > 0$ , where  $Q_a = Q_{a1}$  is the first time  $(X, S)$  enters  $(i, a)$ . Let  $b < a$ , and note that  $Q_a < \infty$  if and only if  $Q_{b_n} < \infty$  and  $R \circ \theta_{Q_{b_n}} > a - b$  for some  $n$ . Hence,

$$\begin{aligned} P\{Q_a < \infty\} &\leq \sum_n P\{Q_{b_n} < \infty, R \circ \theta_{Q_{b_n}} > a - b\} \\ &= \sum_n P\{Q_{b_n} < \infty\}P_{ib}\{R > a - b\} \end{aligned}$$

by the strong Markov property applied at  $Q_{b_n}$  (see (3.2)). Now the positivity of the left side implies that  $n_b(i, a) = P_{ib}\{R > a - b\} > 0$ .

Finally, if  $K_i$  is almost surely empty we define  $L(i, \cdot) = 0$  identically; otherwise, we define

$$(3.9) \quad \begin{aligned} n(i, t) &= 1/n_i(i, a) && \text{if } 0 < t < a, \\ &= n_a(i, t) && \text{if } t \geq a; \end{aligned}$$

and set

$$(3.10) \quad L(i, j, B) = n(i, \varepsilon)L_\varepsilon(i, j, B) \quad \text{if } B \in \mathcal{R}_\varepsilon, \quad \varepsilon > 0.$$

It follows from (3.8) and (3.9) that we have  $n_s(i, t) = n(i, t)/n(i, s)$  for all  $0 < s \leq t$ , and this together with (3.7) shows that the definition (3.10) is unambiguous. Now the desired assertions (3.5) and (3.6) follow from (3.9) and (3.10).

The measure  $L(i, \cdot)$  is the  $i$ -entry of a kernel  $L$  which plays the same role in semimarkov processes as Lévy kernels do in the theories of Hunt processes and Markov additive processes (see Çinlar (1975)). Moreover, the functions  $n(i, \cdot)$  figure as densities for some measure  $N$  which is invariant for  $(X, S)$  (this is to explain the particular choice of letters  $L, n$ , etc.).

(3.11) **EXAMPLE.** Consider the semimarkov process  $X$  of Example (1.1). For the holding state 1 we have

$$L(1, 0, B) = \lambda(B), \quad L(1, 1, B) = 0.$$

(3.12) **EXAMPLE.** Suppose  $X$  is a Chung process and let  $i$  be a holding point. Then  $n(i, t) = c \cdot e^{-\lambda(i)t}$  for some finite constant  $\lambda(i)$  and some constant  $c > 0$ . Therefore,

$$(3.13) \quad L(i, j, dt) = cK(i, j)\lambda(i)e^{-\lambda(i)t} dt$$

for some numbers  $K(i, j) \geq 0, K(i, i) = 0$ . In terms of the generator  $G$  of  $X$  (that is,  $G(i, j)$  is the derivative at  $t = 0$  of the transition function  $P_t(i, j)$ ), we have

$$\lambda(i) = -G(i, i); \quad \lambda(i)K(i, j) = G(i, j), \quad j \neq i.$$

Returning to the general case we note that  $L(i, \cdot)$  and  $n(i, \cdot)$  are defined only up to multiplication by a constant. If the limit  $n(i, 0+)$  of  $n(i, t)$  as  $t \downarrow 0$  is finite, then a convenient normalization is effected by choosing the multiplicative constant so that

$$(3.14) \quad n(i) = \lim_{t \downarrow 0} n(i, t) = 1.$$

Otherwise, if  $n(i, 0+) = \infty$ , we set

$$(3.15) \quad n(i) = \lim_{t \downarrow 0} n(i, t) = +\infty.$$

(3.16) **DEFINITION.** A holding state is said to be stable if  $n(i) = 1$  and unstable if  $n(i) = \infty$ .

It follows from (3.13) that for a Chung process every holding state is stable. The following proposition shows that the behavior of a semimarkov process at a stable holding state is qualitatively the same as that of a Chung process at such a state. We will later see that the converse of this proposition is also true (see Proposition (3.30)).

(3.17) **PROPOSITION.** *Let  $i$  be a stable holding state. Then, for almost every  $\omega$ , the set  $K_i(\omega) \cap B$  has only finitely many connected components for any bounded interval  $B$ .*

**PROOF.** It is enough to show this for  $B = (s, s + t)$  where  $s$  is arbitrary and  $t > 0$  fixed. Choose  $t$  such that  $n(i, t) > 0$ . Define

$$(3.18) \quad T_{\epsilon n} = s + Q_{\epsilon n} \circ \theta_s, \quad U_{\epsilon n} = T_{\epsilon n} + R \circ \theta_{T_{\epsilon n}}.$$

Note that the number of component intervals  $K_i$  has in  $B = (s, s + t)$  is equal to, or one greater than,

$$(3.19) \quad N = \lim_{\epsilon \downarrow 0} \sum_n 1_B(T_{\epsilon n}),$$

and that

$$(3.20) \quad P\{N > k\} = \lim_{\epsilon \downarrow 0} P\{\sum_n 1_B(T_{\epsilon n}) > k\}.$$

So, we need to show that this goes to 0 as  $k \rightarrow \infty$ .



The event figuring on the right side of (3.20) implies the event that the lengths of the intervals  $(T_{\varepsilon 1}, U_{\varepsilon 1}), \dots, (T_{\varepsilon k}, U_{\varepsilon k})$  sum to something less than  $t$ , and this event in turn implies the event that each one of these  $k$  intervals is less than  $t$  in length. On the other hand,  $T_{\varepsilon 1}, T_{\varepsilon 2}, \dots$  are all stopping times. By the strong Markov property applied thereat, the lengths of the  $k$  intervals in question are i.i.d. with the common distribution function  $\varphi$  given by

$$(3.21) \quad \varphi(u) = 1 - n(i, \varepsilon + u)/n(i, \varepsilon), \quad u \geq 0.$$

Hence,

$$(3.22) \quad P\{N > k\} \leq \lim_{\varepsilon \downarrow 0} \left[ 1 - \frac{n(i, \varepsilon + t)}{n(i, \varepsilon)} \right]^k.$$

By hypothesis of stability for  $i$ ,  $n(i, \varepsilon) \rightarrow 1$  as  $\varepsilon \rightarrow 0$ . So, the limit on the right is  $[1 - n(i, t)]^k$ , which goes to 0 as  $k \rightarrow \infty$  since  $n(i, t) > 0$  by choice.

For the fixed state  $i$  under examination, define

$$(3.23) \quad V = \inf M \cap K_i, \quad V_t = t + V \circ \theta_t, \quad t > 0.$$

We call  $V$  the time of first visit to  $i$ ,  $V_t$  is the time of first visit to  $i$  after  $t$ . Note that  $V_t(\omega)$  differs from the first hitting time  $H_t(\omega)$  of  $i$  after  $t$  whenever  $X_t(\omega) = i$ : If  $X_t(\omega) = i$  then  $(t, t + \varepsilon) \subset K_i(\omega)$  for some  $\varepsilon > 0$ , which implies that  $V_t(\omega) \geq t + \varepsilon$  whereas  $H_t(\omega) = t$ .

In interpreting the following recall that  $X_V \circ \theta_t = X_{V_t}$  is the state at the time of first visit to  $i$  after  $t$ .

(3.24) THEOREM. For almost every  $\omega$  and for every  $t > 0$ ,

$$\begin{aligned} X_V \circ \theta_t(\omega) &= i && \text{if } i \text{ is stable,} \\ X_V \circ \theta_t(\omega) &\in \bar{E} \setminus \{i\} && \text{if } i \text{ is unstable.} \end{aligned}$$

PROOF. *Case of stable  $i$ .* By the preceding theorem, for every  $\omega$  in a set  $\Omega'$  of full measure, the left end points of the component intervals of  $K_i(\omega)$  form a discrete set (i.e. every point is isolated). Therefore,  $V_t(\omega)$  must be one of those left end points for  $\omega \in \Omega'$ , and by axiom (2.2 a),  $X(\omega)$  is equal to  $i$  at  $V_t(\omega)$ .

*Case of unstable  $i$ .* Let

$$(3.25) \quad p(s) = P\{X_V \circ \theta_t = i, R \circ \theta_V \circ \theta_t > s\}.$$

By the strong Markov property applied at  $V_t + s$ , recalling that

$$P_{is}\{R > u - s\} = n(i, u)/n(i, s),$$

we have

$$(3.26) \quad p(u) = p(s)n(i, u)/n(i, s), \quad 0 < s < u.$$

Since  $i$  is unstable,  $n(i, s) \rightarrow \infty$  as  $s \rightarrow 0$ . Since  $p(s)$  is bounded, (3.26) can hold only if

$$(3.27) \quad p(s) = 0 \quad \text{for all } s > 0.$$

Since  $i$  is a holding state,

$$R \circ \theta_V \circ \theta_t = R_{V_t} > 0 \quad \text{on } \{X_V \circ \theta_t = i\}.$$

Therefore, (3.27) can hold only if (cf. (3.25) also)

$$(3.28) \quad P\{X_V \circ \theta_t = i\} = 0.$$

This shows that for fixed  $t$  there is a negligible set  $\Omega_t$  such that  $X_V \circ \theta_t \neq i$  outside  $\Omega_t$ . Let

$$(3.29) \quad \Omega_0 = \bigcup_{t \text{ rational}} \Omega_t.$$

Then  $\Omega_0$  is negligible. Pick  $\omega \notin \Omega_0$  and  $t > 0$ . If  $V_t(\omega) = t$  then  $X_V \circ \theta_t(\omega) \neq i$  (since  $X_t(\omega) = i$  implies  $V_t(\omega) > t$ ). If  $V_t(\omega) > t$ , then there are rationals  $r$  in  $(t, V_t(\omega))$ , and for any such  $r$  we have  $V_r(\omega) = V_t(\omega)$ . So,  $X_V \circ \theta_t(\omega) = X_V \circ \theta_r(\omega) \neq i$  since  $\omega \notin \Omega_r$ . This completes the proof.

The following shows that the converse of Proposition (3.17) is also true; a holding state  $i$  is stable if and only if, for almost every  $\omega$ ,  $K_i(\omega)$  has only finitely many component intervals in a bounded time interval.

(3.30) **PROPOSITION.** *Let  $i$  be an unstable holding state. Then, for a.e.  $\omega$ , between any two component intervals of  $K_i(\omega)$  there is a third component interval of  $K_i(\omega)$ .*

**PROOF.** Let  $\Omega_0$  be the negligible set defined by (3.29), and suppose  $\omega \notin \Omega_0$ . Let  $[p, q]$  and  $[u, v]$  be two component intervals of  $K_i(\omega)$ , suppose  $q < u$ . Pick  $t \in (q, u)$ . Clearly,  $V_t(\omega) \leq u$ . But the equality cannot hold, since then  $X(V_t(\omega), \omega) = X_u(\omega) = i$  which contradicts the choice of  $\omega$ . So,  $V_t(\omega) < u$  and therefore there must exist  $r, s \in (V_t(\omega), u)$  such that  $[r, s]$  is a component interval of  $K_i(\omega)$ .

The next corollary follows from Theorem (3.24) and the preceding proposition. Recall that  $L$  is the set of all left end points of the intervals contiguous to  $M$ . Therefore  $L \cap K_i$  is the set of all left end points of the component intervals of  $K_i$ .

(3.31) **COROLLARY.** *Let  $i$  be an unstable holding point. Then, for a.e.  $\omega$  and every  $t > 0$ , if  $(t, t + \epsilon)$  contains one point of  $L(\omega) \cap K_i(\omega)$ , it contains infinitely many points of  $L(\omega) \cap K_i(\omega)$ .*

We have seen in Theorem (3.24) that, at the time of a visit to an unstable state  $i$ ,  $X$  is always at some other state. From the definition of visiting time and the preceding corollary, it follows that at such a time  $X$  must be in an instantaneous state.

(3.32) **COROLLARY.** *Let  $i$  be an unstable holding state. Then, for any  $t > 0$ ,  $X_V \circ \theta_t$  is in the set of instantaneous states almost surely.*

It follows that, if the semimarkov process  $X$  has an unstable state, then it must also have some instantaneous states. This, however, does not require that the state space be infinite. In Example (1.1), for instance, state 1 is an unstable holding state.

(3.33) **REMARK.** As before, consider the transition function  $(P_t)$  of the Markov process  $(X, S)$ , and let  $P_{is}$  be the probability law corresponding to the entrance law  $P_t((i, s), \cdot)$ ,  $t > 0$ . If  $i$  is a holding state and  $s > 0$ , then it is clear that

$$(3.34) \quad \lim_{t \rightarrow 0} (X_t, S_t) = (i, s) \quad \text{a.s. } P_{is}.$$

If  $i$  is a stable holding point, the same is true for  $s = 0$  also. However, if  $i$  is an unstable point, this limit no longer exists for  $s = 0$ . In fact, when  $i$  is unstable, the point  $(i, 0)$  is a branching point for the semigroup  $(P_t)_{t > 0}$  and the branching distribution is concentrated on the set  $\{(j, 0) : j \in \bar{E} \text{ instantaneous}\}$ .

**4. Instantaneous states.** Throughout this section  $i$  will be a fixed instantaneous state in  $E$ . This leaves out the instantaneous point  $\varphi$  about which, for reasons which will become clear, there is not much to report.

Consider the set  $K_i = \{t : X_t = i\}$ . For each  $\omega$ , we define  $K_i^-(\omega)$  to be the set of all limit points of  $K_i(\omega)$  from the right side, that is,  $t \in K_i^-(\omega)$  if and only if there is a sequence  $(t_n) \subset K_i(\omega)$  which is strictly decreasing to  $t$ . Define  $K_i^+(\omega)$  similarly as the set of all limit points from the left. Finally, let

$$(4.1) \quad K_i^m(\omega) = K_i(\omega) \setminus (K_i^-(\omega) \cup K_i^+(\omega));$$

that is,  $K_i^m$  is obtained from  $K_i$  by removing from  $K_i$  those nonisolated points of  $K_i$  which are isolated on the right.

(4.2) **LEMMA.** (a) *The random set  $K_i$  is progressively measurable (relative to  $\mathcal{H}$ ).* (b) *For almost every  $\omega$ ,  $K_i(\omega)$  is right closed, that is,  $K_i^-(\omega) \subset K_i(\omega)$ .* (c) *The random set  $K_i^m$  is progressive.*

**PROOF.** (a) The first statement follows from the progressive measurability of  $X$  itself.

(b) By the definition of an instantaneous state  $K_i \subset M$  almost surely. By Axiom (2.2a),  $L \cap K_i = \emptyset$  almost surely; hence,  $K_i \subset M \setminus L$ . Now Axiom (2.2b) applies to show that  $K_i(\omega)$  is right closed for a.e.  $\omega$ .

(c) Since  $K_i$  is progressive, so are  $K_i^+$  and  $K_i^-$  (see Dellacherie (1972), page 126). Now (4.1) implies the same for  $K_i^m$ .

(4.3) **THEOREM.** *There is a right continuous increasing additive process  $(A_s)_{s \in \mathbb{R}_+}$  (where  $A_0$  is not necessarily zero) such that, for a.e.  $\omega$ ,*

$$(4.4) \quad K_i^m(\omega) = \{t : A_s(\omega) = t \text{ for some } s\}.$$

Proof is immediate from the characterization theorem of Maisonneuve (1971) for regenerative sets whose conditions on  $K_i^m$  are satisfied by the following facts:  $K_i^m$  is right closed and progressive, and is “minimal,” i.e., the left end point of a contiguous interval belongs to  $K_i^m$  only if it is isolated. For every  $t \in K_i^m$ , we have  $X_t = i$  and  $S_t = 0$  (the latter is because  $K_i^m \subset K_i \subset M$ ). Therefore, by the strong Markov property, for any stopping time  $T$  with  $[T] \subset K_i^m$ , the past  $\mathcal{H}_T$  is independent of the future  $\sigma(X_{T+t}; t \geq 0)$  which includes the “future of  $K_i^m$  after  $T$ .”

It follows from the well-known facts about increasing additive processes that

$$(4.5) \quad E[\exp\{-\lambda(A_s - A_0)\}] = \exp[-s\lambda N^\lambda(i)],$$

where

$$(4.6) \quad \lambda N^\lambda(i) = \lambda a(i) + \int_{(0, \infty]} (1 - e^{-\lambda y}) L(i, dy)$$

for some constant  $a(i) \in \mathbb{R}_+$  and some Lévy measure  $L(i, \cdot)$  on  $\bar{\mathbb{R}}_0 = (0, \infty]$ . We define (recall that  $\bar{\mathbb{R}}_t = (t, \infty]$ )

$$(4.7) \quad n(i, t) = L(i, \bar{\mathbb{R}}_t), \quad t \geq 0;$$

then  $N^\lambda(i)$  is the Laplace transform of the measure  $N(i, \cdot)$  which has an atom of weight  $a(i)$  at  $t = 0$  and is absolutely continuous on  $(0, \infty)$  with density  $n(i, \cdot)$ .

(4.8) **DEFINITION.** The instantaneous state  $i$  is said to be *attractive* if  $n(i, 0) = +\infty$  and *repellent* if  $n(i, 0) < \infty$ . If  $i$  is attractive, it is further said to be *light* if  $a(i) = 0$  and *heavy* if  $a(i) > 0$ . State  $i$  is said to be *recurrent* if  $n(i, +\infty) = 0$  and *transient* if  $n(i, \infty) > 0$ .

The next proposition provides the meanings behind these definitions. We omit the proof, which follows from Theorem (4.3), the well-known facts about additive processes, and the observations that  $K_i \setminus K_i^m$  is at most countable and that its every point is a limit point of  $K_i$ .

(4.9) **PROPOSITION.** *The following statements are true for a.e.  $\omega$  in  $\{K_i \neq \emptyset\}$ .*

(a) *If  $i$  is attractive  $K_i(\omega)$  is everywhere dense in itself, and therefore, is perfect (in the right-topology). If  $i$  is repellent, then every point of  $K_i(\omega)$  is isolated, and therefore,  $K_i(\omega)$  is at most countable.*

(b) *If  $i$  is repellent, or attractive and light, then the Lebesgue measure of  $K_i(\omega)$  is zero. If  $i$  is heavy, then the Lebesgue measure of  $K_i(\omega)$  is strictly positive.*

(c) *If  $i$  is recurrent then  $K_i(\omega)$  is unbounded; if  $i$  is transient then  $K_i(\omega)$  is bounded.*

In the case of a repellent state, if  $T_0, T_1, \dots$  are the successive points of  $K_i$ , we see that each  $T_n$  is a stopping time and the process  $(T_n)$  is a delayed (and possibly transient) renewal process. If  $X$  is a Chung process, then every instantaneous state is attractive and heavy. However, repellent and light attractive states do come up within the boundary theory for Chung processes. The behavior at a repellent state is very close to that of a Chung process at a nonsticky boundary point (though not exactly the same, as we shall see below), and the behavior at a light attractive state is the same as that of a Chung process at a sticky boundary point (for the terms see Chung (1968)).

The following clarifies the picture at a repellent state.

(4.10) **THEOREM.** *Let  $i$  be a repellent state. Then, for almost every  $\omega$  and for every  $t \in K_i(\omega)$ , the path  $X(\omega)$  admits  $\varphi$  as a limit point from the right at  $t$ ; moreover, if  $X(\omega)$  admits  $j \in E$  as a limit point from the right side at  $t$ , then  $j$  must be an unstable holding point.*

REMARK. It is possible to have more than one unstable state as limit points from the right.

PROOF. Let  $\Omega'$  be the set on which  $K_i \subset M \setminus L$  and  $K_j \supset K_j^-$  for every stable holding state  $j$  and for every instantaneous state  $j$ . By the definition of an instantaneous state  $i$ , by Proposition (3.17) on  $K_j$  for a stable holding state, and by Lemma (4.2) concerning  $K_j$  for  $j$  instantaneous, we have that  $P(\Omega') = 1$ .

Let  $\omega \in \Omega'$  and let  $t \in K_i(\omega)$ . Since  $t \in M(\omega) \setminus L(\omega)$ , there is a sequence  $(t_n) \subset M(\omega)$  strictly decreasing to  $t$ . If the sequence  $(X_{t_n}(\omega))$  has no limit points in  $E$ , then there is a subsequence  $(X_{s_n}(\omega))$  converging to  $\varphi$  since the state space  $\bar{E}$  is compact.

Suppose that  $(X_{t_n}(\omega))$  admits  $j \in E$  as a limit point for some sequence  $(t_n)$  decreasing to  $t$ . Since  $\omega \in \Omega'$ , the facts that  $t \in K_j^-(\omega)$  and  $t \in K_i(\omega)$  imply that  $j$  must be unstable. Let  $V_s^j$  be defined by (3.23) for the present state  $j$ ; and consider the sequence  $(s_n) = (V_{t_n}^j(\omega))$ . By Corollary (3.32),  $X_{s_n}(\omega)$  is instantaneous for each  $n$ . Hence, the sequence  $(X_{s_n}(\omega))$  cannot have any limit points in  $E$ , because  $(s_n)$  decreases to  $t$  and  $\omega \in \Omega'$ . So,  $(X_{t_n}(\omega))$  must admit  $\varphi$  as a limit point.

The preceding proof goes through for  $t \in K_i(\omega)$  where  $i$  is attractive and  $t$  is isolated on the right. We put this next along with some other supplementary facts about such points  $t$ . Note that Theorem (4.3) specifies the structure of  $K_i^m$  completely, and hence the next proposition completes the picture of  $K_i$ .

(4.11) PROPOSITION. Suppose  $i$  is attractive, and let  $L_i = K_i \setminus K_i^m = K_i^- \setminus K_i^-$ .

- (a) The random set  $L_i$  is progressive.
- (b) If  $T$  is a stopping time such that  $[T] \subset L_i$ , then  $T = +\infty$  almost surely.
- (c) For a.e.  $\omega$ ,  $L_i(\omega)$  is countable; if  $t \in L_i(\omega)$ , then there is  $(t_n) \downarrow t$  such that  $X_{t_n}(\omega) \rightarrow \varphi$ ; if  $t \in L_i(\omega)$  and if there is  $(t_n) \downarrow t$  such that  $X_{t_n}(\omega) \rightarrow j$  for some  $j \in E$ , then  $j$  is unstable.

PROOF. (a) follows from Dellacherie (1972), page 126; the proof of (c) is the same as that of Theorem (4.10). To show (b) let  $T$  be such a stopping time. Then,  $[T] \subset K_i$ , and the strong Markov property at  $T$  together with Proposition (4.9a) imply that  $T(\omega) \in K_i^-(\omega)$  for a.e.  $\omega \in \{T < \infty\}$ . Since  $L_i(\omega) \cap K_i^-(\omega) = \emptyset$ , and since  $T(\omega) \in L_i(\omega)$  for a.e.  $\omega \in \{T < \infty\}$  by hypothesis, we must have  $T = +\infty$  almost surely.

We have seen in the preceding propositions that if  $i$  is attractive and if a left end point  $t$  of an interval contiguous to  $K_i(\omega)$  is such that  $t \in K_i(\omega)$  and  $t \in K_j^-(\omega)$  for some  $j$ , then  $j$  must be unstable. The following strengthens this result by showing that, in that case,  $t \in K_j^-(\omega)$  also: in other words, there are infinitely many component intervals of  $K_j(\omega)$  in any interval  $(t - \epsilon, t)$  with  $\epsilon > 0$ .

(4.12) PROPOSITION. Suppose  $i$  is attractive, let  $L_i = K_i \setminus K_i^m$  as before. Then for almost every  $\omega$ ,

$$L_i(\omega) \cap K_j^-(\omega) \subset L_i(\omega) \cap K_j^-(\omega).$$

PROOF. Let  $V_t^j$  be the time of first visit to  $j$  after  $t$  defined by (3.23), but for state  $j$ . For any fixed  $t$ , this is a stopping time, and therefore,  $[V_t^j] \cap L_i = \emptyset$  almost surely by Proposition (4.11 b). Let

$$(4.13) \quad \Omega' = \bigcup_r \{V_r^j \notin L_i\}$$

where the union is over all rational numbers  $r$ . Then  $P(\Omega') = 1$ .

Choose  $\omega \in \Omega'$  and suppose  $t \in L_i(\omega) \cap K_j^-(\omega)$ . Then, for any  $s < t$ ,  $s \leq V_s^j(\omega) \leq t$ . On the other hand, if  $s$  is rational,  $V_s^j(\omega)$  cannot be  $t$  since  $\omega \in \Omega'$ . Choose a sequence  $r_n$  of rationals strictly increasing to  $t$ . Then,

$$r_n \leq V_{r_n}^j(\omega) < t$$

for every  $n$ , which implies that for each  $n$  there is  $s_n \in [r_n, t)$  such that  $s_n \in K_j(\omega)$ . Since  $r_n \uparrow t$ , this implies that  $t \in K_j^-(\omega)$ .

In view of the ease with which Theorem (4.3) is obtained, it is worth commenting upon its main ingredients. The obvious (and quite reasonable) ones are the progressive measurability and strong Markov property. Other than these, the most important is Axiom (2.2 b), which is equivalent to saying that  $K_i$  is right closed for every instantaneous state  $i$ . For Chung processes this follows from right lower semicontinuity, which in turn is made possible by separability and stochastic continuity. However, for semimarkov processes, it is difficult to obtain the property (2.2 b) by any reasonable hypothesis. The following is an example of a strong Markov process  $X$  which satisfies all the axioms except (2.2 b), and for which Theorem (4.3) fails, and the behavior at any of the three instantaneous states is radically different from that described above.

(4.14) EXAMPLE. Let  $(N_t^1), (N_t^2), \dots$  be independent Poisson processes each of which has parameter 1. Let  $\Omega'$  be the set of all  $\omega$  such that no two paths  $t \rightarrow N_t^n(\omega)$  and  $t \rightarrow N_t^m(\omega)$  have any jump time in common. Then  $P(\Omega') = 1$  as is well known. We now define

$$\begin{aligned} X_t(\omega) &= 1 && \text{if } \omega \in \Omega' \text{ and } t \text{ is a jump time of } N^m(\omega) \\ &&& \text{for some odd integer } m; \\ &= 2 && \text{if } \omega \in \Omega' \text{ and } t \text{ is a jump time of } N^m(\omega) \\ &&& \text{for some even integer } m; \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then  $P\{X_t = 0\} = 1$  for every  $t$ , and  $X$  is stochastically continuous. For almost every  $\omega$ ,  $X(\omega)$  admits all three possible values as limiting values from the right (and left) at any time  $t$ . So,  $X$  is separable trivially. Let  $\mathcal{H}_t^n = \sigma(N_s^n; s \leq t)$ , and set  $\mathcal{H}_t^0 = \mathcal{H}_t^1 \otimes \mathcal{H}_t^2 \otimes \dots$ . Then,  $X$  is strong Markov with respect to  $(\mathcal{H}_t^0)$ . Every state is instantaneous, every  $t$  is a time of discontinuity, Axiom (2.2 a) holds trivially, Axiom (2.2 b) does not hold. Theorem (4.3) is not true for any state.

It should be clear from the foregoing discussion that we cannot say much

about the structure of  $K_\varphi = \{t : X_t = \varphi\}$  even though  $\varphi$  is an instantaneous point. This is because the set  $K_\varphi$  is not necessarily right closed; and because we do not have the strong Markov property at stopping times  $T$  at which  $X$  may be in  $\varphi$ .

**5. Concluding remarks.** We mention the implications of several possible regularity conditions which are commonly introduced.

If every path  $X(\omega)$  is a separable function, then there can be no repellent states (but all other kinds are possible). If  $X$  is right separable, then there can be no repellent states, and  $K_i$  is minimal (i.e.  $L_i = \emptyset$ ) for every instantaneous state  $i$ . If  $X$  is right continuous, then there are no instantaneous states and no unstable states.

The role of “right lower semicontinuity” (RLSC) is somewhat ambiguous. If we define it as usual, i.e. by the condition that

$$(5.1) \quad X_t = \liminf_{s \downarrow t} X_s$$

for some ordering of states in  $E$  so that  $\varphi$  always corresponds to  $+\infty$ , that ordering becomes important. For example, in Example (1.1),  $X$  satisfies (5.1). However, if we interchange the labels of the two states (so that  $0' = 1$  and  $1' = 0$  and  $0' < 1'$ ), then  $X$  is not right lower semicontinuous; and moreover any attempt to make it so would set  $X_t' = 0'$  for all  $t$ .

We may define a property RLSC by requiring that, for any  $t$ ,  $X$  have at most one finite limiting value at  $t$  from the right, and that  $X$  be equal to that finite limit if it exists, and be otherwise  $\varphi$ . In the case of Chung processes this property “RLSC” is simply the right lower semicontinuity. For semimarkov processes, if  $X$  has the RLSC property, then there can be no unstable states and no repellent states.

Finally, we mention the following rectification of Lévy’s assertion alluded to in the introduction. Suppose there are no repellent states, and let  $D$  be the set of all unstable states. Put

$$\hat{D} = D \times \{1, 2, \dots\}, \quad \hat{E} = (E \setminus D) \cup \hat{D}.$$

Define a projection from  $\hat{E}$  onto  $E$  by setting

$$\pi(i) = i \quad \text{if } i \in E \setminus D, \quad \pi(i, n) = i \quad \text{if } (i, n) \in \hat{D}.$$

Then there exist a Chung process  $\hat{X}$  with state space  $\hat{E}$  and a strictly increasing continuous process  $A$  such that

$$X_t = \pi(\hat{X}_{A_t}).$$

Since a strictly increasing continuous time transformation leaves the qualitative structure of a path unaltered, this result singles out the unstable states as the main source of difference between semimarkov and Chung processes; each unstable state  $i$  is obtained by lumping together infinitely many stable states  $(i, n)$  of the Chung process  $\hat{X}$ .

If there are no unstable and no instantaneous states this is the result proved by Yackel (1968), who insured this state of affairs by assuming that the transition semigroup of  $(X, S)$  is "strong." The general result will appear in Çinlar (1975).

**Note added in proof.** In the paragraph containing (3.3) there is an implicit assumption that  $(X, S)$  has a transition semigroup  $(P_t)$ . It is possible that there is no such  $(P_t)$ —we are grateful to Professor Jean Jacod for pointing this out. However, by the strong Markov property at  $Q_s = Q_{s1}$ , there is a probability  $P_{is}$  such that for bounded  $W$

$$E[W \circ \theta_{Q_s} | \mathcal{H}_{Q_s}] = E_{is}[W] \quad \text{on } \{Q_s < \infty\}.$$

If  $Q_s = +\infty$  a.s. then set  $P_{is} = 0$  identically. Using this  $P_{is}$  in the remainder eliminates the implicit assumption mentioned.

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