The Annals of Probability
1976, Vol. 4, No. 4, 667-673

THE ENUMERATION OF COMPARATIVE PROBABILITY RELATIONS

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An attempt is made to enumerate the distinct antisymmetric comparative probability relations on sample spaces of n atoms. The results include an upper bound to the total number of such relations and upper and lower bounds to the size of the subset of the comparative probability relations admitting an agreeing probability measure as representation. The theoretical results are supplemented by computer enumerations for $n \leqq 6$. The upper and lower bounds for the case of agreeing probability measures are both

$$O(3^{\alpha n^2}) \quad \text{for} \quad \log_2(3^{\frac{1}{2}}) \leqq \alpha \leqq 1 .$$

0. Background. Let A and B denote events. By comparative probability (CP) we mean a theory of statements of the following forms:

(i) "A is more probable than B," written "$A > B$";
(ii) "A is as probable as B," written "$A \sim B$";
(iii) "A is at least as probable as B," written "$A \gtrsim B$."

Although we believe that CP is best formulated as a partial ordering between events, for the purposes of this note we assume it to be a complete, antisymmetric ordering of events $A, B, C, \cdots$, represented as subsets of a finite set Ω. For a complete order it suffices to axiomatize $\gtrsim$ since $>$, $\sim$ can be defined from $\gtrsim$ in the usual way. Discussions of complete CP are available in [1], [3], [5]—[12].

The basic axioms for complete CP are the following: For all subsets A, B, C of Ω

CP1. $$A \gtrsim B \quad \text{or} \quad B \gtrsim A .$$

CP2. $$A \gtrsim B \quad \text{and} \quad B \gtrsim C \Rightarrow A \gtrsim C .$$

CP3. It is false that $\varnothing \gtrsim \Omega$, where $\varnothing$ denotes the empty set.

CP4. $$A \gtrsim \varnothing .$$

CP5. If $A \cap (B \cup C) = \varnothing$, then $B \gtrsim C \Rightarrow A \cup B \gtrsim A \cup C$.

Elementary consequences of these axioms include:

(i) $A \supset B \Rightarrow A \gtrsim B$,
(ii) $A \gtrsim B \Rightarrow \bar{B} \gtrsim \bar{A}$, where $\bar{A}$ is the complement of A.

Received December 16, 1974; revised December 29, 1975.

AMS 1970 *subject classification.* Primary 60A05.

Key words and phrases. Comparative probability, qualitative probability, enumeration of orderings.

If $\Omega = \{w_1, w_2, \cdots, w_n\}$ is a sample space containing n atoms, then the indicator vector I_A of an event A is $I_A = \langle x_1, x_2, \cdots, x_n \rangle$, where

$$\begin{aligned} x_i &= 1 \qquad \text{if} \quad w_i \in A \\ &= 0 \qquad \text{if} \quad w_i \notin A . \end{aligned}$$

A useful representation for $\gtrsim$ is obtained by associating the difference vector $I_A - I_B$ to ordered pairs (A, B). Let $\mathbf{3}^n$ denote the space of n-tuples of elements $-1, 0, 1$. Define the subsets $\mathbf{C}^>$, $\mathbf{C}^\sim$ and $\mathbf{C}^<$ of $\mathbf{3}^n$ by

$$\begin{aligned} \mathbf{C}^> &= \{I_A - I_B : A > B\} ; \\ \mathbf{C}^\sim &= \{I_A - I_B : A \sim B\} ; \\ \mathbf{C}^< &= \{I_A - I_B : B > A\} = \{z : -z \in \mathbf{C}^>\} . \end{aligned}$$

For antisymmetric orders, $A \sim B \Rightarrow A = B$. Hence $\mathbf{C}^\sim = \{\mathbf{0}\}$, where $\mathbf{0}$ is the n-tuple of 0's.

An antisymmetric CP order is described by a subset $\mathbf{C}^>$ of $\mathbf{3}^n$ having the defining properties:

(i) if $x > \mathbf{0}$ then $x \in \mathbf{C}^>$ (we write $x > \mathbf{0}$ if $x = x_1 x_2 \cdots x_n$ and $x_i \geqq 0$ for all i and $x_j > 0$ for some j);
(ii) $\mathbf{3}^n = \mathbf{C}^> \cup \mathbf{C}^\sim \cup \mathbf{C}^<$;
(iii) $z \in \mathbf{C}^> \Rightarrow -z \in \mathbf{C}^<$;
(iv) It is false that $x + y + z = \mathbf{0}$ for any $x, y, z \in \mathbf{C}^>$.

We can classify CP orders by their relationship to quantitative probability measures as follows. The CP order $\gtrsim$ is *additive* if there is a probability measure P on Ω such that $A \gtrsim B \Leftrightarrow P(A) \geqq P(B)$ for all events A and B. P is then said to *agree* with $\gtrsim$. The CP order $\gtrsim$ is said to be *nonadditive* if it is not additive.

It is our aim to gain some enumerative knowledge about the basic family of antisymmetric CP orders on finite Ω. Our basic motivation is to advance the study of the largely neglected subject of comparative probability. We are particularly interested in the new class of models of uncertainty and chance phenomena given by the nonadditive relations. The novel properties of the nonadditive relations are discussed in [3, 4, 5, 7]. The object of our enumeration attempt is to assess the richness and complexity of the family of CP models and to prove (and here we were unsuccessful) that the additive subclass is of asymptotically (in the number of atoms) negligible size in comparison to the nonadditive subclass.

1. Upper bound to the number of antisymmetric CP orders. We derive an upper bound $\bar{\mu}_n$ to the number μ_n of complete antisymmetric CP relations defined on an n-atom sample space Ω.

THEOREM 1. $\mu_n < (n!)^2 2^{2^n - \frac{3}{4}n(n+1) - \frac{1}{2} + n \ln(2\pi^{-\frac{1}{2}})} = \bar{\mu}_n$.

PROOF. Without loss of generality assume the atoms $\{w_i\}$ are ordered so that $w_i < w_{i+1}$. To derive a recursive formula for an upper bound consider any antisymmetric complete CP ordering of the subsets $\{A_i\}$ of $\{w_1, \cdots, w_{n-1}\}$ enumerated in order of increasing likelihood,

$$\emptyset = A_0 < A_1 < \cdots < A_{2^{n-1}-1} = \{w_1, \cdots, w_{n-1}\} .$$

The ordering on n atoms derivable from $\{A_i\}$ arises by appropriately interleaving $\{A_i\}$ and $\{\{w_n\} \cup A_i\}$. We bound above the number of possible interleavings as follows. Consider a possible interleaving $\{B_i\}$ of $\{A_i\}$ and $\{\{w_n\} \cup A_i\}$, say, $\emptyset = B_0 = A_0 < B_1 = A_1 < \cdots < B_{2^n-1} = \{w_1, \cdots, w_n\}$. Define a corresponding binary random walk $\{S_i\}$ by

$$S_i = \sum_{j=1}^{i} x_j , \qquad \text{where} \quad x_j = 1 \quad \text{if} \quad w_n \notin B_{j-1}$$
$$= -1 \quad \text{if} \quad w_n \in B_{j-1} .$$

Observe the following:

(i) $S_j = j$ for $j < n$, since $w_n > w_j$ for $j < n$;
(ii) $S_i > 0$ when $0 < i < 2^n - 1$, since $\{w_n\} \cup A_i > A_i$;
(iii) $S_j = S_{2^n-j}$ for every j, since $C > D \Rightarrow \bar{D} > \bar{C}$.

Thus the number of possible interleavings is no more than the number of binary random walks $\{S_j\}$ beginning at $(n-1, n-1)$ that are positive for $j \leqq 2^{n-1}$. In fact this yields a strict upper bound because some of the walk paths correspond to interleavings inconsistent with the CP axioms. Taking into account the initial linearly rising portion, we are interested in the number λ_n of positive paths of $2^{n-1} - n + 1$ steps that start at $(0, n-1)$.

Let us denote $2^{n-1} - n + 1$ by N. Then, employing the reflection principle ([2], pages 69–70), we obtain for odd n

$$\lambda_n = \sum_{k>-(n-1);k\,\text{even}} \binom{N}{\frac{1}{2}(N+k)} - \sum_{k\geqq(n-1);k\,\text{even}} \binom{N}{\frac{1}{2}(N+k+(n-1))} \tag{1}$$

as the number of paths, with a similar result for n even. The summations in (1) collapse, yielding

$$\lambda_n = \sum_{k=2;k\,\text{even}}^{n-1} \binom{N}{\frac{1}{2}(N+k)} < \frac{1}{2}n\binom{N}{\frac{1}{2}N} .$$

By Stirling's formula ([2], page 52) we conclude that

$$\lambda_n < (2\pi^{-\frac{1}{2}})n2^{(2^{n-1}-\frac{3}{2}n)} . \tag{2}$$

We now have an upper bound λ_n on the number of n-atom antisymmetric complete CP relations that can be derived from any given $(n-1)$-atom relation by adding an nth and largest atom. Noting that $\mu_1 = 1$, a recursive application of the above argument yields

$$\mu_n < (n!) \prod_{i=2}^{n} \lambda_i ,$$

where the factor of $n!$ accounts for the $n!$ distinct relations on n-atoms derivable by permutations of the atoms. Using our upper bound to λ_n in (2) yields the desired

$$\mu_n < (n!)^2 2^{2^n - \frac{3}{4}n(n+1) - \frac{1}{2} + n \ln(2\pi^{-\frac{1}{2}})} .$$

□

The accuracy of our bound $\bar{\mu}_n$ to μ_n is open to doubt. Computer enumeration has revealed that

n	2	3	4	5	6
$\mu_n/n!$	1	2	14	546	169,444

Our upper bound to $\mu_n/n!$ for $n = 6$ is approximately 10^{12} and off by a factor of about 10^7. Nevertheless the dominant factor of 2^{2^n} may be of the right form. Complete computer evaluation of the case $n = 7$ appears to be out of the question and thus cannot be used to suggest the rate of growth.

The absence of a control on $\bar{\mu}_n$, as might have been provided by an interesting lower bound $\underline{\mu}_n$ to μ_n, is of concern to us. While we believe it likely that $\bar{\mu}_n$ can be trivially improved upon (i.e., the addition of terms of $O(n)$ to the exponent of $O(2^n)$), we are pessimistic about achieving a significant improvement. This pessimism is based upon exploration of several unsuccessful arguments and studies of a list of all 5-atom CP relations, as well as data on the number of 6-atom relations that are extensions of each of the 5-atom relations. We cannot survey our unsuccessful efforts here, and it is always possible that we overlooked the obvious. However, it is a poor prognosis for an easy solution that our computer generated exact enumeration yields an integer sequence that is not in Sloane [13].

2. Upper bound to the number of antisymmetric additive relations. Let σ_n denote the number of distinct antisymmetric CP relations admitting an agreeing probability distribution. To derive an upper bound $\bar{\sigma}_n$ to σ_n we invoke a result of Schäfli used in the study of linear threshold circuits.

LEMMA 1 ([14]). *An upper bound to the number of disjoint regions formed by passing m hyperplanes through the origin of n-space is*

$$2 \sum_{i=0}^{n-1} \binom{m-1}{i} .$$

To apply this result we note that the space of all probability distributions on n atoms is a compact subset of $(n - 1)$-dimensional space given by

$$S_{n-1} = \{(x_1 \cdots x_{n-1}) : x_i \geqq 0, \sum_{i=1}^{n-1} x_i \leqq 1\} .$$

We wish to identify all distributions that give rise to the same CP relation. Note that two distributions π, π', give rise to the same CP relation if and only if $\pi(A) > \pi(B) \Leftrightarrow \pi'(A) > \pi'(B)$ and $\pi(A) = \pi(B) \Leftrightarrow \pi'(A) = \pi'(B)$. Under the hypothesis of antisymmetry, $\pi(A) = \pi(B) \Leftrightarrow A = B$.

It is convenient at this point to shift to the representation of $\gtrsim$ as the subset $\mathbf{C}^>$ of $\mathbf{3}^n$. The order $\gtrsim$ is additive if for some vector $\pi \in [0, 1]^n$ we have $x \in \mathbf{C}^> \Leftrightarrow \pi \cdot x > 0$. Two vectors π and π' define the same order if and only if for all $x \in \mathbf{3}^n$, $\pi \cdot x > 0 \Leftrightarrow \pi' \cdot x > 0$.

There are $\frac{1}{2}(3^n - 1)$ points in $\mathbf{C}^>$. Two vectors π and π' in S_{n-1} give rise to the same order if they lie in the same region of S_{n-1} defined by the hyperplanes with normals the points in $\mathbf{C}^>$. Taking into account the reduction by one dimension to account for the irrelevant scale of the probability distribution, we see that we are interested in knowing into how many disjoint simplices the $\frac{1}{2}(3^n - 1)$ hyperplanes with normals having elements drawn from $\{-1, 0, 1\}^n$ divide the simplex S_{n-1} of probability distribution on n-atoms. Any two points in the same simplex correspond to probability vectors inducing the same CP relation, whereas points in different simplices correspond to probability vectors inducing different CP relations. An upper bound to the number of regions into which $\mathrm{C}^>$ divides S_{n-1} is given by the upper bound to the number of regions into which the hyperplanes defined by $\mathrm{C}^>$ divide R^{n-1}. Invoking Lemma 1 yields

THEOREM 2.

$$\sigma_n < 2 \sum_{i=0}^{n-2} \binom{\frac{1}{2}(3^n-1)-1}{i} .$$

Simplification of the upper bound yields the simpler but larger expression [14]

$$\sigma_n < 2\binom{\frac{1}{2}(3^n-1)-1}{n-1} \frac{\frac{1}{2}(3^n - 1) - n + 1}{\frac{1}{2}(3^n - 1) - 2(n - 1)} = \bar{\sigma}_n .$$

Since our interest is not in very small n, we have the simpler approximation

$$\bar{\sigma}_n \approx \frac{3^{n(n-1)}}{2^{n-2}(n-1)!} .$$

This upper bound to the number of additive antisymmetric CP relations, while very large, is negligible in comparison with our upper bound to the number of all antisymmetric CP relations; albeit in the absence of a lower bound to μ_n of the same order of magnitude as the upper bound, this comparison may be spurious.

3. Lower bound to the number of antisymmetric additive relations. Our lower bound $\underline{\sigma}_n$ is a consequence of

LEMMA 2. *The number of linearly separable n-argument* $\{0, 1\}$*-valued functions specified on some* $m \geqq 2$ *points of the n-cube* $\{-1, 0, 1\}^n$ *is at least*

$$4m^{(\log_2 m - 1)/2} .$$

PROOF. Theorem 1 of Winder [15] is Lemma 2 with $\{-1, 0, 1\}^n$ replaced by $\{0, 1\}^n$. Winder's proof of his Theorem 1 is by induction on m. We first evaluate a lower bound $\underline{R}_n{}^m$ to the minimum number of linearly separable subsets of $\{-1, 0, 1\}^n$ specified on m points for $m \leqq 4$. Two points X_1, X_2 can be separated in 4 ways as follows: X_1, X_2 on the negative side; X_1 on the negative side and X_2 on the positive side; X_1 on the positive side and X_2 on the negative side; X_1, X_2 both on the positive side. Hence $\underline{R}_n{}^2 = 4$, as was the case for Winder in $\{0, 1\}^n$. For $m = 3$ we have the least flexible possibility, not available in $\{0, 1\}^n$, that the three points are collinear. We now find $\underline{R}_n{}^3 = 6$. For $m = 4$ we find that $\underline{R}_n{}^4 = 8$. These three values satisfy the bound of Lemma 2. Hence the initial conditions of the induction process are validated. The induction step itself centers around equation (7) of Winder [15] and is easily seen to hold in $\{-1, 0, 1\}^n$ as well as in $\{0, 1\}^n$. These observations when combined with Winder's proof establish Winder's lower bound as valid in $\{-1, 0, 1\}^n$. □

The desired lower bound $\underline{\sigma}_n$ is given by

THEOREM 3. $\underline{\sigma}_n \geqq 2^{2-n} 3^{(n-1)((n-1)\log_2 3-1)/2}$.

PROOF. To use Lemma 2 we observe that if in the representation in $\mathbf{3}^n$ via indicators we delete the hyperplane corresponding to, say, the first coordinate equalling 0 then the remaining $2(3^{n-1})$ points are arranged on two parallel hyperplanes in n-space. Furthermore the $2(3^{n-1})$ are arrangeable as 3^{n-1} pairs of the form $(x, -x)$. Since only one of x, $-x$, can be in $\mathbf{C}^>$ we may take one representative from each pair. We now ask for a lower bound on the number of partitions by a hyperplane of the resulting set of 3^{n-1} points. Invoking Lemma 2 yields

$$4 \cdot 3^{(n-1)^2 \log_2 (3)^{\frac{1}{2}} - \frac{1}{2}(n-1)} .$$

However, some of the linear separations being counted are by hyperplanes with normals not all of whose coordinates are nonnegative, as required by the CP axioms. Since there are at most 2^n linear separations generable by changing signs in the coordinates of a given normal, the desired lower bound results after division by 2^n. □

We observe from a comparison of $\bar{\sigma}_n$ and $\underline{\sigma}_n$ that $\sigma_n = O(3^{an^2})$ for $\log_2 (3)^{\frac{1}{2}} \leqq a \leqq 1$.

Computer enumeration has revealed that

n	2	3	4	5	6
$\sigma_n/n!$	1	2	14	516	$<140{,}000$

These results are compatible with our bounds.

4. Conclusions. It is clear that much remains to be done in enumerating CP relations. Our results suggest, but do not prove, that the usual additive CP relations constitute an asymptotically negligible fraction of the large class of CP relations on n atoms. Of greatest immediate interest would be a lower bound to the number of all CP relations that would confirm the rate of growth as $O(2^{2^n})$.

We wish to acknowledge useful conversations with T. M. Cover.

REFERENCES

[1] DOMOTOR, Z. (1964). Probabilistic relational structures and their applications. Technical Report No. 144, Inst. for Math. Studies in Social Sciences, Stanford Univ.

[2] FELLER, W. (1957). *An Introduction to Probability Theory and Its Applications* **1**, 2nd ed. Wiley, New York.

[3] FINE, T. (1973). *Theories of Probability: An Examination of Foundations.* Academic Press, New York.

[4] FINE, T. (1976). An argument for comparative probability. In *Proc. Fifth International Congress of LMPS* (R. Butts and J. Hintikka, eds.). Reidel, Dordrecht.

[5] FINE, T. and KAPLAN, M. A. (1976). Extensions of comparative probability orders. To appear in *Ann. Probability.*

[6] FISHBURN, P. (1969). Weak qualitative probability on finite sets. *Ann. Math. Statist.* **40** 2118-2126.

[7] KAPLAN, M. A. (1974). Extensions and limits of comparative probability orders. Ph. D. thesis, Cornell Univ.

[8] KRAFT, C., PRATT, J. and SEIDENBERG, A. (1959). Intuitive probability on finite sets. *Ann. Math. Statist.* **30** 408–419.

[9] KRANTZ, D., LUCE, R., SUPPES, P. and TVERSKY, A. (1971). *Foundations of Measurement* I. Academic Press, New York.

[10] LUCE, R. (1967). Sufficient conditions for the existence of a finitely additive probability measure. *Ann. Math. Statist.* **38** 780–786.

[11] SAVAGE, L. (1954). *Foundations of Statistics* 30–38. Wiley, New York.

[12] SCOTT, D. (1964). Measurement structures and linear inequalities. *J. Math. Psychology* **1** 233–247.

[13] SLOANE, N. (1973). *A Handbook of Integer Sequences.* Academic Press, New York.

[14] WINDER, R. O. (1963). Bounds on threshold gate realizability. *IEEE Trans. Electronic Computers EC*-12 561–564.

[15] WINDER, R. O. (1970). Threshold logic asymptotes. *IEEE Trans. Computers* 350.

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