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# WEAK MARTINGALES AND STOCHASTIC INTEGRALS IN THE PLANE[1]

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This paper continues the development of a stochastic calculus for two-parameter martingales, and particularly for the two-parameter Wiener process. Whereas in an earlier paper we showed that two types of stochastic integrals were necessary for representing functionals and martingales of a Wiener process, introduction of two mixed area integrals is necessary to complete the stochastic calculus. These mixed integrals are weak martingales in the sense of Cairoli and Walsh, and are necessary in a general representation for weak martingales and transformations of weak martingales.

Stopping times are introduced for two-parameter processes, and a characterization of strong martingales in terms of stopping times is given.

**0. Introduction.** This paper continues recent work toward the development of a stochastic calculus in the plane (i.e., for the case where the time parameter is two dimensional) for continuous martingales in general and for the two parameter Wiener process in particular.

The basic references for this work are the fundamental paper by Cairoli and Walsh [3] and a previous paper by the present authors [4]. The reader is referred to [3] and [4] for further references.

In order to describe the contents of this paper we give, first, an incomplete definition for two parameter martingales, weak, 1- and 2-martingales. Precise definitions and references will be given in the next section. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space, $\mathscr{F}_{s,t}$, $0 \leqq s \leqq s_0$, $0 \leqq t \leqq t_0$, sub $\sigma$-fields of $\mathscr{F}$ such that $\mathscr{F}_{s_1,t_1} \subset \mathscr{F}_{s_2,t_2}$ if $s_1 \leqq s_2$ and $t_1 \leqq t_2$. In what follows assume $0 \leqq s_1 \leqq s_2 \leqq s_0$, $0 \leqq t_1 \leqq t_2 \leqq t_0$, and $X_{s,t}$ to be $\mathscr{F}_{s,t}$-measurable. Then $X_{s,t}$ is a martingale if $E(X_{s_2,t_2} | \mathscr{F}_{s_1,t_1}) = X_{s_1,t_1}$. $X_{s,t}$ is an adapted 1-martingale if for all fixed $t$ $E(X_{s_2,t} | \mathscr{F}_{s_1,t}) = X_{s_1,t}$ and an adapted 2-martingale if for all fixed $s$ $E(X_{s,t_2} | \mathscr{F}_{s,t_1}) = X_{s,t_1}$ (there is some difference between the definition of 1- and 2-martingales used in this paper and that of [3] as will be pointed out in the next section). $X_{s,t}$ is a weak martingale if

$$E\{X_{s_2,t_2} + X_{s_1,t_1} - X_{s_2,t_1} - X_{s_1,t_2} | \mathscr{F}_{s_1,t_1}\} = 0 .$$

In Section 2 we show that $X_{s,t}$ is a weak martingale if and only if it is the sum of a martingale, a 1-martingale and a 2-martingale (a discrete version of this

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result appears in [1]). A one (or two) martingale $X_{s,t}$ is said to be proper if for a fixed $s$ (resp. $t$) it is of bounded variation in $t$ (resp. $s$). It is shown that weak martingales satisfying certain restrictions can be decomposed into the sum of a martingale, a proper 1-martingale and a proper 2-martingale. In Section 3 we introduce a mixed area integral $\iint \psi(z, z')\, dM_z\, d\mu(z')$ where $\mu(z)$ is a (possibly random) function of bounded variation and $M_z$ is a martingale. It is shown that such integrals are proper 1 or 2 martingales. In some special cases this integral reduces to the mixed integral introduced by Cairoli and Walsh [3]. In Section 4 it is shown that every proper 1- or 2-martingale of the Wiener process satisfying a suitable differentiability condition can be represented as a mixed area integral.

Stopping times are introduced in Section 5 and used to give a characterization of strong martingales of the Wiener process.

**1. Preliminaries and notation.** Let $z = (s, t)$, $0 \leqq s \leqq s_0$, $0 \leqq t \leqq t_0$ denote points on a rectangle in the positive quadrant of the plane. $z_1 \prec z_2$ will denote $s_1 \leqq s_2$ and $t_1 \leqq t_2$. $R_{z_0}$ will denote the rectangle $\{z : 0 \prec z \prec z_0\}$. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space and $\{\mathscr{F}_z, z \in R_{z_0}\}$ be a family of sub $\sigma$-fields of $\mathscr{F}$ such that [3]:

($F_1$) $z \prec z'$ implies $\mathscr{F}_z \subset \mathscr{F}_{z'}$,

($F_2$) $\mathscr{F}_0$ contains all the null sets of $\mathscr{F}$,

($F_3$) for all $z$, $\mathscr{F}_z = \bigcap \mathscr{F}_{z'}$, $s' > s$, $t' > t$,

($F_4$) for each $z$, $\mathscr{F}_z^1$ and $\mathscr{F}_z^2$ are conditionally independent given $\mathscr{F}_z$, where $\mathscr{F}_z^1 = \mathscr{F}_{s,t_0}$, $\mathscr{F}_z^2 = \mathscr{F}_{s_0,t}$.

DEFINITION. A process $\{M_z, z \in R_{z_0}\}$ is a martingale if (1) $M_z$ is $\mathscr{F}_z$ adapted, (2) for each $z$, $M_z$ is integrable, (3) for each $z \prec z'$, $E(M_{z'} \mid \mathscr{F}_z) = M_z$.

Let $z = (s, t)$, $z' = (s', t')$, the condition $s < s'$, $t < t'$ will be denoted by $z \lll z'$. If $z \lll z'$, $(z, z']$ will denote the rectangle $(s, s'] \times (t, t']$ and if $X_z$ is a random process, $X(z, z']$ will denote $X_{s',t'} + X_{s,t} - X_{s',t} - X_{s,t'}$.

Several other notions of martingales were introduced in [3]. We follow here these definitions with the exception of the definition of *adapted* 1- *and* 2-*martingales* which differ from the definition of 1- and 2-martingales given in [3], as will be pointed out later. In the following definitions $X = \{X_z, z \in R_{z_0}\}$ is assumed, for each $z \in R_{z_0}$, to be integrable and $\mathscr{F}_z$ adapted.

DEFINITIONS. (a) $X_z$ is a *weak martingale* if $E\{X(z, z'] \mid \mathscr{F}_z\} = 0$ for every $z \lll z' \prec z_0$.

(b) $X_z$ is an adapted 1-martingale (2-martingale) if $X_z$ is $\mathscr{F}_z$ adapted and $\{X_{s,t}, \mathscr{F}_{s,t}\}$ is a martingale in $s$ for each fixed $t$ (in $t$ for each fixed $s$).

(c) $X_z$ is a strong martingale if it vanishes at the axes and $E\{X(z, z'] \mid \mathscr{F}_z^1 \vee \mathscr{F}_z^2\} = 0$ for every $z \lll z'$.

REMARK. $X_z$ was defined in [3] to be a 1-martingale if $X_z$ is $\mathscr{F}_z^1$ adapted and $E\{X(z, z'] \mid \mathscr{F}_z^1\} = 0$, $z \lll z'$, therefore $X_z$ is an adapted 1-martingale if and only if it is a 1-martingale, $\mathscr{F}_z$ adapted and $X_{s,0}$ is an $\mathscr{F}_{s,0}$-martingale.

*Some additional notational conventions.*

(a) The letters $z, \zeta, \eta$ will be used to denote points in $R_{z_0}$ whenever these letters appear with or without primes. It will always be assumed that $z_0 = (s_0, t_0)$, $0 < s_0 < \infty$, $0 < t_0 < \infty$ is a fixed point in the plane.

(b) We say that $z_1 \wedge z_2$ if $s_1 \leqq s_2$ and $t_2 \leqq t_1$ and that $z_1 \not\wedge z_2$ if $s_1 < s_2$ and $t_2 < t_1$, in either of these cases $z_1 \wedge z_2$ will denote the point $(s_1, t_2)$.

(c) $z_1 \vee z_2$ will denote the point $(\max(s_1, s_2), \max(t_1, t_2))$.

(d) The function $h(z, z')$ is defined as $h(z, z') = 1$ if $z \wedge z'$, and 0 otherwise.

(e) The region of integration for a stochastic integral is usually understood from the context and in such cases will be omitted from the notation. For example, if we write

$$X_z = \int \psi(\zeta, \zeta')\, dM_\zeta\, dM_{\zeta'}$$

it will be understood that the region of integration is $R_z \times R_z$.

## 2. The decomposition of weak martingales.

PROPOSITION 2.1. *$X_z$ is a weak martingale on $R_{z_0}$ if and only if it is expressible as $X_z = M_z^1 + M_z^2$ where $M_z^1$ is an adapted 1-martingale, $M_z^2$ is an adapted 2-martingale.*

PROOF. It follows directly from the definitions that every adapted 1- or 2-martingale is a weak martingale. Let

$$M_{s,t}^1 = E(X_{s_0,t} \mid \mathscr{F}_{s,t}) .$$

Note that $E(X_{s_0,t} \mid \mathscr{F}_{s,t}) = E(X_{s_0,t} \mid \mathscr{F}_{s,t_0})$ by assumption $(F_4)$ on the conditional independence property of the $\sigma$-fields. Therefore $M_{s,t}^1$ is an adapted 1-martingale.

Let $Y_z = X_z - M_z^1$. Then for $h > 0$, $(s, t+h) \prec z_0$,

$$\begin{aligned} E(Y_{s,t+h} &- Y_{s,t} \mid \mathscr{F}_{s_0,t}) \\ &= E(Y_{s,t+h} - Y_{s,t} \mid \mathscr{F}_{s,t}) \\ &= E\{X_{s,t+h} - X_{s,t} - E(X_{s_0,t+h} \mid \mathscr{F}_{s,t+h}) + E(X_{s_0,t} \mid \mathscr{F}_{s,t}) \mid \mathscr{F}_{s,t}\} \\ &= E\{X_{s,t+h} - X_{s,t} - X_{s_0,t+h} + X_{s_0,t} \mid \mathscr{F}_{s,t}\} \\ &= 0 \end{aligned}$$

since $X_{s,t}$ is a weak martingale. Therefore $Y_z = M_z^2$ is an adapted 2-martingale. □

REMARKS. (a) If the $\sigma$-fields $\mathscr{F}_{0,\infty}$ and $\mathscr{F}_{\infty,0}$ are trivial and $X_{0,0} = 0$ then $M_{0,t}^1 = M_{s,0}^1 = M_{0,t}^2 = 0$. (b) The decomposition of Proposition 1 is not unique. However, if $X_z = M_z^1 + M_z^2$ and also $X_z = N_z^1 + N_z^2$ then $M_z^1 - N_z^1$ and $M_z^2 - N_z^2$ are both 1- and 2-martingales. Therefore, by the converse to Proposition 1.1 of [3] (see the proof of Proposition 1.1 of [3]), $M_z^1 - N_z^1 = N_z^2 - M_z^2$ is a martingale.

Let Var $(X_{s,\cdot})$ denote the variation in the $t$ direction of $X_{s,t}$ over the interval $[0, t_0]$, similarly Var $(X_{\cdot,t})$ will denote the variation in the $s$ direction of $X_{s,t}$ over $[0, s_0]$.

DEFINITION. A weak martingale, in particular an adapted 1- or 2-martingale, will be said to be *regular* on $R_{z_0}$ if it satisfies the following conditions:

(a) For every fixed $t$, $X_{s,t}$ is a one parameter semimartingale in the parameter $s$ (i.e., the sum of a one parameter martingale relative to $\mathscr{F}_{s,t}$ and a function of bounded variation).

(b) For every fixed $s$, $X_{s,t}$ is a one parameter semimartingale in the $t$ parameter.

(c) Let $X_{s,t_0} = m(s) + \lambda(s)$ where $m(s)$ is an $\mathscr{F}_{s,t_0}$ martingale and $\lambda(s)$ is of bounded variation then $E\,\mathrm{Var}\,\lambda(\cdot) < \infty$.

(d) Let $X_{s_0,t} = n(t) + \rho(t)$ where $n(t)$ is an $\mathscr{F}_{s_0,t}$ martingale and $\rho(t)$ is of bounded variation then $E\,\mathrm{Var}\,\rho(\cdot) < \infty$.

DEFINITION. An adapted 1-martingale $M_z^1$ (2-martingale $M_z^2$) is said to be a *proper* 1- (2-) martingale if $E\,\mathrm{Var}\,(M^1_{s,\cdot}) < \infty$ for all $s \leqq s_0$ ($E\,\mathrm{Var}\,(M^2_{\cdot,t}) < \infty$ for all $t \leqq t_0$).

PROPOSITION 2.2. *Let $M_z^1$ be an adapted* 1*-martingale on $R_{z_0}$. If* $E\,\mathrm{Var}\,(M_{s_0,\cdot}) < \infty$ *then $M^1_{s,t}$ is proper on $R_{z_0}$ and, moreover,* $\mathrm{Var}\,(M_{s,\cdot})$ *is a one parameter positive submartingale relative to $\mathscr{F}_{s,t_0}$.*

PROOF. Let $\lambda(t) = M^1_{s_0,t}$, $\lambda(t) = \lambda(0) + \lambda^+(t) - \lambda^-(t)$ where $\lambda^+(t)$ and $\lambda^-(t)$ are nondecreasing and nonnegative and $\lambda^+(0) = \lambda^-(0) = 0$. Then

$$\begin{aligned} M^1_{s,t} &= E(\lambda_0 + \lambda^+(t) - \lambda^-(t) \mid \mathscr{F}_{s,t}) \\ &= E(\lambda_0 + \lambda^+(t) - \lambda^-(t) \mid \mathscr{F}_{s,t_0}) . \end{aligned}$$

Note that since $\lambda_t^+$ and $\lambda_t^-$ are increasing functions, so are $E(\lambda_t^+ \mid \mathscr{F}_{s,t_0})$ and $E(\lambda_t^- \mid \mathscr{F}_{s,t_0})$. Therefore

$$\mathrm{Var}\,(M^1_{s,\cdot}) \leqq E(\mathrm{Var}\,(M^1_{s_0,\cdot}) \mid \mathscr{F}_{s,t_0})$$

which proves the proposition. □

PROPOSITION 2.3. *Let $M_z^1$ be a regular* 1*-martingale; then $M_z^1 = M_z^{1,P} + M_z$ where $M_z^{1,P}$ is a proper* 1*-martingale and $M_z$ is a martingale.*

PROOF. Let $M^1_{s_0,t} = \lambda(t) + M(t)$ where $\lambda(t)$ is of bounded variation and $m(t)$ is a one parameter martingale. Let

$$X_z = E(\lambda(t) \mid \mathscr{F}_{s,t}) , \qquad Y_z = E(m(t) \mid \mathscr{F}_{s,t}) .$$

Then $X_z$ is a proper 1-martingale, $Y_z$ is a martingale and $M_z^1 = X_z + Y_z$. □

THEOREM 2.4 *Every regular weak martingale $X_z$ can be decomposed as*

$$X_z = M_z^{1,P} + M_z^{2,P} + M_z$$

*where $M_z^{1,P}$ is a proper* 1*-martingale, $M^{2,P}$ is a proper* 2*-martingale and $M_z$ is a martingale.*

PROOF. Let $X_{s_0,t} = \lambda_t + m_t$ where $\lambda$ is of bounded variation and $m_t$ is a one parameter martingale. Let

$$X_z^a = E(\lambda_t \mid \mathscr{F}_{s,t}) , \qquad X_z^b = E(m_t \mid \mathscr{F}_{s,t}) .$$

Let $Y_z = X_z - X_z^a - X_z^b$, note that $Y_{s_0,t} = 0$ for all $t \leqq t_0$. Let $Y_{s,t_0} = \rho_s + h_s$ where $\rho_s$ is of bounded variation and $h_s$ is a one parameter martingale. Such a decomposition is possible since $X^a_{s,t_0} + X^b_{s,t_0}$ is a one parameter martingale and $X_z$ is regular. Let

$$X_z^c = E(\rho_s \mid \mathscr{F}_{s,t}), \qquad X_z^d = E(h_s \mid \mathscr{F}_{s,t}).$$

Note that $X_z^e = X_z - X_z^a - X_z^b - X_z^c - X_z^d$ is a weak martingale, $X^e_{s_0,t} = 0$ for all $t \leqq t_0$ and $X^e_{s,t_0} = 0$ for all $s \leqq s_0$. It follows from the definition of weak martingales that $X_z^e = 0$ for all $z \prec z_0$. Setting $M^{1,P} = X^a$, $M^{2,P} = X^c$ and $M = X^b + X^d$ completes the proof. □

THEOREM 2.5. *If $M_z^{1,P}$ is a proper and continuous* 1-*martingale with $M^{1,P}_{s,0} \equiv 0$, then for $q > 1$*

$$E(\sup_{z \in R_{z_0}} |M_z^{1,P}|)^q \leqq \left(\frac{q}{q-1}\right)^q E(\operatorname{Var}(M^{1,P}_{s_0,\cdot}))^q .$$

*Similarly for a proper and continuous* 2-*martingale $M_z^{2,P}$ with $M^{2,P}_{0,t} \equiv 0$,*

$$E(\sup_{z \in R_{z_0}} |M_z^{2,P}|)^q \leqq \left(\frac{q}{q-1}\right)^q E(\operatorname{Var}(M^{2,P}_{\cdot,t}))^q$$

*for $q > 1$.*

PROOF. Since $M^{1,P}_{s,0} = 0$,

$$\sup_{t \leqq t_0} |M^{1,P}_{s,t}| \leqq \operatorname{Var}(M_{s,\cdot}) ;$$

therefore

$$\sup_{z < z_0} |M^{1,P}_{s,t}| \leqq \sup_{s \leqq s_0} \operatorname{Var}(M_{s,\cdot}) .$$

Since, by Proposition 2.2, $\operatorname{Var}(M_{s,\cdot})$ is a positive submartingale, Doob's maximal inequality yields for $q > 1$

$$\begin{aligned} E^{1/q}\{\sup_{z<z_0} |M_z^{1,P}|^q\} &\leqq E^{1/q}(\sup_{s \leqq s_0} \operatorname{Var}(M^{1,P}_{s,\cdot}))^q \\ &\leqq \frac{q}{q-1} E^{1/q}(\operatorname{Var}(M^{1,P}_{s_0,\cdot}))^q \end{aligned}$$

which proves the theorem. There is, obviously, a corresponding inequality for $q = 1$. □

REMARK. The original version of Proposition 2.2 did not include explicitly the conclusion that $\operatorname{Var}(M_{s,\cdot})$ is a submartingale. A reviewer called our attention to this fact and also pointed out that our proof of Theorem 2.5 can be replaced by the simplified proof given here.

**3. Mixed area integrals.** In [4] we introduced a stochastic integral over $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ $\iint \psi(z, z')\, dW(z)\, dW(z')$ (see also [3]). It seems that for the full development of a stochastic calculus in the plane still another integral is necessary. This integral will be of the form $\iint \psi(z, z')\, dW(z')\, dz$ where $\psi(z, z') = 0$ unless $z \curlywedge z'$ (or $(z' \curlywedge z)$) and will be a proper 1-martingale (2-martingale). A related integral has been introduced by Cairoli and Walsh in [3] and termed a mixed integral. The relation between the mixed integral of Cairoli and Walsh and the mixed area integral so defined in this section will be pointed out later.

Let $\mu_z$, $z \in R_{z_0}$ be a continuous random function of bounded variation adapted to $\mathscr{F}_z$, and let $\mu(A)$ be the signed measure induced on the Borel sets $A$ of $R_{z_0}$ by $\mu_z$. Let $|\mu|(A)$ denote the variation of the $\mu$ measure. That is, if $\mu(A) = \mu^+(A) - \mu^-(A)$ is the Jordan decomposition of $\mu$ then $|\mu|(A) = \mu^+(A) + \mu^-(A)$. We assume that the total variation of $\mu$ is bounded by a constant $\mu_0 < \infty$, i.e., $|\mu|(R_{z_0}) \leqq \mu_0$ a.s.

Let $M_z$ be a continuous martingale and let $A = (z_1, z_1']$, $B = (z_2, z_2']$ be rectangles such that if $z \in B$ and $z' \in A$, then $z \wedge z'$. Define, now, the process

$$X_z = \alpha M(A \cap R_z)\mu(B \cap R_z) \tag{3.1}$$

where $\alpha$ is $\mathscr{F}_{z_1 \vee z_2}$-measurable. Then

(a) $X_z$ is a continuous proper 1-martingale,

(b) The variation of $X_z$ is $|\alpha| \cdot |M(A)| \cdot \int_0^{t_0} |d_t \mu(B \cap R_{s_0,t})| \leqq |M(A)| \cdot |\alpha| \cdot |\mu|(B)$.

Let

$$\begin{aligned} \phi(z, z') &= \alpha \qquad \text{if } \ z \in B\,, \quad z' \in A \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and define

$$\iint \phi(\zeta, \zeta')\, dM_{\zeta'}\, d\mu_\zeta = X_z \tag{3.2}$$

where $X_z$ is as defined by (3.1).

To simplify notation assume $z_0 = (1, 1)$. Fix an integer $n$ and introduce a grid on $R_{z_0}$

$$z_{ij} = (2^{-n}i, 2^{-n}j)$$

where $i, j$ are integers $0 \leqq i, j \leqq 2^n$. Define the rectangle $\Delta_{ij} = (z_{ij}, z_{i+1,j+1}]$. Let $I_{\Delta_{ij}}(z)$ denote the indicator function of $\Delta_{ij}$. Define

$$\begin{aligned} \phi_{ij,kl}(z, z') &= \alpha I_{\Delta_{ij}}(z) I_{\Delta_{kl}}(z') \qquad \text{if } \ z_{ij} \curlywedge z_{kl} \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and $\alpha$ is bounded and $\mathscr{F}_{z_{ij} \vee z_{kl}}$ measurable. A function $\phi(z, z')$ is said to be a simple function if it is a finite sum of functions of the form $\phi_{ij,kl}(z, z')$ for some $n$. The extension of (3.2) to simple functions is obvious, and the resulting $X_z$ is a proper 1-martingale. Let $\phi$ be a simple function and for $\Delta_{ij} = (z_{ij}, z_{i+1,j+1}]$, let $M(\Delta_{ij}) = z_{i+1,j+1} + z_{ij} - z_{i+1,j} - z_{i,j+1}$. Then

$$X_{z_0} = \textstyle\sum_{ij,kl} \phi_{ij,kl}\, \mu(\Delta_{ij}) M(\Delta_{kl})\,. \tag{3.3}$$

If $M_z$ is a strong martingale then we have

$$\begin{aligned} EX_{z_0}^2 &= E\{\textstyle\sum_{ij,kl,i'j'} \phi_{ij,kl}\phi_{i'j',kl}\, \mu(\Delta_{ij})\mu(\Delta_{i'j'}) M^2(\Delta_{kl})\} \\ &= E \iiint_{R_{z_0}\times R_{z_0}\times R_{z_0}} \phi(z, z')\phi(\eta, z')\, d\mu_\eta\, d\mu_z\, d[M]_{z'}^1 \\ &= E \int_{R_{z_0}} \left(\int_{R_{z_0}} \phi(z, z')\, d\mu_z\right)^2 d[M]_{z'}^1 \end{aligned} \tag{3.4}$$

where $[M]_z^1$ is the unique $\mathscr{F}_{st}^1$ predictable process such that $\{M_z^2 - [M]_z^1, \mathscr{F}_{st}^1\}$

is a martingale in $s$ for $t$ fixed, and the passage from (3.3) to (3.4) follows from Proposition 1.7 of [3].

The variation of $X_{s_0,\theta}$, $0 \leqq \theta \leqq t_0$ is upper bounded by

$$\text{(3.5)} \qquad \operatorname{Var}(X_{s_0,\theta}, 0 \leqq \theta \leqq t_0) \leqq \sum_{ij} |\mu|(\Delta_{ij}) \cdot |\sum_{k,l} \psi_{ij,kl} M(\Delta_{k,l})| .$$

Setting $|\mu|(\Delta_{ij}) = (|\mu|)^{\frac{1}{2}} \cdot (|\mu|)^{\frac{1}{2}}$ we have by the Schwarz inequality

$$\begin{aligned}\text{(3.6)} \qquad & E(\operatorname{Var}(X_{s_0,\theta}, 0 \leqq \theta \leqq t_0))^2 \\ & \leqq E\{\sum_{ij} |\mu|(\Delta_{ij}) \cdot \sum_{ij} |\mu|(\Delta_{ij})(\sum_{k,l} \psi_{ijkl} M(\Delta_{kl}))^2\} .\end{aligned}$$

And since $M_z$ is a square integrable strong martingale, we have by 1.7 of [3]

$$\begin{aligned}\text{(3.7)} \qquad & E(\operatorname{Var}(X_{s_0,\theta}, 0 \leqq \theta \leqq t_0))^2 \leqq \mu_0 E \sum_{ij} |\mu|(\Delta_{ij})(\sum_{kl} \psi^2_{ij,kl} M^2(\Delta_{kl})) \\ \text{(3.8)} \qquad & = \mu_0 E \iint_{R_{z_0} \times R_{z_0}} \psi^2(z, z') d|\mu|(z) d[M]^1_{z'} .\end{aligned}$$

Consider now the special case where $\mu(z)$ is a product measure $\mu(s, t) = \mu^{(1)}(s)\mu^{(2)}(t)$. For simplicity we will assume that $\mu$ is a positive measure, $\mu^{(1)}(d_i)$ will denote $\mu^{(1)}(2^{-n}(i + 1)) - \mu^{(1)}(2^{-n}i)$ and similarly for $\mu^{(2)}(d_j)$. In this case we can write instead of (3.5)

$$\operatorname{Var}(X_{s_0,\theta}, 0 \leqq \theta \leqq t_0) \leqq \sum_j u^{(2)}(d_j) |\sum_{i,kl} \psi_{ijkl} \mu^{(1)}(d_i) M(\Delta_{kl})| .$$

Setting $\mu^{(2)} = (\mu^{(2)})^{\frac{1}{2}}(\mu^{(2)})^{\frac{1}{2}}$ yields

$$\begin{aligned}\text{(3.9)} \qquad & E(\operatorname{Var} X)^2 \leqq E\{\sum_j \mu^{(2)}(d_j) \sum_j \mu^{(2)}(d_j)(\sum_{ikl})^2\} \\ & \leqq \mu_0{}^{(2)} E \int_0^{t_0} (\int_0^{s'} \psi(\sigma, \tau, z') \, d\mu^{(1)}(\sigma))^2 d[M]^1_{z'} \, d\mu^{(2)}(\tau) .\end{aligned}$$

If $\mu$ is not positive, then (3.9) holds with $\mu^{(2)}(t)$ replaced by $|\mu^{(2)}|(t)$.

The requirement that $M_z$ be a strong martingale was needed to pass from (3.7) to (3.8); in the following particular case this is not necessary. Let $\psi(z, z')$ be a corner function, i.e., $\psi(z, z') = h(z, z')\pi(z \vee z')$ where $h(z, z') = 1$ whenever $z \wedge z'$ and zero otherwise. Then

$$\text{(3.10)} \qquad \psi_{ij,kl} = \pi_{k,j} \cdot I(i < k) \cdot I(l < j)$$

where $I(\ )$ denotes the indicator function. Substituting (3.10) in (3.3) and summing over $l$ we have

$$\text{(3.11)} \qquad X_{z_0} = \sum_{ij} \mu(\Delta_{ij}) \sum_{k>i} \pi_{kj}(M(k + 1, j) - M(k, j)) .$$

Setting $\mu = \mu^{\frac{1}{2}}\mu^{\frac{1}{2}}$ we have

$$\begin{aligned} EX^2_{z_0} & \leqq \mu_0 E\{\sum_{ij} |\mu|(\Delta_{ij}) \sum_{k>i} \pi^2_{kj}(M(k + 1, j) - M(k, j))^2\} \\ & = \mu_0 E\{\int_{R_{z_0}} d|\mu|(s, t) \int_s^{s_0} \pi^2_{\theta,t} d_\theta [M]^1_{\theta,t}\}\end{aligned}$$

where $[M]_z^1$ is as in (3.4) and is chosen to be measurable in $(s, t)$. Integration by parts with respect to $s$ yields

$$\text{(3.12)} \qquad EX^2_{z_0} \leqq \mu_0 E \int_0^{t_0} \int_0^{s_0} \pi^2_{s,t} d_s[M]^1_{s,t} d_t|\mu|(s, t) .$$

Furthermore

$$\operatorname{Var}(X_{s_0,\theta}, 0 \leqq \theta \leqq t_0) \leqq \sum_{ij} |\mu|(\Delta_{ij}) \,|\, \sum_{k>i} \pi_{kj}(M(k + 1, j) - M(k, j)) .$$

Therefore by the same arguments as those leading from (3.11) to (3.12) we have

$$E(\mathrm{Var}\, X_{s_0,\theta},\ 0 \leqq \theta \leqq t_0)^2 \leqq \mu_0 E\{\textstyle\int_0^{t_0}\int_0^{s_0} \pi_{s,t}^2\, d_s[M]_{st}^1\, d_t|\mu|(s, t)\}\,. \tag{3.13}$$

In addition to (3.10) assume, now, that $\mu$ is a product measure: namely $\mu(s, t) = \mu^{(1)}(s)\mu^{(2)}(t)$ where, for simplicity, we assume that $\mu^{(1)}$ and $\mu^{(2)}$ are positive measures. Then

$$X_{z_0} = \textstyle\sum_j \mu_j^{(2)}(\sum_i \mu_i^{(1)}(\sum_{k>i} \pi_{kj}(M_{k+1,j} - M_{k,j})))\,.$$

Let

$$a_j = \textstyle\sum_i \mu_i^{(1)}(\sum_{k>i} \pi_{kj}(M_{k+1,j} - M_{k,j}))\,;$$

then $\mathrm{Var}\,(X_{s_0,\theta},\ 0 \leqq \theta \leqq t_0) \leqq \sum_j \mu_j^{(2)}|a_j|$.

Setting $\mu_j^{(2)} = (\mu_j^{(2)})^{\frac12}(\mu_j^{(2)})^{\frac12}$,

$$E(\mathrm{Var}\, X)^2 \leqq \mu^{(2)}(t_0)E(\textstyle\sum_j \mu_j^{(2)}a_j^2)\,.$$

Now, $a_j$ can also be written as

$$a_j = \textstyle\sum_k (\pi_{kj}(M_{k+1,j} - M_{k,j}) \sum_{i>k} \mu_i^{(1)})\,.$$

Therefore

$$E(\mathrm{Var}\, X)^2 \leqq \mu^{(2)}(t_0) \textstyle\int_0^{t_0}\int_0^{t_0} (\mu^{(1)}(s))\pi_{s,t}^2\, d_s[M]_{s,t}^1\, d_t\mu^{(2)}(t)\,. \tag{3.14}$$

Let $M_z$ be a square integrable strong martingale and let $B_a$ be the class of all processes $\{\psi(\zeta, \zeta'),\ \zeta, \zeta' \prec z_0\}$ satisfying

(1) $\psi$ is predictable as defined in Section 2 of [3],

(2) $\psi(\zeta, \zeta') = 0$ unless $\zeta \wedge \zeta'$,

(3) $E\int\int_{R_{z_0}\times R_{z_0}} \psi^2(\zeta, \zeta')d|\mu|d[M]_z^1 < \infty$, or if $\mu_z$ is a product measure, the right-hand side of (3.9) is finite.

Since simple functions are dense in $B_a$, the mixed area integral $\int\int \psi\, d\mu\, dM$ can be extended by continuity to all $\psi$ in $B_a$. In view of Theorem 3 of Section 2 the integral will be a continuous proper 1 martingale satisfying (3.4) and (3.8). Similarly, let $M_z$ be a square integrable martingale and let $B_b$ be the class of all corner functions $\psi(\zeta, \zeta') = h(\zeta, \zeta')\pi(\zeta \vee \zeta')$ satisfying

(1) $\pi(\zeta)$ is $F_\zeta$ predictable,

(2) $E\{\int_0^{t_0}\int_0^{s_0} \pi_{s,t}^2 d_s[M]_{st}^1 d_t|\mu|(s, t)\} < \infty$, or if $\mu$ is a product measure, the right-hand side of (3.9) is finite.

Then the mixed surface integral can be extended to $B_b$. To summarize:

THEOREM 3.1. (1) *Let $\mu_z$ satisfy the assumptions made at the beginning of this section, let $M_z$ be a continuous strong square integrable martingale, and assume $\psi \in B_a$. Then*

(a) $\int\int \psi(\zeta, \zeta')\, d\mu(\zeta)\, dM_{\zeta'}$ *is a proper square integrable continuous* 1-*martingale*;

(b) *the integral is linear in* $\psi$;

(c) $EX_z^2$ *is as given by* (3.4) *and* $E(\text{Var } X_{s,\theta}, 0 \leqq \theta \leqq t)^2$ *satisfies the upper bound* (3.8).

(d) *Furthermore, if* $\mu$ *is a product measure,* (3.9) *holds.*

(2) *Let* $\mu_z$ *and* $M_z$ *be as in part* 1 *and* $\pi \in B_b$ *then* (a) *and* (b) *hold with* $\psi(\zeta, \zeta') = h(\zeta, \zeta')\pi(\zeta \vee \zeta')$. $EX_z^2$ *and* $E(\text{Var } X_{s,\theta}, 0 \leqq \theta \leqq t)^2$ *satisfy the bounds* (3.12) *and* (3.13) *respectively. If* $\mu$ *is a product measure then* (3.14) *is satisfied.*

REMARKS. (a) In [3] Cairoli and Walsh introduced the mixed integral

$$\int_0^{t_0} \int_0^{s_0} \pi(s, t)\, \partial_s M_{s,t}\, dt\,.$$

We now show that the mixed area integral of this section includes the mixed integral of [3] when $\pi_z$ is $\mathscr{F}_z$-predictable. Let $\mu(t) = st$. Approximate $\psi$ by simple functions. It follows that the area integral $\int\int \pi\, dz\, dM_{z'}$ can be expressed as

$$\int\int_{R_{z_0} \times R_{z_0}} \pi(z \vee z')\, dz\, dM_{z'} = \int_0^{t_0} \int_0^{s_0} s\pi(s, t)\, \partial_s M_{st}\, dt$$

and conversely if $E \int_0^{t_0} \int_0^{s_0} \pi^2(s, t)\, dt\, d_s[M]_{s,t}^s < \infty$, then

$$\int_0^{t_0} \int_0^{s_0} \pi(s, t)\, \partial_s M_{s,t}\, dt = \int\int \frac{1}{s'} \pi(z \vee z')\, dz\, dM_{z'}$$

and the integrand $\pi(z \vee z')/s'$ is admissible by (3.14). Note that $\pi(z \vee z')/s'$ is also a corner function since we integrate over $z \vee z'$, and $z \vee z' = (s', t)$.

(b) Let $X_z = \int\int \psi(\zeta, \zeta')\, d\mu_\zeta\, dM_{\zeta'}$ then, in view of (3.4), $X_z = 0$ for all $z \in R_{z_0}$ does not imply that $\psi(\zeta, \zeta') = 0$ in $R_{z_0} \times R_{z_0}$. In particular, for $\zeta = (\sigma, \tau)$, $d\mu_\zeta = d\sigma\, d\tau$, if

$$\psi(\zeta, \zeta') = \sin \frac{2\pi(\sigma - \sigma')}{\sigma'}\, \psi(\zeta')h(\zeta, \zeta')$$

then $X_z = 0$ for all $z$ in $R_{z_0}$. For any $\psi(\zeta, \zeta')$ define

$$\psi(\bar{\zeta}, \zeta') = \frac{1}{\sigma'} \int_0^{\sigma'} \psi(\sigma, \tau; \zeta')\, d\sigma$$

and $\psi(\tilde{\zeta}, \zeta') = \psi(\zeta, \zeta') - \psi(\bar{\zeta}, \zeta')$, and similarly

$$\psi(\zeta, \bar{\zeta}') = \frac{1}{\tau} \int_0^{\tau} \psi(\zeta, \sigma', \tau')\, d\tau'\,.$$

Then

$$\int\int \psi(\zeta, \tilde{\zeta}')\, dW_\zeta\, d\zeta' = 0$$
$$\int\int \psi(\tilde{\zeta}, \zeta')\, d\zeta\, dW_{\zeta'} = 0\,.$$

We can also define $\psi(\bar{\zeta}, \bar{\zeta}')$, $\psi(\tilde{\zeta}, \bar{\zeta})$, etc., since the bar and $\sim$ operations on the $\zeta$ and $\zeta'$ variables commute. Note that $\psi(\bar{\zeta}, \bar{\zeta}') = \pi(\sigma', \tau)$ (a corner function) and $\int\int \psi_1(\tilde{\zeta}, \zeta')\psi_2(\bar{\zeta}, \zeta')\, d\zeta\, d\zeta' = 0$.

(c) The stochastic integral of the second type [4] was generalized in [3] to $\int\int \psi(z, z')\, dM_z\, dM_{z'}$, where $M_z$ is a strong martingale and $EM_{z_0}^4 < \infty$. By an argument similar to the one given here $\iint \psi \, dM \, dM$ can be defined for martingales which are not strong provided that $\psi(z, z')$ is a corner function ($\psi(z, z') = \pi(z \vee z')h(z, z')$) as follows:

Let $A = (z, z']$ be a rectangle, $z = (s, t)$, $z' = (s', t')$. Let $A_1 = ((s, 0), (s', t)]$, $A_2 = ((0, t'), (s, t')]$. Define $X_z^A = \alpha M(A_1 \cap R_z) M(A_2 \cap R_z)$ as in Proposition 2.4 of [3]. Note that in this case, since $M_z$ is not strong, $X_z^A$ need not be orthogonal to $M_z$ but $X_z^A$ is a martingale and we still have as in Proposition 2.4 of [3]

$$\langle X^A \rangle_z = \alpha^2 \iint I_{A_2}(\zeta) I_{A_2}(\zeta') \, d[M]_\zeta^2 \, d[M]_{\zeta'}^1 .$$

If $A = (z_1, z_1']$, $B = (z_2, z_2']$ and $A \cap B = \varnothing$ then $X_z^A$ and $X_z^B$ are orthogonal. It follows, by standard arguments, that for corner functions, Proposition 2.5 of [3] holds without the requirement that $M_z$ be strong except that in this case $\iint \psi \, dM_\zeta \, dM_{\zeta'}$ need not be orthogonal to $M$.

**4. The representation of some weak martingales of the Wiener process.** Let $X_z \in \mathscr{H}_{z_0}^2$ be a proper 1-martingale of the Wiener process and assume that almost all the sample functions of $\lambda(t) = X_{s_0, t}$ are absolutely continuous with respect to some fixed (nonrandom) positive finite measure, i.e.,

$$\lambda(t) = \int_0^t \rho(\theta) \, dv(\theta) . \tag{4.1}$$

Furthermore, we will assume that

$$E \int_0^{t_0} \rho^2(\theta) \, dv(\theta) < \infty . \tag{4.2}$$

It will be shown in this section that 1-martingales satisfying the above conditions can be represented as mixed area integrals. The Wiener process assumption is not used in the following proposition but will be needed later.

PROPOSITION 4.1. *Let $\{f_i\}$ be a complete orthogonal set with respect to the $v$ measure on* $[0, t_0]$ (i.e., $\int_0^{t_0} f_i(t') f_j(t') \, dv(t') = \delta_{ij}$). *Under the above conditions on $X_z$ there exists a sequence of martingales $M_i(z)$ such that for $z \prec z_0$*

$$E\Big(X_z - \sum_{i=1}^N \int_0^t f_i(\theta) M_i(s, \theta) \, dv_\theta\Big)^2 \to_{N \to \infty} 0 . \tag{4.3}$$

PROOF.

$$X_{s,t} = \mathrm{E}(\lambda_t \mid \mathscr{F}_{s,t}) = \int_0^t E(\rho_\theta \mid \mathscr{F}_{s,t}) \, dv_\theta , \tag{4.4}$$

hence, by $F_4$ of Section 1

$$X_{s,t} = \int_0^t E(\rho_\theta \mid \mathscr{F}_{s,\theta}) \, dv_\theta .$$

Let

$$\alpha_i = \int_0^{t_0} \rho(t) f_i(t) \, dv_t .$$

Therefore $\alpha_i$ are $\mathscr{F}_{z_0}$-measurable and

$$\begin{aligned} E\Big(\lambda_t - \sum_i^N \alpha_i \int_0^t f_i(\theta) \, dv_\theta\Big)^2 &= E\Big(\int_0^t \Big(\rho(\theta) - \sum_1^N \alpha_i f_i(\theta)\Big) dv_\theta\Big)^2 \\ &\leqq K \int_0^{t_0} E\Big(\rho(\theta) - \sum_1^N \alpha_i f_i(\theta)\Big)^2 dv_\theta \end{aligned}$$

which converges to zero by dominated convergence. Let $M_i(z) = E(\alpha_i \mid \mathscr{F}_z)$. Then $M_i^2(z) \leqq E\alpha_i^2$, and by (4.4)

$$E(X_{s,t} - \int_0^t \sum_1^N M_i(s,\theta) f_i(\theta)\, dv_\theta)^2 = E(\int_0^t E\{\rho_\theta - \sum_1^N \alpha_i f_i(\theta) \mid \mathscr{F}_{s,\theta}\}\, dv_\theta)^2$$
$$\leqq K \int_0^t E(\rho_\theta - \sum_1^N \alpha_i f_i(\theta))^2\, dv_\theta$$

which converges to zero as $N \to \infty$, thus proving (4.3). □

THEOREM 4.2. *Under the above conditions on $X_z$, $X_z$ can be written as*

$$X_z = \iint_{R_z \times R_z} \psi(\zeta, \zeta')\, d\mu(\zeta)\, dW_{\zeta'} \tag{4.5}$$

*where $d\mu(z) = ds\, dv(t)$.*

PROOF. Let $M_i(z)$ be the martingales of Proposition 4.1. Then, by the corollary to Theorem (6.1) of [4]

$$M_i(z) = \int \phi_i(\zeta)\, dW_\zeta + \iint \psi_i(\zeta, \zeta')\, dW_\zeta\, dW_{\zeta'}\,,$$

and by (4.4)

$$E \sum_1^\infty M_i^2(z_0) = E \sum_1^\infty \int_{R_{z_0}} \phi_i^2(\zeta)\, d\zeta + E \sum_1^\infty \int_{R_{z_0}} \int_{R_{z_0}} \psi_i^2(\zeta, \zeta')\, d\zeta\, d\zeta'\,. \tag{4.6}$$

Let $M_{a,i}(z) = \int \phi_i(\zeta)\, dW_\zeta$, and approximate $\phi$ and $f$ by simple functions. It follows that

$$\int_0^t f_i(\theta) M_{a,i}(s,\theta)\, dv(\theta) = \iint \psi_{a,i}(\zeta, \zeta')\, d\mu_\zeta\, dW_{\zeta'}$$

where $\zeta = (\sigma, \theta)$, $d\mu_\zeta = d\sigma\, dv(\theta)$, and

$$\psi_{a,i}(\zeta, \zeta') = h(\zeta, \zeta') \frac{f_i(\theta)}{\sigma'} \phi_i(\zeta')\,.$$

Now, by the orthogonality of $f_i(\theta)$

$$E \iint_{R_{z_0} \times R_{z_0}} (\sum_N^{N+K} f_i(\theta)\phi_i(\zeta'))^2\, d\mu_\zeta\, d\zeta' \leqq K_1 E \sum_N^{N+K} \int_{R_{z_0}} \phi_i^2(\zeta')\, d\zeta'\,,$$

where $K_1$ is independent of $N$ and $K$. Therefore, by (4.6) $\sum_1^N f_i(\theta)\phi_i(\zeta')$ converges to a function $\phi^\infty(\theta, \zeta')$. Set

$$\Phi_1(\zeta, \zeta') = \frac{1}{\sigma'} \phi^\infty(\theta, \zeta')\,;$$

then

$$\sum_1^N \int_0^t f_i(\theta) M_{a,i}(s,\theta)\, dv(\theta) \to_{\text{q.m.}} \iint \Phi_1(\zeta, \zeta')\, d\mu_\zeta\, dW_{\zeta'}\,. \tag{4.7}$$

Similarly, let

$$M_{b,i}(z) = \iint \psi_i(\zeta, \zeta')\, dW_\zeta\, dW_{\zeta'}$$

and approximate $f$ and $\psi$ by simple functions. It follows that

$$\int f_i(\theta) M_{b,i}(s,\theta)\, dv(\theta) = \iint \psi_{b,i}(\zeta, \zeta')\, d\mu_\zeta\, dW_{\zeta'}$$

where $\mu_\zeta$ is as before and

$$\psi_{b,i}(\zeta, \zeta') = \frac{f_i(\theta)}{\sigma'} (\int_{R_{\zeta \vee \zeta'}} \psi_i(\zeta, \eta)\, dW_\eta)$$

(cf. Theorem 2.6 of [3]). The convergence of $1/\sigma' \sum_1^N \sigma'\psi_{b,i}$ to a function $\psi$ follows as in the previous case. Hence, by Proposition 4.1.

$$X_z = \int\int (\Phi(\zeta, \zeta') - \psi(\zeta, \zeta'))\, d\mu_\zeta\, dW_{\zeta'}$$

which is the desired result. □

**5. A characterization of strong martingales of the Wiener process.** It was shown by Cairoli and Walsh [3] that a martingale $M_z$ of the Wiener process $W_z$ is a strong martingale if and only if it is a type-one integral, i.e., $M_z = \int \phi_\zeta\, dW_\zeta$. A characterization in terms of stopping times will be given here.

DEFINITIONS.

1. $T(z, \omega)$ is a stopping time if

(a) $T(z, \omega)$ is a measurable and adapted random process;

(b) for almost all $\omega$, $T(z, \omega)$ as a function of $z$ is nonincreasing ($z \succ z' \Rightarrow T_z \leqq T_{z'}$) and takes only the values zero or one.

2. $T(z, \omega)$ is a predictable stopping time if it is a stopping time and a predictable process.

3. Let $Y_z$ be a square integrable martingale (or a function of bounded variation) and let $T$ be a predictable stopping time. Then $Y_{z\wedge T}$ ($Y$ stopped at $T$)

$$Y_{z\wedge T} = \int_{R_z} T(\zeta, \omega)\, dY(\zeta, \omega) .$$

More generally, let $Y_z$ be any adapted process such that

$$\int_{R_z} T_\zeta\, dY_\zeta$$

is defined and adapted, then $Y_{z\wedge T}$ is defined in the same way.

In order to point out the difference between stopping in the one-parameter and the two-parameter cases, let $T$ be defined as

$$\begin{aligned} T(z) &= 0 \quad \text{if } s \geqq \tfrac{1}{2} \text{ and } t \geqq \tfrac{1}{2} \\ &= 1 \quad \text{otherwise;} \end{aligned}$$

then if $(s, t)$ is in the region where $T = 0$, $M_{(s,t)\wedge T} = M_{\frac{1}{2},\frac{1}{2}} + (M_{s,\frac{1}{2}} - M_{\frac{1}{2},\frac{1}{2}}) + (M_{\frac{1}{2},t} - M_{\frac{1}{2},\frac{1}{2}})$. Therefore in the stopped region $M_{z\wedge T}$ is $M_{\frac{1}{2},\frac{1}{2}}$ plus the sum of two one-parameter martingales.

PROPOSITION 5.1. *Let $M_z$ be a right continuous square integrable martingale, $T$ a predictable stopping time and let*

$$X_z = \int \phi_\zeta\, dM_\zeta$$

*where*

$$E \int_{R_{z_0}} \phi_z^2\, d[M]_z < \infty .$$

*Also if $M_z$ is a right continuous strong martingale, and $EM_{z_0}^4 < \infty$, let*

$$Y_z = \int\int \psi(\zeta, \zeta')\, dM_\zeta\, dM_{\zeta'}$$

*where*

$$E \int\int_{R_{z_0}\times R_{z_0}} \psi^2(z, z')\, d[M]_z\, d[M]_{z'} < \infty\,.$$

*Then*

$$X_{z\wedge T} = \int_{R_z} T_\zeta \phi_\zeta\, dM_\zeta \tag{5.1}$$

*and*

$$Y_{z\wedge T} = \int\int_{R_z\times R_z} T(\zeta \vee \zeta')\psi(\zeta, \zeta')\, dM_\zeta\, dM_{\zeta'}\,. \tag{5.2}$$

PROOF. We prove, first, (5.1). It follows from Theorem 2.2 of [3] that

$$\begin{aligned} E(X_{z\wedge T} - \int T_\zeta \phi_\zeta\, dM_\zeta)^2 &= E(\int T_\zeta\, dX_\zeta - \int T_\zeta\phi_\zeta\, dM_\zeta)^2 \\ &= E\{\int T_\zeta\, d\langle X\rangle_\zeta + \int T_\zeta \phi_\zeta^2\, d\langle M\rangle_\zeta \\ &\qquad - 2\int T_\zeta\phi_\zeta\, d\langle X, M\rangle_\zeta\} \end{aligned}$$

where $\langle\cdot\rangle$ is an increasing function as defined in [3]. Equation (5.1) follows since $\langle X\rangle_z = \int \phi_\zeta^2\, d\langle M\rangle_\zeta$, $\langle X, M\rangle = \int \phi_\zeta\, d\langle M\rangle_\zeta$. Turning now to the proof of (5.2), let $\psi^n$ be such that

$$E\int\int (\psi_{z,z'} - \psi^n_{z,z'})^2\, d[M]_z^2\, d[M]^1_{z'} \to_{n\to\infty} 0 \tag{5.3}$$

and let

$$Y_z^n = \int\int \psi^n_{\zeta,\zeta'}\, dM_\zeta\, dM_{\zeta'}\,.$$

Also let $T_\zeta^n$ be such that $|T_\zeta| \leqq 1$ and

$$E\int (T_z^n - T_z)^2\, d\langle Y\rangle_z \to_{n\to\infty} 0\,. \tag{5.4}$$

By (2.19) of [3]

$$E\int (T_z^n - T_z)^2\, d\langle Y\rangle_z = E\int\int (T^n_{z\vee z'} - T_{z\vee z'})^2 \psi^2_{z,z'}\, d[M]_z^2\, d[M]^1_{z'}\,. \tag{5.5}$$

Therefore

$$\begin{aligned} E(\int T_\zeta^n\, dY_\zeta^n - \int T_\zeta\, dY_\zeta)^2 &\leqq E(\int T^n d(Y - Y^n))^2 + E(\int (T - T^n)\, dY)^2 \\ &\leqq E\langle Y^n - Y\rangle_{z_0} + E\int (T_z - T_z^n)^2\, d\langle Y\rangle_z \end{aligned}$$

which tends to zero as $n \to \infty$. Therefore

$$E(\int T_z^n\, dY_z^n - \int T_z\, dY_z)^2 \to_{n\to\infty} 0\,. \tag{5.6}$$

Let $n$, $z_{ij}$, $\Delta_{ij}$, $I_{\Delta_{ij}}(z)$, $\psi_{ij,kl}$ be as defined in Section 3 (the lines between equations (3.2) and (3.3)). Let $T_z^n$ be a sequence of simple function approximations to $T_z$ on the partition defined by $n$ satisfying (5.4); for $z \in \Delta_{ij}$, $T^n_{ij}$ will denote $T_z^n$. Then

$$\int T_\zeta^n\, dY_\zeta^n = \sum_{i,j} T^n_{ij} Y^n(R_z \cap \Delta_{ij})\,. \tag{5.7}$$

Approximating $\psi_{\zeta,\zeta'}$ by simple functions $\psi^n_{\zeta,\zeta'}$

$$\begin{aligned} \psi^n_{\zeta,\zeta'} &= \sum_{ij,kl} \psi^n_{ij,kl} I_{\Delta_{ij}}(\zeta) I_{\Delta_{kl}}(\zeta') \\ Y_t^n &= \sum_{ij,kl} \psi^n_{ij,kl} M(\Delta_{ij} \cap R_z) M(\Delta_{kl} \cap R_z)\,. \end{aligned}$$

Substituting $Y_n$ into (5.7) yields

$$\int T_\zeta^n \, dY_\zeta^n = \int\int T^n_{\zeta,\zeta'} \phi^n_{\zeta,\zeta'} \, dM_\zeta \, dM_{\zeta'}$$
$$= \int\int [T^n_{\zeta,\zeta'}(\phi^n_{\zeta,\zeta'} - \phi_{\zeta,\zeta'}) + (T^n_{\zeta,\zeta-} - T_{\zeta,\zeta'})\phi_{\zeta,\zeta'} + T_{\zeta,\zeta'}\phi_{\zeta,\zeta'}] \, dM_\zeta \, dM_{\zeta'}$$

and (5.2) follows by (5.3), (5.4), (5.5) and (5.6). □

From now on we consider the Wiener process case; in this case every stopping time is predictable. Let $\mathscr{F}_z$ be the $\sigma$-fields generated by the Wiener process $W_\zeta$, $\zeta \prec z$, let $T$ be a stopping time and let $\mathscr{F}_{z \wedge T}$, be the $\sigma$-fields generated by $W_{\zeta \wedge T}$, $\zeta \prec z$.

PROPOSITION 5.2. *Let $\phi_\zeta$ be $\mathscr{F}_\zeta$ adapted and $E \int_{R_{z_0}} \phi_z^2 \, dz < \infty$. Let $T_\zeta$ be a stopping time; then*, a.s.

$$E\{\int_{R_z} \phi_\zeta \, dW_\zeta \mid \mathscr{F}_{z_0 \wedge T}\} = \int_{R_z} T_\zeta \phi_\zeta \, dW_\zeta \, .$$

PROOF. Let $T_\zeta^-$ be a left continuous modification of $T_\zeta$. Then, by Proposition 5.1, $W_{\zeta \wedge T} = W_{\zeta \wedge T-}$ and therefore $\mathscr{F}_{\zeta \wedge T} = \mathscr{F}_{\zeta \wedge T-}$. Given a sample $W_{\zeta \wedge T}$, $\zeta \prec z$, we can determine whether $T_z^- = 1$ or $T_z^- = 0$ by examining the quadratic variation of $W_{\zeta \wedge T}$ along an increasing path from (0, 0) to $z$; this follows from Proposition 7.1 of [3]. Therefore $T_\zeta^-$ is $\mathscr{F}_{\zeta \wedge T}$-measurable and so is $\phi_\zeta T_\zeta^-$. Therefore

$$\int_{R_z} \phi_\zeta T_\zeta^- \, dW_{\zeta \wedge T} = \int_{R_z} \phi_\zeta T_\zeta \, dW_\zeta$$

is $\mathscr{F}_z$-measurable.

It remains to be shown that $E\{\int_{R_z} (1 - T_\zeta)\phi_\zeta \, dW_\zeta \mid \mathscr{F}_{z_0 \wedge T}\} = 0$. Let $n$, $z_{i,j}$, $\Delta_{ij}$ be as defined in Section 3 (after equation 3.2). Let $[z] = ([s \cdot 2^n], [t \cdot 2^n])$ where $[s \cdot 2^n]$ is the largest integer $k$ satisfying $k \leqq s \cdot 2^n$. Set $T_\zeta^n = (T_{[\zeta]}^-)^-$. Note that the number of different samples functions of the random function $T_\zeta^n$, $\zeta \prec z_0$ is *finite*. Consider now $\int (1 - T_\zeta)\phi_\zeta \, dW_\zeta$. Since $T_\zeta^n \geqq T_\zeta$ and $T_\zeta^n \searrow T_\zeta$ as $n \to \infty$, it follows by dominated convergence:

$$E(\int (T_\zeta^n - T_\zeta)\phi_\zeta \, dW_\zeta)^2 = E \int (T_\zeta^n - T_\zeta)\phi_\zeta^2 \, d\zeta \to_{n \to \infty} 0 \, . \tag{5.8}$$

Let $\phi_\zeta$ be a simple function:

$$\phi_\zeta = \sum_{i,j} \alpha_{ij} I_{\Delta_{ij}}(\zeta)$$

where $\alpha_{ij}$ is $\mathscr{F}_{[\zeta]}$-measurable. Then

$$\int_{R_z} (1 - T_\zeta^n)\phi_\zeta \, dW_\zeta = \sum_{ij} \alpha_{ij}(1 - T^n_{ij}) W(\Delta_{ij} \cap R_z)$$

and

$$E(\alpha_{ij}(1 - T^n_{ij}) W(\Delta_{ij} \cap R_z) \mid \mathscr{F}_{T^n}) = 0$$

since if $T^n_{ij} = 1$ then $1 - T^n_{ij} = 0$ and if $T^n_{ij} = 0$ then

$$E(\alpha_{ij}(1 - T^n_{ij}) W(\Delta_{ij} \cap R_z) \mid \mathscr{F}_{T^n} \vee \mathscr{F}^1_{z_{ij}} \vee \mathscr{F}^2_{z_{ij}}) = 0 \, .$$

Let $\phi_\zeta^n$ be a sequence of simple function approximations to $\phi$; then

$$\int (1 - T)\phi \, dW = \int (1 - T^n)\phi^n \, dW + \int (1 - T^n)(\phi - \phi^n) \, dW$$
$$+ \int (T^n - T)\phi \, dW \, .$$

The last two terms converge to zero in quadratic mean as $n \to \infty$. Therefore, since $\mathscr{F}_{T^n} \supset \mathscr{F}_T$, $E(\int (1 - T)\phi \, dW \mid \mathscr{F}_{T \wedge z_0}) = 0$ which completes the proof. □

Let $\mathscr{F}_{T+} = \bigcap_n \mathscr{F}_{T^n}$ where $T^n$ is as defined in the proof of the previous theorem (i.e., $T^n = (T^-_{[\zeta]})^-$ and $T^-$ is the left continuous version of $T$). We will assume that $T_z \equiv 0$ for $z \succ z_0$ and denote $X_{z_0 \wedge T}$ by $X_T$.

PROPOSITION 5.3. $\mathscr{F}_{T+} = \mathscr{F}_{T-}$.

PROOF. In the proof of Proposition 5.2 we showed that $\mathscr{F}_{T-} = \mathscr{F}_T$. Let $g(\zeta)$ be square integrable and nonrandom. Let

$$Y = \exp \int_{R_{z_0}} g(\zeta) \, dW_\zeta \, .$$

Since the number of different samples of $T^n$ is finite,

$$E(Y \mid \mathscr{F}_{T^n}) = \exp \int_{R_{z_0}} g(\zeta) T_\zeta^n \, dW_\zeta \cdot \exp \tfrac{1}{2} \int_{R_{z_0}} (1 - T^n) g^2(\zeta) \, d\zeta \, .$$

By the (reversed) martingale convergence theorem

$$\begin{aligned} E(Y \mid \mathscr{F}_{T+}) &= \lim_{n \to \infty} E(Y \mid \mathscr{F}_{T^n}) \\ &= \exp \int_{R_{z_0}} g(\zeta) T_\zeta^- \, dW_\zeta \cdot \exp \tfrac{1}{2} \int_{R_{z_0}} (1 - T_\zeta^-) g^2(\zeta) \, d\zeta \end{aligned}$$

which is $\mathscr{F}_T$-measurable. Since random variables of the form of $Y$ with $g(\zeta) = \sum_1^N \alpha_i g_i(\zeta)$, where $g_i(\zeta)$ are orthonormal on $R_{z_0}$, generate the Hermite polynomials which are dense in the space of square integrable functionals of $W$, it follows that $\mathscr{F}_{T+} = \mathscr{F}_{T-}$. □

THEOREM 5.4. *Let $T_1$ and $T_2$ be stopping times and $T_3 = T_1(\zeta) \cdot T_2(\zeta) = \min(T_1, T_2)$; then $\mathscr{F}_{T_1}$ and $\mathscr{F}_{T_2}$ are conditionaly independent given $\mathscr{F}_{T_3}$.*

PROOF. Since the number of different samples of $T^n$ is finite, it follows by the independence of $W(A)$ and $W(B)$, where $A$ and $B$ are Borel sets in $R_{z_0}^+$, $A \cap B = \varnothing$, that $\mathscr{F}_{T_1^n}$ and $\mathscr{F}_{T_2^n}$ are conditionaly independent given $\mathscr{F}_{T_3^n}$. Therefore, if $Y$ is a bounded $\mathscr{F}_{T_1+}$-measurable random variable and since $T_1^n \cdot T_2^n = (T_1 \cdot T_2)^n$

$$\begin{aligned} E\{Y \mid \mathscr{F}_{T_2+} \vee \mathscr{F}_{T_3+}\} &= E\{E[Y \mid \mathscr{F}_{T_2^n} \vee \mathscr{F}_{T_3^n}] \mid \mathscr{F}_{T_2+} \vee \mathscr{F}_{T_3+}\} \\ &= E\{E[Y \mid \mathscr{F}_{T_3^n}] \mid \mathscr{F}_{T_2+} \vee \mathscr{F}_{T_3+}\} \, . \end{aligned}$$

Since $E[Y \mid \mathscr{F}_{T_3^n}] \to_{\text{a.s.}} E[Y \mid \mathscr{F}_{T_3+}]$ as $h \to \infty$, it follows by the smoothing property of conditional expectations that

$$E\{Y \mid \mathscr{F}_{T_2+} \vee \mathscr{F}_{T_3+}\} = E\{Y \mid \mathscr{F}_{T_3+}\} \, .$$

By Proposition 5.3 $\mathscr{F}_{T+} = \mathscr{F}_{T-}$ and the proof is complete. □

PROPOSITION 5.5. *If $\phi_\zeta^{(1)}$ is $\mathscr{F}_\zeta^1$ adapted and $\psi_{\zeta,\zeta'}$ is $\mathscr{F}_{\zeta \vee \zeta'}$ adapted, $E \int (\phi_z^{(1)})^2 \, dz < \infty$, $E \iint \psi_{z,z'}^2 \, dz \, dz' < \infty$ then* a.s.

$$E\{\int \phi_\zeta^{(1)} \, dW_\zeta \mid \mathscr{F}_T\} = \int E\{\phi_\zeta^{(1)} \mid \mathscr{F}_\zeta^1 \wedge \mathscr{F}_T\} T_\zeta \, dW_\zeta \tag{5.9}$$

*and*

$$E\{\iint \psi_{\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} \mid \mathscr{F}_T\} = \iint E(\psi_{\zeta,\zeta'} \mid \mathscr{F}_{T_{\zeta \vee \zeta'}}) T_\zeta \cdot T_{\zeta'} \, dW_\zeta \, dW_{\zeta'} \, . \tag{5.10}$$

PROOF. Equation (5.9) follows easily from Theorem 5.4 for the case where $\phi_\zeta^{(1)}$ are simple functions and the extension to general $\phi^{(1)}$ is straightforward. Equation (5.10) follows from (5.9) by the stochastic Fubini theorem (Theorem 2.6 of [3]). □

Let $T_\lambda(z, \omega)$ $0 \leqq \lambda < \infty$, be a one-parameter collection of stopping times such that for almost all $\omega$, $T_{\lambda_2}(z, \omega) \geqq T_{\lambda_1}(z, \omega)$ whenever $\lambda_1 \leqq \lambda_2$. We will call such a collection an increasing collection of stopping times. Let $M_z$ be a martingale of the Wiener process and let $z_0$ be fixed. We will denote

$$\mathscr{F}_\lambda = \mathscr{F}_{z_0 \wedge T\lambda}$$
$$X_\lambda = M_{z_0 \wedge T\lambda} .$$

THEOREM 5.6. *Let $M_z$ be a square integrable martingale of the Wiener process, then $M_z$, $z \prec z_0$ is a strong martingale if and only if $\{X_\lambda, \mathscr{F}_\lambda\}$ is a martingale for all increasing families of stopping times.*

PROOF. If $M_z$ is a strong martingale then $M_z = \int_{R_z} \phi_\zeta \, dW_\zeta$ [3],

$$X_\lambda = \int_{R_{z_0}} T_\lambda(\zeta)\phi_\zeta \, dW_\zeta$$

by Proposition 5.1 and therefore $(X_\lambda, \mathscr{F}_\lambda)$ is a martingale by Proposition 5.2. Conversely, let $\alpha < \beta$ and define

$$A = \{z : s + t \leqq \alpha\}$$
$$B = \{z : \alpha < s + t \leqq \beta\} .$$

Let $T_1$ and $T_2$ be the following deterministic stopping times.

$$\begin{aligned} T_1(z, \omega) &= 1 \quad \text{if } z \in A \\ &= 0 \quad \text{otherwise;} \\ T_2(z, \omega) &= 1 \quad \text{if } z \in A \cup B \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Let $M_z = \iint \psi(\zeta, \zeta') \, dW_\zeta \, dW_{\zeta'}$; then

$$X_{\lambda_2} - X_{\lambda_1} = \iint_{R_{z_0} \times R_{z_0}} (T_2(\zeta \vee \zeta') - T_1(\zeta \vee \zeta'))\psi(\zeta, \zeta') \, dW_\zeta \, dW_{\zeta'} .$$

Divide the above integral into five integrals. $I_1$ is the above integral over $\zeta \vee \zeta' \in A$ hence this integral is zero. $I_2$ is the above integral over $\zeta \in A$, $\zeta' \in B$, (and $\zeta \vee \zeta' \in B$), $I_3$ is the above integral over $\zeta' \in A$, $\zeta \in B$, $I_4$ is over $\zeta \vee \zeta' \in B$, $\zeta \in A$, $\zeta' \in A$, $I_5$ is over $\zeta' \in B$, $\zeta \in B$. Since $\mathscr{F}_{T_1} = \sigma\{W_\zeta, \zeta \in A\}$, it follows by simple function approximation that $E(I_i \mid \mathscr{F}_{T_1}) = 0$ for all $i$ with the exception of $i = 4$. Consider now $E(I_4 \mid \mathscr{F}_{T_1})$. If $X$ is to be a martingale, we must have a.s.

$$E\{\iint_{z \in A, z' \in A, z \vee z' \in B} \psi(z, z') \, dW_{z \wedge T_1} \, dW_{z' \wedge T_1} \mid \mathscr{F}_{T_1}\} = 0 .$$

And, by Proposition 5.5

$$\iint E\{\psi(z, z') \mid \mathscr{F}_{T_1}\} \, dW_{z \wedge T_1} \, dW_{z' \wedge T_1} = 0$$

where the region of integration is the same as the previous integral.

Thus $\int\int (E\{\psi \mid \mathscr{F}_{T_1}\})^2 d(z \wedge T_1)d(z' \wedge T_1) = 0$, and

$$E(\psi(\zeta, \zeta') \mid \mathscr{F}_{T_1}) = 0 \quad \text{a.s.}$$

For $\zeta \vee \zeta'$ fixed let $\alpha \nearrow (\zeta \vee \zeta')$. By the continuity of the $\mathscr{F}_\lambda$ $\sigma$-fields

$$\psi(\zeta, \zeta') = \lim_{\alpha \to \zeta \wedge \zeta'} E\{\psi(\zeta, \zeta') \mid \mathscr{F}_\alpha\} = 0$$

which completes the proof. □

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