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AN ERGODIC THEOREM FOR THE SQUARE OF A WIDE-SENSE STATIONARY PROCESS[1]

By A. Larry Wright

University of Arizona

Let $\{X(t), -\infty < t < \infty\}$ be a stochastic process which is stationary in the wide sense with spectral representation $X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\xi(\lambda)$, where the ξ process is centered and has independent increments with $E\xi(\lambda) \equiv 0$, $E|\xi(\lambda)|^2 < \infty$. It is shown that under weak conditions

$$P - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(t)|^2\, dt$$

exists and is equal to $\sigma^2 + \sum J_t^2 + \sum \xi_n^2$, where σ^2 is equal to the variance of the Gaussian component of the continuous part of the ξ process, $\sum J_t^2$ is the sum of the squares of the jumps of the Gaussian component of the ξ process, and $\xi_N = \xi(\lambda_N + 0) - \xi(\lambda_N - 0)$, where $\{\lambda_N\}$ are the fixed discontinuities of the ξ process.

1. Introduction. If $\{X(t), -\infty < t < \infty\}$ is an L^2-continuous stochastic process which is stationary in the wide sense, then it admits a spectral representation of the form

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\xi(\lambda)\,, \tag{1}$$

where the process $\{\xi(\lambda), -\infty < \lambda < \infty\}$ is a process with orthogonal increments. Corresponding to such a process are two functions defined over R^1, namely, $r(t) = E(X(u + t)\overline{X(u)})$ and $F(\lambda)$ determined by $F(v) - F(u) = E|\xi(v) - \xi(u)|^2$ for $u < v$ and $F(-\infty) = 0$. F is a bounded nondecreasing function, and $r(\cdot)$ is a nonnegative definite function; they are related by the relation

$$r(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, dF(\lambda)\,,$$

It is assumed that the expectations $EX(t)$ and $E\xi(\lambda)$ are zero. When ξ has independent increments and is centered, a separable version of ξ is such that except at a countable number of fixed points $\{\lambda_n\} \subset R^1$, almost all sample functions are continuous and the limits $\{\xi(\lambda_n + 0) - \xi(\lambda_n - 0), n = 1, 2, \cdots\}$ exist and are independent random variables. These properties of ξ are often useful in studying the properties of X.

An infinitely divisible stochastic process is a process $\{Z(t), t \in T\}$ such that every finite dimensional marginal distribution is an infinitely divisible distribution function. An infinitely divisible process which is stationary in the wide sense can be constructed as follows. Let $\{\xi(\lambda), -\infty < \lambda < \infty\}$ be a process with independent increments, is centered and has no fixed points of discontinuity;

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assume that it satisfies $E|\xi(\lambda)|^2 < \infty$ for all $\lambda \in R^1$, and that the limits $\xi(+\infty)$ and $\xi(-\infty)$ exist in quadratic mean (and hence almost surely). Then the stochastic process in (1) has the above described properties. Possibly the first study of the general infinitely divisible process was made by P. M. Lee [4]. A deeper study of such processes was made by G. Maruyama [6], who also in the same paper obtained results concerning those that are stationary in the wide sense. This present paper concerns itself with a slightly wider class of processes, which will be referred to as extended infinitely divisible processes which are stationary in the wide sense. Such a process is constructed in the same way as is X above, except now ξ may have fixed discontinuities which are necessarily countable in number. For this class of random spectral measures ξ, the processes X defined by (1) have a rich structure and provide a natural setting for an extension of a theorem due to N. Wiener.

The theorem due to Wiener referred to is this: if g is the characteristic function of a distribution function G, then the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |g(u)|^2 \, du \tag{2}$$

exists and is equal to the sum of the squares of the jumps of G. With this theorem and some rather general results of variational sums in mind, one is led to consider the following. Let ξ be a process with independent increments, centered but possibly with (necessarily countable) fixed discontinuities at $\{\lambda_n\}$ of (random) sizes $\{\xi_n\}$, with $E|\xi(\lambda)|^2 < \infty$, all λ, $E\xi(\lambda) \equiv 0$, and the limit $\xi(+\infty) - \xi(-\infty)$ existing in quadratic mean. Then ξ can be written as the sum of two independent processes

$$\xi = \xi_c + \xi_d ,$$

where ξ_c has no fixed discontinuities and ξ_d is defined by

$$\xi_d(\lambda) = \sum_{\{n:\lambda_n \leq \lambda\}} \xi_n .$$

We may, and do, assume that ξ_c is separable, so that almost all sample functions are continuous from the right at each point, and have left limits, thus having a countable number of discontinuities (referred to as the *mobile* discontinuities by P. Lévy). With these assumptions, the aim here is to prove that the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(t)|^2 \, dt \tag{3}$$

exists in probability and equals

$$\sigma^2 + \sum J_t^2 + \sum \xi_n^2 , \tag{4}$$

where $\sigma^2 \geq 0$ is the variance of the Gaussian component of the continuous part ξ_c of ξ, and $\{J_t(\omega)\}$ are the jumps of the sample function $\xi_c(\cdot, \omega)$.

In the special case where ξ is a stationary Gaussian process, then the limit as given in (4) reduces to

$$\sigma^2 + \sum \xi_n^2 . \tag{5}$$

2. Preliminary theorems and lemmas. In this section we provide the material necessary for the proof of the main theorem.

Let $X(t)$ be a separable process with independent increments over the finite time interval $T = [t_1, t_2]$. When X is continuous in law it is known to have the following representation for its characteristic function

$$f_{X(t)-X(t_1)}(u) = \exp\{\psi_t(u)\}\,, \qquad \text{where}$$

$$\psi_t(u) = iu\alpha_t + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\,d\phi_t(x)\,.$$

We refer to this characteristic function by (α_t, ϕ_t), and we set $X_{(s,t)} = X(t) - X(s)$, $t_1 \leqq s < t \leqq t_2$. For each n we divide the interval T into the k_n subintervals (*refinement*) $P_{n,k}$, $k = 1, 2, \cdots, k_n$, where $P_{n,k} = [t_{n,k-1}, t_{n,k}]$, with $\max\{t_{n,k} - t_{n,k-1}, 1 \leqq k \leqq k_n\} \to 0$. We next set

$$X_{n,k} = X(t_{n,k}) - X(t_{n,k-1})\,, \qquad 1 \leqq k \leqq k_n\,.$$

Cogburn and Tucker [1] proved that if $\{X(t), t \in [t_1, t_2]\}$ is a separable process, continuous in law, and with law (α_t, ϕ_t), where α_t is a function of bounded variation on T, then

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} X_{n,k}^2 = \sigma^2 + \sum J_t^2 \quad \text{a.s.}\,,$$

where σ^2 is the variance of the Gaussian component of $X_{(t_1,t_2)}$ and $\sum J_t^2$ is the sum of the squares of the jumps of X. We shall need a generalization of this theorem in the case where the process $X(t)$ is not necessarily continuous in law. Let $\{X(t), 0 \leqq t \leqq 1\}$ be a centered real stochastic process with independent increments, satisfying $EX(t) \equiv 0$, and $X(t) \in L^2$ for all $t \in [0, 1]$. Let the countable set of fixed discontinuities of X in $[0, 1]$ occur at the points $\{t_i, i = 1, 2, \cdots\}$. Letting $V_n = X(t_n + 0) - X(t_n - 0)$, we see that $EV_n = 0$, the V_n's are independent random variables, and $V_n \in L^2$, $n = 1, 2, \cdots$.

We can write $X(t) = U(t) + V(t)$, where the $U(t)$ process is continuous in law, with characteristic function (α_t, ϕ_t), and $V(t) = \sum_{\{n: t_n \leqq t\}} V_n$. Both processes have independent increments, and they are independent of each other.

For each $n = 1, 2, \cdots$ and $k = 1, \cdots, k_n$, set

$$X_{n,k} = X(t_{n,k}) - X(t_{n,k-1})$$
$$U_{n,k} = U(t_{n,k}) - U(t_{n,k-1})$$
$$V_{n,k} = V(t_{n,k}) - V(t_{n,k-1})\,.$$

Using the above notation, we now prove the following extension of the theorem of Cogburn and Tucker referred to above.

THEOREM 1. *If $\{X(t), 0 \leqq t \leqq 1\}$ is a separable stochastic process with independent increments which is centered, and such that the continuous in law part of the process has law (α_t, ϕ_t), where α_t is a function of bounded variation on $T = [0, 1]$, then $\sum X_{n,k}^2 \to_P \sigma^2 + \sum J_t^2 + \sum V_n^2$, where σ^2 is the variance of the Gaussian component of $U_{(0,1)}$,* and *$\{J_t^2, 0 \leqq t \leqq 1\}$ are the squares of the jumps of U.*

The proof follows from writing

$$\sum X_{n,k}^2 = \sum U_{n,k}^2 + 2 \sum U_{n,k} V_{n,k} + \sum V_{n,k}^2 ,$$

applying the Cogburn–Tucker theorem to the first term on the right side, and verifying that

$$\sum U_{n,k} V_{n,k} \to_{L^1} 0 \quad \text{and} \quad \sum V_{n,k}^2 \to_P \sum V_r^2 .$$

Let us now consider the following case. Let $\xi(\lambda)$ be a separable and a centered complex process over $(-\infty, \infty)$ with independent increments and a countable number of fixed discontinuities, $E\xi(\lambda) \equiv 0$, $\xi(\infty) = \lim_{T\to\infty} \xi(T)$ and $\xi(-\infty) = \lim_{T\to-\infty} \xi(T)$ exist, and Var $(\xi(+\infty) - \xi(-\infty)) < \infty$, $F(\lambda+) = F(\lambda)$. We will also assume that in the infinitely divisible representation of the continuous-inlaw part of the process, as given by (α_t, ϕ_t), that α_t is of bounded variation over every finite interval. Using the notation of the last theorem, and setting $\xi_{n,k} = \xi(k/2^n) - \xi((k-1)/2^n)$, we have the following result:

THEOREM 2.

$$P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} |\xi_{n,k}|^2 = \sigma_1^2 + \sigma_2^2 + \sum |J_t|^2 + \sum_{s=1}^{\infty} |V_s|^2 ,$$

where σ_1^2 *is the variance of the Gaussian component of* Re $(\xi(+\infty) - \xi(-\infty))$, σ_2^2 *is the variance of the Gaussian component of* $I_m(\xi(+\infty) - \xi(-\infty))$, *and* $\{|J_t|^2, t = 1, 2, \cdots\}$ *are the squares of the magnitudes of the jumps of* U.

The proof follows from writing $\xi(t) = Z(t) + iW(t)$ and applying Theorem 1 to the Z and W processes over intervals of the form $[-T, T]$, and letting $T \to \infty$.

3. Proof of the main theorem. In this section we prove a result which allows us to weaken the hypotheses of the ergodic theorem to include a class of processes which are stationary in the wide sense but not necessarily strictly stationary. In addition, an explicit representation of the limiting random variable is given. In the following, we continue to use the notation of the last section, and the $\xi(\lambda)$ process will be as in Theorem 2.

THEOREM 3. *Let* $\{X(t), -\infty < t < \infty\}$ *be a measurable extended infinitely divisible process, stationary in the wide sense with representation*

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\xi(\lambda) .$$

Then

$$P - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(t)|^2\, dt = \sigma_1^2 + \sigma_2^2 + \sum |J_t|^2 + \sum_{s=1}^{\infty} |V_s|^2 ,$$

where σ_1^2 *is the variance of the Gaussian component of* $R_e(\xi(+\infty) - \xi(-\infty))$, σ_2^2 *is the variance of the Gaussian component of* $I_m(\xi(+\infty) - \xi(-\infty))$, $\sum |J_t|^2$ *is the sum of the squares of the magnitudes of the jumps of the sample function* ξ_c, *and* $\{V_n\}$ *are the sizes of the jumps at the fixed discontinuities of* ξ.

PROOF. We shall prove the theorem by showing that for each T,

$$\frac{1}{2T} \int_{-T}^{T} |X(t)|^2\, dt = P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} |\xi_{n,k}|^2 + D_T ,$$

where $D_T \to 0$ in probability as $T \to \infty$. For fixed $t \in (-\infty, \infty)$,

$$|X(t)|^2 = |P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} e^{itk/2n}\xi_{n,k}|^2$$
$$= P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} |\xi_{n,k}|^2$$
$$+ P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} \sum_{r=-n2^n;r\neq k}^{n2^n} e^{it(k-r)/2^n}\xi_{n,k}\overline{\xi_{n,r}} .$$

It follows that for fixed T,

$$\frac{1}{2T} \int_{-T}^{T} |X(t)|^2\,dt$$
$$= P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} |\xi_{n,k}|^2 \tag{1}$$
$$+ \frac{1}{2T} \int_{-T}^{T} \{P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} \sum_{r=-n2^n;r\neq k}^{n2^n} e^{it(k-r)/2^n}\xi_{n,k}\overline{\xi_{n,r}}\}\,dt .$$

The second term above is D_T, which we will show converges to zero in probability as $T \to \infty$, which will prove the theorem. We shall begin by justifying an interchange of the P-limit and integral sign in D_T.

From the definition of X, it follows ([10], pages 353–357) that there exists a sequence of nondecreasing right-continuous step functions $\{\theta_r, r = 1, 2, \cdots\}$ on $[-T, T]$, with $\theta_r(t) \to t$ uniformly in t, such that the sequence of measurable processes $\{X_r, r = 1, 2, \cdots\}$ defined by $X_r(t, \omega) = X(\theta_r(t), \omega)$ for $t \in [-T, T]$, $\omega \in \Omega$ satisfies $X_r(t, \cdot) \in L_2(\Omega)$ for $t \in [-T, T]$ and

$$\lim_{r\to\infty} \int_{[-T,T]\times\Omega} |X_r(t, \omega) - X(t, \omega)|^2(m_L \times P)(d(t, \omega)) = 0 .$$

It follows from Fubini's theorem that we may choose a subsequence $\{r_m\}$ of $\{r\}$ such that a.e. $[P]$,

$$\frac{1}{2T} \int_{[-T,T]} |X_{r_m}(t, \omega)|^2\,dt \to \frac{1}{2T} \int_{[-T,T]} |X(t, \omega)|^2\,dt \tag{2}$$

as $m \to \infty$. Hence it remains to show that

$$\lim_{k\to\infty} \frac{1}{2T} \int_{[-T,T]} |X_{r_k}(t, \cdot)|^2\,dt$$
$$= P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} |\xi_{n,k}|^2 \tag{3}$$
$$+ P - \lim_{n\to\infty} \sum_{k=n2^n}^{n2^n} \sum_{r=-n2^n;r\neq k}^{n2^n} \left[\frac{1}{2T} \int_{-T}^{T} e^{it(k-r)/2^n}\,dt\right]$$
$$\times\ \xi_{n,k}\overline{\xi_{n,r}} \quad \text{a.s.},$$

and that the last term converges to zero in probability as $T \to \infty$. We note that

$$X_{r_m}(t, \cdot) = P - \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} \exp[i\theta_{r_m}(t)k/2^n]\xi_{n,k} . \tag{4}$$

Since the set $\{\theta_{r_m}(t), m = 1, 2, \cdots, t \in [-T, T]\}$ is countable, by proper choice of a subsequence, we may and do assume in the following that the above limits as well as the limit of $\sum_{k=-n2^n}^{n2^n} |\xi_{n,k}|^2$ are almost sure in n. By the definition of $\{\theta_{r_m}\}$, for each m there exists a partition $-T = t_{0,m} < t_{1,m} < \cdots < t_{N_m,m} = T$ of $[-T, T]$ such that $\theta_{r_m}(t) = \theta_{r_m}(s)$ if $t_{j-1,m} \leqq s < t < t_{j,m}$, and also,

$\lim_{m\to\infty}[\max_j \Delta t_{j,m}] = 0$, where $\Delta t_{j,m} = t_{j,m} - t_{j-1,m}$. Thus for fixed m,

$$\frac{1}{2T}\int_{[-T,T]} |X(\theta_{r_m}(t,\cdot))|^2\,dt$$

$$(5)\qquad = \lim_{n\to\infty}\sum_{k=-n2^n}^{n2^n}|\xi_{n,k}|^2 + \lim_{n\to\infty}\sum_{k=-n2^n}^{n2^n}\sum_{r=-n2^n;r\neq k}^{n2^n}\Bigg(\sum_{j=1}^{N_m}\frac{\Delta t_{j,m}}{2T} \times \exp[it_{j-1,m}(k-r)/2^n]\xi_{n,k}\overline{\xi_{n,r}}\Bigg).$$

By comparing (3) and (5), and defining

$$\psi_{m,n} = \sum_{k=-n2^n}^{n2^n}\sum_{r=-n2^n;r\neq k}^{n2^n}\left[\frac{1}{2T}\left\{\int_{[-T,T]} e^{it(k-r)/2^n}\,dt - \sum_{j=1}^{N_m}\exp[it_{j-1,m}(k-r)/2^n]\Delta t_{j,m}\right\}\right]\xi_{n,k}\overline{\xi_{n,r}},$$

it can be shown ([9], pages 60–62) that given $\varepsilon > 0$,

$$(6)\qquad \lim_{m\to\infty}\{\sup_{n=1,2,\cdots} P[|\psi_{m,n}| \geqq \varepsilon]\} = 0,$$

and that this in turn implies that (3) holds. From (3) it follows that the proof of the theorem will be complete if we can show that

$$P - \lim_{T\to\infty}\lim_{n\to\infty}\phi_{n,T} = 0,$$

where

$$\phi_{n,T} = \sum_{k=-n2^n}^{n2^n}\sum_{r=-n2^n;r\neq k}^{n2^n}\left(\frac{2^n}{T(k-r)}\right)\sin\left(\frac{T(k-r)}{2^n}\right)\xi_{n,k}\overline{\xi_{n,r}}.$$

Since $E\phi_{n,T} = 0$, we only need show that $\lim_{T\to\infty}\sup_n \operatorname{Var}\phi_{n,T} = 0$. This can be shown by noting that

$$\begin{aligned}\operatorname{Var}\phi_{n,T} &= E|\phi_{n,T}|^2 \\ &\leqq 4\sum_{k=-n2^n}^{n2^n} F_{n,k}\left\{\sum_{r=-n2^n;r\neq k}^{n2^n}\left(\frac{2^n}{T(k-r)}\right)^2\sin^2\left(\frac{T(k-r)}{2^n}\right)F_{n,r}\right\},\end{aligned}$$

where $F_{n,k} = E|\xi(k/2^n) - \xi((k-1)/2^n)|^2$, and that $|(\sin x)/x| \leqq 1$ and $\lim_{T\to\infty}E|\xi(\infty) - \xi(T)|^2 = 0$, $\lim_{T\to-\infty}E|\xi(T) - \xi(-\infty)|^2 = 0$. The calculation is rather involved, and may be found in [9], pages 63–69.

In the following results we assume that the ξ processes involved satisfy the conditions as given in Theorem 2.

The following corollary is an immediate consequence of Theorem 3.

COROLLARY 1. *Let $X(t)$, $-\infty < t < \infty$ be a stationary and measurable Gaussian process with spectral representation*

$$X(t) = \int_{-\infty}^{\infty} e^{it\lambda}\,d\xi(\lambda),$$

where the ξ process necessarily has independent Gaussian increments and satisfies $E\xi(\lambda) \equiv 0$, $E\{|\xi(+\infty) - \xi(-\infty)|^2\} = F(+\infty) - F(-\infty) < \infty$, $F(\lambda+) = F(\lambda)$. Then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|X(t)|^2\,dt = \sigma_1^2 + \sigma_2^2 + \sum_{s=1}^{\infty}V_s^2.$$

Corollary 1 allows us to give a quick proof of the following result.

COROLLARY 2. *If $X(t)$ is a stationary Gaussian process satisfying the conditions of Corollary* 1, *then $X(t)$ is ergodic if and only if $|X(t)|^2$ is ergodic.*

PROOF. Suppose $X(t)$ is ergodic. Since every invariant set for the $|X(t)|^2$ process is an invariant set for the $X(t)$ process, it follows that $|X(t)|^2$ is ergodic.

Conversely, suppose $|X(t)|^2$ is ergodic. By the ergodic theorem,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(t)|^2 \, dt \quad \text{is a constant a.s.}$$

Hence by Corollary 1, $\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |X(t)|^2 = \sigma_1^2 + \sigma_2^2 + \sum_{s=1}^{\infty} V_s^2$ is a constant a.s. Hence there are no fixed points of discontinuity of $\xi(\lambda)$, which implies that F is continuous. By Maruyama's theorem ([5]), $X(t)$ is ergodic.

In the same way that Corollary 1 was proved, the following may be proved.

COROLLARY 3. *Let $X(n)$, $n = 0, \pm 1, \pm 2, \cdots$ be a real stationary Gaussian sequence with spectral representation $X(n) = \int_{-\pi}^{\pi} e^{in\lambda} \, d\xi(\lambda)$, where $E\xi(\lambda) \equiv 0$, $E\{|\xi(+\pi) - \xi(-\pi)|^2\} = F(+\pi) - F(-\pi) < \infty$, $F(\lambda +) = F(\lambda)$. Then*

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{k=-N}^{N} |X(k)|^2 = \sigma_1^2 + \sigma_2^2 + \sum_{s=1}^{\infty} V_s^2 \quad \text{a.s.}$$

REFERENCES

[1] COGBURN, R. and TUCKER, H. G. (1961). A limit theorem for a function of the increments of a decomposable process. *Trans. Amer. Math. Soc.* **99** 278–284.

[2] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[3] HEWITT, E. and STROMBERG, K. (1969). *Real and Abstract Analysis.* Springer-Verlag, New York.

[4] LEE, P. M. (1967). Infinitely divisible stochastic processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **7** 147–160.

[5] MARUYAMA, G. (1949). The harmonic analysis of stationary Gaussian processes. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **4** 45–106.

[6] MARUYAMA, G. (1970). Infinitely divisible processes. *Theor. Probability Appl.* **15** 3–23.

[7] TUCKER, H. G. (1967). *A Graduate Course in Probability.* Academic Press, New York.

[8] WINTNER, A. (1947). *The Fourier Transforms of Probability Distributions.* Baltimore.

[9] WRIGHT, A. L. (1974). Statistical inference for stationary Gaussian processes. Ph. D. thesis, Univ. of California, Irvine.

[10] YEH, J. J. (1973). *Stochastic Processes and the Wiener Integral.* Dekker, New York.

[11] YEH, J. J. (1970). Unpublished real analysis lecture notes.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
TUCSON, ARIZONA 85721