The Annals of Probability
1977, Vol. 5, No. 2, 315–317

NOTES

CORRECTION TO "RADON-NIKODYM DERIVATIVES OF GAUSSIAN MEASURES"

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Introduction. J. R. Klauder kindly pointed out that the first statement of Theorem 11 of my paper [2] is incorrect. It was claimed incorrectly that if $h = h(t)$, $0 \leqq t \leqq T$ is a (strictly) increasing absolutely continuous function with $h(0) = 0$, then a necessary and sufficient condition that the Gauss–Markov process

$$X(t) = \frac{1}{(h'(t))^{\frac{1}{2}}} W(h(t)) , \qquad 0 \leqq t \leqq T \tag{1}$$

is equivalent to the Wiener process W, $X \sim W$, is that

$$\int_0^T \left[\frac{d}{dt}(1/(h'(t))^{\frac{1}{2}})\right]^2 dt < \infty . \tag{2}$$

The case

$$h(t) = t + t^{\frac{3}{2}} , \qquad 0 \leqq t \leqq T = 1 \tag{3}$$

gives an example where (2) fails although $X \sim W$. We will prove that the condition

$$\int_0^T h(t)\left[\frac{d}{dt}(1/(h'(t))^{\frac{1}{2}})\right]^2 dt < \infty \tag{4}$$

is necessary and sufficient for $X \sim W$. Note that (3) satisfies (4) but not (2). Theorem 1 of [2] gives a general condition for a Gaussian process to be equivalent to W but the condition is difficult to apply in this case. Instead we use the elegant results of M. Hitsuda [1]. Note that [4] gives necessary and sufficient conditions among a restricted class of h for $X \sim W$. Of course the exact scale normalization $1/(h'(t))^{\frac{1}{2}}$ in (1) is necessary for $X \sim W$ (e.g., note that $cW \sim W$ only for $c = 1$).

The error in the argument in [2] that $X \sim W$ implies (2) occurs in the ninth line from the bottom of page 344 where it is incorrectly claimed that $v' \in L^2 [0, T]$ if $u'(\min(s, t))v'(\max(s, t)) \in L^2[0, T] \times [0, T]$.

The argument given for the converse assertion, that (2) implies $X \sim W$, tacitly assumes that h is bounded and under this assumption is correct since then (2) implies (4) which implies that $X \sim W$. However for unbounded h, i.e., $h(T) = \infty$, e.g.,

$$h(t) = t/(1 - t) , \qquad 0 \leqq t \leqq T = 1 , \tag{5}$$

if (1) is defined by continuity at $t = 1$ so that X is the pinned Wiener process with $X(1) = 0$, then (2) holds but $X \sim W$ is false since $W(1) \neq 0$ w.p.1. Thus the assertion "$1 \notin sp(K)$ holds automatically" on page 344 of [2] tacitly assumes bounded h. Of course, Hitsuda's method avoids the spectral condition altogether and has other advantages [1, page 299].

Proof that (4) is necessary and sufficient that $X \sim W$. If (4) holds then

$$l(s, u) = -(h'(u))^{\frac{1}{2}}(1/(h'(s))^{\frac{1}{2}})' ; \quad s > u \\ = 0 ; \quad s \leqq u \tag{6}$$

is a Volterra kernel in $L^2[0, T] \times [0, T]$ the primes denoting differentiation with respect to s or u as indicated in each term by the variable in parentheses. By Theorem 2 of [1], $Y \sim \mathbf{W}$ where Y is defined in terms of a Wiener process $\mathbf{W}$ by

$$\begin{aligned} Y(t) &= \mathbf{W}(t) - \textstyle\int_0^t \int_0^s l(s, u)\, d\mathbf{W}(u)\, ds \\ &= \mathbf{W}(t) - \textstyle\int_0^t \int_u^t l(s, u)\, ds\, d\mathbf{W}(u) \\ &= \mathbf{W}(t) - \textstyle\int_0^t (h'(u))^{\frac{1}{2}}((h'(u))^{-\frac{1}{2}} - (h'(t))^{-\frac{1}{2}})\, d\mathbf{W}(u) \\ &= \frac{1}{(h'(t))^{\frac{1}{2}}} \textstyle\int_0^t (h'(u))^{\frac{1}{2}}\, d\mathbf{W}(u) \end{aligned} \tag{7}$$

where we have used the argument on the top of page 306 of [1] to interchange the integrals in the second line of (7), and (6) in the third line. Since the last line of (7) is a Gaussian process with the same covariance as X in (1), it follows that X and Y are the same process (induce the same measure). Since $Y \sim \mathbf{W}$ and $\mathbf{W}$ is a Wiener process we have proved that (4) implies $X \sim W$.

To prove that $X \sim W$ implies (4), note that the process

$$\mathbf{X}(t) = \frac{1}{(h'(t))^{\frac{1}{2}}} \textstyle\int_0^t (h'(u))^{\frac{1}{2}}\, dW(u) \tag{8}$$

is the same process as X in (1) as observed above. Since $\mathbf{X}$ is equivalent to a Wiener process, by Theorem 1 of [1] there exists on the same space as $\mathbf{X}$ and W in (8), another Wiener process $\mathbf{W}$ for which

$$\mathbf{X}(t) = \mathbf{W}(t) - \textstyle\int_0^t (\int_0^s \mathbf{l}(s, u)\, d\mathbf{W}(u))\, ds \tag{9}$$

where $\mathbf{l}$ is a (unique) L^2 Volterra kernel. Moreover $\mathbf{W}$ is a Wiener process with respect to the same σ-fields $\mathscr{F}_t$ as W.

Since $(h'(t))^{\frac{1}{2}}\mathbf{X}(t) = \int_0^t (h'(u))^{\frac{1}{2}}\, d\mathbf{W}(u)$ is a martingale with respect to $\mathscr{F}_t$, we have for any $\tau < t$

$$E[X(t)(h'(t))^{\frac{1}{2}} \mid \mathscr{F}_\tau] = X(\tau)(h'(\tau))^{\frac{1}{2}} . \tag{10}$$

From (9) and (10) with $s \wedge \tau = \min(s, \tau)$, for $\tau < t$

$$\begin{aligned} &(h'(t))^{\frac{1}{2}}\mathbf{W}(\tau) - (h'(t))^{\frac{1}{2}} \textstyle\int_0^t (\int_0^{s\wedge\tau} \mathbf{l}(s, u)\, d\mathbf{W}(u))\, ds \\ &\quad = (h'(\tau))^{\frac{1}{2}}\mathbf{W}(\tau) - (h'(\tau))^{\frac{1}{2}} \textstyle\int_0^\tau (\int_0^s \mathbf{l}(s, u)\, d\mathbf{W}(u))\, ds . \end{aligned} \tag{11}$$

Interchanging integrals as before since $\mathbf{l} \in L^2\,[0, T] \times [0, T]$ we obtain

$$(12)\qquad \mathbf{W}(\tau)((h'(t))^{\frac{1}{2}} - (h'(\tau))^{\frac{1}{2}}) = \int_0^\tau ((h'(t))^{\frac{1}{2}} \int_u^t \mathbf{l}(s, u)\, ds - (h'(\tau))^{\frac{1}{2}} \int_u^\tau \mathbf{l}(s, u)\, ds)\, d\mathbf{W}(u)\,.$$

Considering τ and t as fixed and noting that $\int_a^b \varphi\, d\mathbf{W} = 0$ for an L^2 function φ implies $\varphi \equiv 0$ a.e., we obtain that for each $0 < u < \tau < t$, a.e.

$$(13)\qquad (h'(t))^{\frac{1}{2}} - (h'(\tau))^{\frac{1}{2}} = (h'(t))^{\frac{1}{2}} \int_u^t \mathbf{l}(s, u)\, ds - (h'(\tau))^{\frac{1}{2}} \int_u^\tau \mathbf{l}(s, u)\, ds\,.$$

Setting $\tau = u$ we obtain easily that h is twice differentiable and $\mathbf{l} = l$ in (6). Thus $l \in L^2\,[0, T] \times [0, T]$, and since $\int_0^T \int_0^T l^2(s, u)\, ds\, du$ is the left side of (4), we have shown that (4) holds.

We remark that since $X \sim W$ implies the scale changed processes $\mathbf{X}$ and $\mathbf{W}$ where, for any Y,

$$(14)\qquad \mathbf{Y}(t) = \frac{1}{(g'(t))^{\frac{1}{2}}}\, Y(g(t))$$

are also equivalent, we have $\mathbf{X} \sim \mathbf{W}$, for any increasing differentiable function g with $g(0) = 0$. Taking g to be h^{-1} and noting that $\mathbf{X} = W$ in this case we see that $X \sim W$ and only if $\mathbf{X} \sim W$, i.e., the condition (4) must be invariant under the change from h to h^{-1}. A direct proof of this fact is given in [3].

Other corrections in [2].

1. Israel Bar-David pointed out that (16.2), page 347, should include the additional term:

$$-\tfrac{1}{2}X^2(0)[R(0, 0)]^{-1}$$

on the right-hand side.

2. In footnote 3, page 332, the name referred to should be I. M. Golosov.
3. (18.19), page 352: change X_j to x_j.
4. First line of display below (18.19), page 352: change T to T^2.
5. Change (18.21), page 352 to read

$$(18.21)\qquad \Delta^2 g_k = \frac{T^2}{n^2} f_k g_{k+1}\,.$$

REFERENCES

[1] HITSUDA, M. (1968). Representation of Gaussian processes equivalent to Wiener process. *Osaka J. Math* **5** 299–312.

[2] SHEPP, L. A. (1966). Radon-Nikodym derivatives of Gaussian measures. *Ann. Math. Statist.* **37** 321–354.

[3] SHEPP, L. A. *SIAM Review* problem section. To appear.

[4] VARBERG, D. E. (1964). On Gaussian measures equivalent to Wiener measure. *Ann. Math. Statist.* **35** 262–273.

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