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REPRESENTATIONS OF INVARIANT MEASURES ON MULTITYPE GALTON–WATSON PROCESSES

BY FRED M. HOPPE

University of Alberta, Edmonton and Centre de Recherches Mathématiques, Montréal

We show that there is a one-to-one correspondence between invariant measures for the noncritical multitype Galton–Watson process and invariant measures for the single type process with a *linear* offspring probability generating function. Two corollaries emerge as simple applications, the first being Spitzer's Martin boundary representation, the second giving the asymptotic behaviour of the measures. Both require no extra moment assumptions and are valid for the multitype theory.

We consider a d-type Galton–Watson process $\{Z_n\}$, positively regular, subcritical (maximal eigenvalue $\rho < 1$), with offspring probability generating function (p.g.f.) $F(x) = (F^{(1)}(x), \cdots, F^{(d)}(x))$, $x \in [0, 1]^d$. An invariant measure is a nonnegative solution of the system

$$\pi(k) = \sum \pi(i)P(i, k)$$

where $P(i, k) = \Pr[Z_1 = k \mid Z_0 = i]$ and i, k are d-tuples of nonnegative integers excluding zero. There is a one-to-one correspondence between invariant measures and generating function (g.f.) solutions of Abel's equation

$$P(F(x)) = P(x) + 1\,, \qquad x \in [0, 1)^d,\ P(0) = 0\,, \tag{1}$$

this correspondence being valid under the normalization $P(F(0)) = 1$. For $d = 1$ a proof may be found in Harris (1963) and for $d \geqq 2$ in Hoppe (1975a).

Let $A(x)$ denote the p.g.f. of the conditional Yaglom limit distribution (Joffe and Spitzer (1967)). It satisfies the Schröder equation

$$1 - A(F(x)) = \rho(1 - A(x))\,, \qquad x \in [0, 1]^d,\ A(0) = 0\,. \tag{2}$$

Associate with $\{Z_n\}$ the single type Galton–Watson process with linear offspring p.g.f.

$$1 - \rho + \rho s$$

so that the g.f. of an invariant measure on this process satisfies

$$H(1 - \rho + \rho s) = H(s) + 1\,, \qquad s \in [0, 1),\ H(0) = 0\,. \tag{3}$$

THEOREM. *There is a one-to-one correspondence between* g.f. *solutions of* (1) *and* (3) *given by*

$$P(x) = H(A(x))\,. \tag{4}$$

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This means that if H satisfies (3) *then* (4) *defines a solution of* (1) *and, conversely, if P is a solution of* (1) *then there is a solution H of* (3) *giving the representation* (4).

PROOF OF THEOREM. The direct part follows by substitution. The converse will obtain after we prove the following lemma. F_n denotes the nth functional iterate of F.

LEMMA. *The function*

$$H(s) = \lim_{n\to\infty} P(F_n(0) + s(1 - F_n(0))) - n\,, \qquad s \in [0, 1) \tag{5}$$

exists, is a g.f., *and satisfies* (3).

PROOF OF LEMMA. Let

$$H_n(s) = P(F_n(0) + s(1 - F_n(0))) - n\,, \qquad s \in [0, 1),\ n = 0, 1, 2, \cdots. \tag{6}$$

Writing the argument of P as $(1 - s)F_n(0) + s1$ shows that it lies in $[0, 1)^d$ and so (6) is well defined. We first establish that $\{H_n\}$ is a monotonic nondecreasing sequence of functions. By convexity of F,

$$\begin{aligned} F(F_n(0) + s(1 - F_n(0))) &\equiv F((1 - s)F_n(0) + s1) \\ &\leqq (1 - s)F_{n+1}(0) + s1 \\ &= F_{n+1}(0) + s(1 - F_{n+1}(0))\,, \end{aligned}$$

which implies

$$\begin{aligned} P(F_{n+1}(0) + s(1 - F_{n+1}(0))) &\geqq P(F(F_n(0) + s(1 - F_n(0)))) \\ &= P(F_n(0) + s(1 - F_n(0))) + 1 \quad (\text{by } (1))\,, \end{aligned}$$

and then subtracting $n + 1$ from both sides

$$H_{n+1}(s) \geqq H_n(s)$$

proving monotonicity. Next we show that for each s, $\{H_n(s)\}$ is bounded above. According to the recipe for A given by Joffe and Spitzer, for any x, given $\varepsilon > 0$, for all n sufficiently large

$$\begin{aligned} (1 - \varepsilon)(1 - A(x))(1 - F_n(0)) &\leqq 1 - F_n(x) \\ &\leqq (1 + \varepsilon)(1 - A(x))(1 - F_n(0))\,. \end{aligned} \tag{7}$$

Given $s \in [0, 1)$ choose x so close to 1 that

$$0 < \tfrac{3}{2}(1 - A(x)) < 1 - s\,. \tag{8}$$

Rearrange part of (7), add and subtract $F_n(0)$, obtaining

$$F_n(0) + [1 - (1 + \varepsilon)(1 - A(x))](1 - F_n(0)) \leqq F_n(x)\,,$$

and then letting $\varepsilon = \frac{1}{2}$ and using (8),

$$F_n(0) + s(1 - F_n(0)) \leqq F_n(x)\,.$$

Applying P to this inequality, subtracting n, and since iteration of (1) shows

$P(F_n(x)) - n = P(x)$, we get

$$H_n(s) \leqq P(x)$$

for all n sufficiently large, proving the asserted boundedness.

Hence $H(s) = \lim (n \to \infty) H_n(s)$ exists and since for each n, $H_n(s)$ is a g.f., $H(s)$ is also a g.f. by the continuity theorem. It remains only to show that H satisfies (3). Clearly $H(0) = 0$. Again from Joffe and Spitzer, given any $\varepsilon > 0$, for all sufficiently large n,

$$(1 - \varepsilon)\rho(1 - F_n(0)) \leqq 1 - F_{n+1}(0) \leqq (1 + \varepsilon)\rho(1 - F_n(0)) . \tag{9}$$

Therefore

$$\begin{aligned} F_n(0) + [1 - (1 + \varepsilon)\rho](1 - F_n(0)) \\ \leqq F_{n+1}(0) \leqq F_n(0) + [1 - (1 - \varepsilon)\rho](1 - F_n(0)) , \end{aligned} \tag{10}$$

and multiplying (9) by s and adding to (10),

$$\begin{aligned} F_n(0) + [1 - (1 + \varepsilon)\rho + s(1 - \varepsilon)\rho](1 - F_n(0)) \\ \leqq F_{n+1}(0) + s(1 - F_{n+1}(0)) \\ \leqq F_n(0) + [1 - (1 - \varepsilon)\rho + s(1 + \varepsilon)\rho](1 - F_n(0)) . \end{aligned} \tag{11}$$

For any fixed $s \in [0, 1)$ whenever ε is sufficiently small the coefficients of $1 - F_n(0)$ in (11) lie in $[0, 1)$. Thus

$$H_n(1 - (1 + \varepsilon)\rho + s(1 - \varepsilon)\rho) \leqq H_{n+1}(s) + 1 \leqq H_n(1 - (1 - \varepsilon)\rho + s(1 + \varepsilon)\rho) ,$$

and taking limits as $n \to \infty$ then letting $\varepsilon \downarrow 0$ we see that H satisfies (3). □

Returning to the proof of the theorem, let $P(x)$ be any g.f. solution of (1). Let x be chosen and fixed. Suppose first that $A(x) \neq 0$. Then from (7),

$$\begin{aligned} F_n(0) + [1 - (1 + \varepsilon)(1 - A(x))](1 - F_n(0)) \\ \leqq F_n(x) \leqq F_n(0) + [1 - (1 - \varepsilon)(1 - A(x))](1 - F_n(0)) , \end{aligned} \tag{12}$$

where, as before, for ε sufficiently small the coefficients of $1 - F_n(0)$ lie in $[0, 1)$. Consequently

$$H_n(1 - (1 + \varepsilon)(1 - A(x))) \leqq P(x) \leqq H_n(1 - (1 - \varepsilon)(1 - A(x))) ,$$

and letting $n \to \infty$ followed by $\varepsilon \downarrow 0$,

$$H(A(x)) = P(x) .$$

If $A(x) = 0$ we proceed analogously but using the inequalities

$$F_n(0) \leqq F_n(x) \leqq F_n(0) + \varepsilon(1 - F_n(0))$$

in place of (11) to deduce that $P(x) = 0 = H(0) = H(A(x))$. Thus in all cases we obtain (4). □

Example. $H(s) = \log (1 - s)/\log \rho$ is obviously a g.f. solution of (3) so we immediately have $P(x) = \log(1 - A(x))/\log \rho$ as a solution of (1), a result previously established (Hoppe (1975a)) by a direct calculation. This example is a special case of the following much more general result.

COROLLARY 1. *For every probability measure* ν *on* $[0, 1)$, *the function*

$$P(x) = \int_0^1 \{\sum_{n=-\infty}^{\infty} [\exp\{-(1 - A(x))\rho^{n-t}\} - \exp\{-\rho^{n-t}\}]\}\nu(dt) \tag{13}$$

is the g.f. *of an invariant measure and conversely every invariant measure has a representation* (13) *for some probability measure* ν *on* $[0, 1)$.

PROOF. Spitzer (1967) has shown that in the case of a Galton–Watson process with a linear offspring p.g.f., for every probability measure ν on $[0, 1)$, the function

$$H(s) = \int_0^1 U(s, t)\nu(dt)$$

is the g.f. of an invariant measure, where

$$U(s, t) = \sum_{n=-\infty}^{\infty} [\exp\{-(1 - s)\rho^{n-t}\} - \exp\{-\rho^{n-t}\}] .$$

Conversely, every invariant measure has such a representation for some ν. Our corollary follows as an immediate consequence. □

REMARKS. When $\rho > 1$ there is a similar representation obtained by employing the usual fixed point transformation to reduce the supercritical case to the subcritical case. In case $\rho = 1$, invariant measures exist and are unique (Hoppe (1975b)). In fact, the uniqueness proof in the critical case depends on a lemma similar in spirit to the one used here.

COROLLARY 2. *If* $\{\pi(k)\}$ *is an invariant measure for* $\rho < 1$ *then*

$$\sum_{k: k\cdot u \leq y} \pi(k) \sim \frac{-\log y}{\log \rho} \qquad \text{as} \quad y \to \infty . \tag{14}$$

PROOF. Let $H(s) = \sum \nu_j s^j$ be the g.f. defined by (3). Iterate (3) and set $s = 0$ obtaining $H(1 - \rho^n) = n$. For each s, let n be such that $1 - \rho^n \leqq s \leqq 1 - \rho^{n+1}$. This yields the bounds

$$\frac{n}{n+1} \leqq \frac{H(s) \log \rho}{\log(1 - s)} \leqq \frac{n+1}{n} .$$

We conclude that

$$\lim_{s \to 1-} \frac{H(s) \log \rho}{\log(1 - s)} = 1 .$$

Define $\phi(t) = 1 - A(1 - tu)$ for small positive t where u is the right eigenvector corresponding to the eigenvalue ρ of the offspring expectation matrix for the process $\{Z_n\}$. It is shown in Hoppe (1975a) that

$$\phi(t) = tL(t) \tag{15}$$

where $L(\cdot)$ is slowly varying at 0. Thus

$$\frac{(\log \rho)P(1 - tu)}{\log t} = \frac{H(1 - \phi(t))}{\log \phi(t)} \left[1 + \frac{\log L(t)}{\log t}\right] \log \rho$$

and so

$$\lim_{t\to 0} \frac{(\log \rho)P(1 - tu)}{\log t} = 1 . \tag{16}$$

Setting $e^{-tu} = (e^{-tu_1}, \cdots, e^{-tu_d})$ it follows from (16) that

$$\lim_{t\to 0} \frac{(\log \rho)P(e^{-tu})}{\log t} = 1 .$$

However $P(e^{-tu})$ is the Laplace–Stieltjes transform of the measure μ on $[0, \infty)$ where

$$\mu\{[0, y]\} = \sum_{k:k\cdot u \leqq y} \pi(k) .$$

Karamata's Tauberian theorem (Feller, page 445) then yields (14). □

REMARKS. Lipow (1971) (see Athreya and Ney, page 89 for the proof) first obtained this corollary for $d = 1$ under the logarithmic moment condition $E[Z_1 \log Z_1] < \infty$. Seneta (1971) also had shown (14), without this moment restriction, for the special invariant measure whose g.f. is

$$G(x) = \frac{\log (1 - A(x))}{\log \rho} . \tag{17}$$

$G(x)$ was characterized as the unique g.f. satisfying a regular variation condition and then (14) obtained directly from analogous properties of the p.g.f. $A(x)$. Seneta's result also holds for multitype processes and the precise condition on $G(x)$ is

$$\lim_{t\to 0} G(1 - \lambda tu) - G(1 - tu) = \frac{\log \lambda}{\log \rho}, \quad \lambda > 0 . \tag{18}$$

Thus in this case, (16) may be strengthened to

$$G(1 - tu) = \frac{\log t}{\log \rho} + \frac{\log L(t)}{\log \rho} . \tag{19}$$

(We note in passing that $L(0) < \infty$ iff the logarithmic moment condition holds.)

A little further analysis will now give relations similar to (18) and (19) valid for all g.f. $P(x)$. In fact it is easy to see that

$$H(1 - t) - \frac{\log t}{\log \rho} = \gamma_H(t)$$

where $|\gamma_H(t)| \leqq 1$ for all t. Now invoke the representation (4) to obtain

$$\begin{aligned} P(1 - tu) &= H(A(1 - tu)) \\ &= \frac{\log [1 - A(1 - tu)]}{\log \rho} + \gamma_H(1 - A(1 - tu)) . \end{aligned}$$

Thus by (17)

$$P(1 - tu) - G(1 - tu) = \gamma_H(1 - A(1 - tu)) \tag{20}$$

and therefore

$$P(1 - tu) = \frac{\log t}{\log \rho} + \frac{\log L(t)}{\log \rho} + \gamma_H(1 - A(1 - tu)) .$$

This generalizes (19). Notice also that if $P_1(t)$ and $P_2(t)$ are any g.f. of invariant measures then

$$|P_1(1 - tu) - P_2(1 - tu)| \leqq 2 \qquad \text{for all} \quad t .$$

Equation (20) shows that any g.f. "stays close" to the special one $G(x)$ uniformly as $t \to 0$ (i.e., as the functions blow up). This suggests on an intuitive level why there is one and only one $G(x)$ satisfying the property (18). Specifically, from (3),

$$H(1 - \rho t) \doteq H(1 - t) + 1 ,$$

so that

$$\begin{aligned} \gamma_H(\rho t) &\equiv H(1 - \rho t) - \left[\frac{\log \rho t}{\log \rho}\right] \\ &= H(1 - t) + 1 - \frac{\log \rho}{\log \rho} - \frac{\log t}{\log \rho} \\ &= \gamma_H(t) . \end{aligned}$$

Also, $\gamma_H(1) = 0$. Thus, unless $\gamma_H(t)$ is identically a constant (in which case the constant is zero) it repeats itself over each interval $[\rho^{n+1}, \rho^n]$ and since these intervals decrease to zero, as $t \to 0$, $\gamma_H(t)$ oscillates wildly near $t = 0$. This is what causes the regularity manifested by (18) to break down in general.

Finally, consider the difference

$$\begin{aligned} (21) \qquad P(1 - 2tu) - P(1 - tu) = {} & [G(1 - 2tu) - G(1 - tu)] \\ & + [\gamma_H(1 - A(1 - 2tu)) - \gamma_H(1 - A(1 - tu)) . \end{aligned}$$

Define $r(t) = \exp P(1 - tu)$. Then from (18) and (21) it follows that

$$c_1 2^{1/\rho} \leqq \frac{r(2t)}{r(t)} \leqq c_2 2^{1/\rho} , \qquad t > 0 ,$$

for positive constants c_1 and c_2. In the terminology of Feller (definition, page 289) the function $r(t)$ is said to vary dominatedly at zero. This appears to be the best universal regularity condition analogous to (18) possessed by arbitrary g.f. $P(x)$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, CANADA T6G 2G1