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A RATIO LIMIT THEOREM FOR SUBTERMINAL TIMES[1]

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Consider a recurrent random walk $\{X_n\}$ with state space $S \subseteq R^d$ ($d \leqq 2$). A stopping time T is called subterminal if it satisfies a technical condition which essentially states that it is the first time a path possesses some property which does not depend on how long the process has been running. Suppose T is a subterminal time which can occur only when $\{X_n\}$ is in a bounded set; then under an additional assumption a ratio limit theorem (as $n \to \infty$) is obtained for $P(T > n \mid X_0 = x)$ ($x \in S$). The theorem applies in particular to the hitting time of a bounded set with nonempty interior in the general case, and to the hitting time of a bounded set with nonzero Haar measure in the nonsingular case.

1. Introduction and statement of results. Let μ be a probability measure on $\mathbb{R}^d$ ($d \leqq 2$) and let S be the closed subgroup of $\mathbb{R}^d$ generated by the support of μ. We assume that S is equal to $\mathbb{R}^d$, $\mathbb{Z}^d$, or $\mathbb{R} \times \mathbb{Z}$. Let $\{X_n\}$ be the random walk generated by μ, and assume $\{X_n\}$ is recurrent (i.e., $\sum_n \mu^{(n)}(G) = \infty$ for all nonempty open sets $G \subseteq S$, where $\mu^{(n)}$ denotes the n-fold convolution of μ with itself). For any Borel set $B \subseteq S$ let $T_B = \inf\{n > 0 : X_n \in B\}$ ($= \infty$ if no such n exists) be the hitting time of B, and set $R_n = P^D(T_D > n) \equiv \int_D P^x(T_D > n)\,dx$ where D is the unit ball in S. Here $P^x(\cdot)$ denotes $P(\cdot \mid X_0 = x)$ and dx denotes Haar measure. We assume given shift operators θ_n, which are transformations on the underlying probability space Ω on which $\{X_n\}$ is defined and which satisfy for all m and n

$$X_m(\theta_n \omega) = X_{m+n}(\omega) \qquad (\omega \in \Omega)\,.$$

Recall that random variables composed with θ_n should be interpreted as "starting from time n."

Our main result concerns the asymptotic behavior as $n \to \infty$ of $P^x(T > n)$, where T is a member of a class $\mathscr{U}$ of stopping times which will be defined in a moment. Specifically, the following result will be proved:

THEOREM 1.1. *Let $T \in \mathscr{U}$. Then for any $x \in S$*

$$\lim_{n\to\infty} \frac{P^x(T > n)}{R_n} = L_T(x)\,.$$

Moreover, the convergence is uniform for x in compacts and the limit is bounded on compacts.

In order to define $\mathscr{U}$ we first define the following classes of sets:

$$\mathscr{A} = \{\text{bounded Borel subsets of } S\}$$

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and

$$\mathscr{B} = \{B \in \mathscr{A}: \text{for each } C \in \mathscr{A}, \ \exists\, n \text{ such that } \inf_{x \in C} P^x(T_B \leqq n) > 0\}\,.$$

We shall see that sets in $\mathscr{B}$ have properties (in the general state space case) similar to properties of points in the lattice case. $\mathscr{U}$ is then defined to be the set of all stopping times T (relative to the sigma fields $\mathscr{F}_n = \sigma(X_1, \cdots, X_n)$) which possess properties (i)—(iii) of the following definition.

DEFINITION 1.2. A stopping time T

(i) is *subterminal* if for all n, $T \leqq n + T \circ \theta_n$ a.s.;

(ii) has the *uniform lower bound* property if for some $C \in \mathscr{B}$ there exists n such that $\inf_{x \in C} P^x(T \leqq n) > 0$; and

(iii) has *bounded support* if there exists some $B \in \mathscr{A}$ such that $T < \infty$ implies $X_T \in B$ a.s.

Some comments on (i)—(iii) are in order. Since $T \circ \theta_n$ is the time of T's occurrence for the process starting from time n, (*i*) essentially means that T is the first time the path of the process possesses some property which does not depend on how long the process has been running. Any terminal time (T satisfying $T = n + T \circ \theta_n$ a.s. on $\{T > n\}$) is subterminal, and in particular $T_B \in \mathscr{U}$ for any $B \in \mathscr{B}$. More examples of subterminal times will be given at the end of Section 3.

(iii) is self-explanatory. Condition (ii) allows certain estimates to be made in the general state space case which use point recurrence in the lattice case. We shall show in Proposition 3.1 that for general μ, if $B \in \mathscr{A}$ has nonempty interior then $B \in \mathscr{B}$; while for nonsingular random walks, $|B| > 0$ ($|\cdot|$ denotes the Haar measure) is sufficient for $B \in \mathscr{A}$ to be in $\mathscr{B}$. In particular, in the lattice case the uniform lower bound property reduces to the requirement that $P^x(T < \infty) > 0$ for some $x \in S$.

We also prove the following result, which shows that the set used in defining R_n is of no special importance:

THEOREM 1.3. *Let $A, B \in \mathscr{B}$. Then*

$$\lim_{n \to \infty} \frac{P^A(T_A > n)}{P^B(T_B > n)} = 1\,.$$

Note that this result shows that $P^A(T_A > n)$ and $P^B(T_B > n)$ are asymptotically equal, while in the lattice case they would in fact be equal for all n if A and B were one point sets.

Theorem 1.3 together with the fact that $T_B \in \mathscr{U}$ if $B \in \mathscr{B}$ results in the following special case of Theorem 1.1. Although in fact a corollary, it is stated as a theorem because of its independent interest.

THEOREM 1.4. *If $B \in \mathscr{B}$, then for any $x \in S$*

$$\lim_{n \to \infty} \frac{P^x(T_B > n)}{P^B(T_B > n)} = L_B(x) < \infty\,. \tag{1.1}$$

Moreover, the convergence is uniform on compacts and the limit is bounded on compacts.

Theorem 1.4 was proved for the lattice case $S = \mathbb{Z}^d$ by Kesten and Spitzer in [1]. In [3], Ornstein generalized these results to show that, for μ an arbitrary probability measure on $\mathbb{R}$ with infinite variance, (1.1) holds for any bounded interval B and any $x \in S$. His proofs actually are valid for any bounded set B with nonempty interior and $|\partial B| = 0$, and work in the two-dimensional case as well. We shall briefly outline these extensions in Section 6. Using Tauberian theorems, Port and Stone ([4]) obtained (1.1) for μ on $\mathbb{R}^d$ in the domain of attraction of a stable law. Their theorem requires B to have nonempty interior in the general case and positive Haar measure in the nonsingular case. They also obtained weaker ratio limit theorems, including a version of Theorem 1.3, for arbitrary μ on $\mathbb{R}^d$.

Theorem 1.4 generalizes these previous results on T_B in that there are no restrictions on ∂B and that B need only have positive Haar measure in the nonsingular case. Even in the singular case (1.1) may hold for sets with empty interior, since examples may be constructed of singular random walks which have sets in $\mathscr{B}$ with empty interior.

Our stopping times are quite similar to some which Port and Stone considered for stable processes in [5], where they obtained a version of Theorem 1.1 by Tauberian methods. Our methods are similar to those of Kesten and Spitzer and of Ornstein.

The outline of the paper is as follows: In Section 2 the necessary notation is introduced. Section 3 contains examples and some elementary properties of $\mathscr{U}$. In Section 4 we define the function L_T which arises as the limit in Theorem 1.1, and give it a characterization in terms of the behavior at infinity of the "Green's operator" G_T of T. Section 5 develops and states some analytical properties of R_n and $\mu^{(n)}(B)$, using in part some of Ornstein's techniques to adapt Kesten and Spitzer's lattice case arguments to the case of general S. The methods of Section 5 also can be used to prove $\sup_n P^B(T_B > n)/P^B(T_B > 2n) < \infty$, which was proved by Ornstein ([3]) for B an interval. The proofs of Theorems 1.1 and 1.3 are then given in Section 6.

2. Notation. We introduce here some general definitions and notation which were not covered in the introduction. For events A and B we denote $(A \cap B)$ by $(A; B)$. If Y is a random variable with range in S and f a measurable function on S, then we denote $E[f(Y)]$ by $\int P(Y \in dy)f(y)$ and more generally $E[f(Y) \cdot 1_A]$ by $\int P(A; Y \in dy)f(y)$ (provided the integrals are defined, of course), where 1_A is the indicator function of A.

If $B \subseteq S$ is a Borel set (we shall only consider Borel sets), we define

$$\begin{aligned} T_B^{(0)} &\equiv 0 \quad \text{and} \\ T_B^{(k)} &= \inf\{n > T_B^{(k-1)} : X_n \in B\} \quad (k \geqq 1)\,; \\ &= \infty \qquad \text{if no such } n \text{ exists} \quad (k \geqq 1)\,. \end{aligned}$$

$T_B^{(k)}$ is thus the time of the kth visit to B; $T_B^{(1)}$ will be generally written as T_B. In addition to the classes $\mathscr{A}$ and $\mathscr{B}$ defined in the introduction, we shall have occasion to consider

$$\mathscr{B}_0 = \{B \in \mathscr{A} \colon \operatorname{int}(B) \neq \emptyset\} .$$

For a stopping time T we define the operator G_T on $S \times$ {Borel subsets of S} by

$$G_T(x, dy) = \sum_{n=1}^{\infty} P^x(T > n; X_n \in dy) ;$$

$G_T(x, A)$ is thus the expected number of visits to A before T occurs. If $T = T_B$ then we shall write G_B for G_{T_B}. We remark here that $P^x(X_0 \in \cdot) = \delta_x(\cdot)$ where δ_x is the unit point mass at x. For $B \in \mathscr{A}$ and $x \in S$ we also define

$$r_n(x, B) = P^x(T_B > n) .$$

By an abuse of notation, we will occasionally write $\int_A f(x)\, dx$ as $f(A)$; in particular

$$P^A(\cdot) \equiv \int_A P^x(\cdot)\, dx$$

and

$$r_n(A, B) \equiv \int_A r_n(x, B)\, dx .$$

Let $\hat{\mu}$, defined by $\hat{\mu}(A) = \mu(-A)$, be the dual measure to μ; $\hat{\mu}$ also defines a (recurrent) random walk on S, which is $\{X_n\}$ "reversed." Quantities referring to the reversed random walk will be denoted by $\hat{}$; e.g., $\hat{P}^x$, $\hat{G}_B$. It follows easily from

$$\int_A \mu(B - x)\, dx = \int_B \hat{\mu}(A - y)\, dy$$

that

$$P^A(X_n \in B) = \hat{P}^B(X_n \in A)$$

and

$$P^A(X_n \in B;\, T_C \geqq n) = \hat{P}^B(X_n \in A;\, T_C \geqq n)$$

for any sets A, B, C and any $n \geqq 0$.

Finally, $\{X_n\}$ is said to be nonsingular if some convolution of μ has a nontrivial absolutely continuous component with respect to Haar measure.

3. Subterminal times: elementary properties and examples. We first determine sufficient conditions for a set to be in the class $\mathscr{B}$ defined in the introduction.

PROPOSITION 3.1. *If $B \in \mathscr{A}$ and* int $(B) \neq \emptyset$ *then $B \in \mathscr{B}$. In the nonsingular case, if $B \in \mathscr{A}$ and $|B| > 0$ then $B \in \mathscr{B}$.*

REMARK. It follows from the second part of this proposition that in the nonsingular nonlattice case, $\mathscr{B}$ strictly contains the set of all bounded open sets. Examples may also be constructed of singular random walks where this occurs.

PROOF OF THE PROPOSITION. For the first statement let I be a ball centered at 0 sufficiently small that $B_0 + I \subseteq B$ for some open set B_0, and fix $x \in S$. Then for any $u \in I$, $P^{x+u}(T_B \leqq n) = P^x(T_{B-u} \leqq n) \geqq P^x(T_{B_0} \leqq n) \geqq \delta > 0$ for n sufficiently large. A covering argument then yields the desired result.

For the second statement fix $x \in S$. As in [4], we may find n such that $\mu^{(n)}$ may be decomposed as $\varphi_n + \nu_n$, where φ_n has a density f which is bounded away from 0, say $f \geqq \delta$, on some ball $N - x$. Let then N_0 be a ball such that $N_0 + I \subseteqq N$ for some ball I centered at 0, and pick m such that $P^{N_0}(X_m \in B) = \hat{P}^B(X_m \in N_0) = \varepsilon > 0$. Then for any $u \in I$

$$\begin{aligned} P^{x+u}(X_{n+m} \in B) &\geqq \int \mu^{(n)}(dz) 1_{N_0}(z + x + u) P^{z+x+u}(X_m \in B) \\ &\geqq \delta\varepsilon , \end{aligned}$$

and thus $\inf_{y \in x+I} P^y(T \leqq n + m) \geqq \delta\varepsilon$. A covering argument again completes the proof. □

Observe that, by writing $\{x \in S: P^x(T < \infty) > 0\}$ as $\bigcup_{\delta_n \downarrow 0} \{x \in S: P^x(T < \infty) > \delta_n\}$, one has from Proposition 3.1 that in the nonsingular case the uniform lower bound property reduces to the requirement that $|\{x \in S: P^x(T < \infty) > 0\}| > 0$. In particular, in the lattice case T has the uniform lower bound property if $P^x(T < \infty) > 0$ for some $x \in S$.

The following result gives some useful consequences of the uniform lower bound property. Note that a subterminal time satisfies Definition 1.2(i) with n replaced by any nonnegative integer-valued random variable.

PROPOSITION 3.2. *Let T be a subterminal time.*

(i) *If T has the uniform lower bound property, then for each $A \in \mathscr{A}$ there exists n such that* $\inf_{x \in A} P^x(T \leqq n) > 0$.

(ii) *If T has the uniform lower bound property, then for any $A, C \in \mathscr{A}$ there exist $\delta < 1$ and j such that for all $k \geqq 1$,* $\sup_{x \in A} P^x(T > T_C^{(jk)}) \leqq \delta^k$.

(iii) *T has the uniform lower bound property if and only if* $\lim_{n \to \infty} P^x(T \leqq n) = 1$ *uniformly for x in compacts.*

PROOF. To verify (i) let $A \in \mathscr{A}$ and pick $C \in \mathscr{B}$ and n such that $\inf_{y \in C} P^y(T \leqq n) = \varepsilon > 0$. Since $C \in \mathscr{B}$, m may be found such that $\inf_{x \in A} P^x(T_C \leqq m) = \delta > 0$. Then for $x \in A$, T's subterminality guarantees that

$$\begin{aligned} P^x(T \leqq m + n) &\geqq P^x(T_C \leqq m;\ T_C + T \circ \theta_{T_C} \leqq m + n) \\ &\geqq \int_C P^x(T_C \leqq m;\ X_{T_C} \in dy) P^y(T \leqq n) \\ &\geqq \delta\varepsilon > 0 . \end{aligned}$$

For (ii) we use an argument similar to one in [5]. Let $A, C \in \mathscr{A}$ and use part (i) of this proposition to pick j sufficiently large that $\inf_{x \in A \cup C} P^x(T \leqq j) = 1 - \delta > 0$. Since $T_C^{(j)} \geqq j$, this proves (ii) for $k = 1$. Now assume (ii) true for $k \leqq l$, and use T's subterminality to conclude that for $x \in A$

$$\begin{aligned} P^x(T > T_C^{(j(l+1))}) &= P^x(T > T_C^{(jl)};\ T > T_C^{(jl)} + T_C^{(j)} \circ \theta_{T_C^{(jl)}}) \\ &\leqq P^x(T > T_C^{(jl)};\ T \circ \theta_{T_C^{(jl)}} < T_C^{(j)} \circ \theta_{T_C^{(jl)}}) \\ &= \int_C P^x(T > T_C^{(jl)};\ X_{T_C^{(jl)}} \in dy) P^y(T > T_C^{(j)}) \\ &\leqq \delta^{l+1} . \end{aligned}$$

This proves (ii).

The if part of (iii) is immediate. For the only if part, let A be compact and $\rho > 0$ be given. Let C be a ball large enough that $C - A \supseteqq C_0$ for some other ball C_0. By part (ii) of this proposition, pick k so large that

$$\sup_{x \in A} P^x(T > T_C^{(k)}) < \frac{\rho}{2} .$$

Then pick N so large that for $n \geqq N$ and $x \in A$ one has $P^x(T_C^{(k)} > n) = P^0(T_{C-x}^{(k)} > n) \leqq P^0(T_{C_0}^{(k)} > n) < \rho/2$. Then for all such x and n one has

$$\begin{aligned} P^x(T > n) &\leqq P^x(T > T_C^{(k)}) + P^x(T_C^{(k)} > n) \\ &< \rho , \end{aligned}$$

completing the proof of (iii). □

As an immediate consequence we obtain the following useful result:

PROPOSITION 3.3. *Let $T \in \mathscr{U}$ and $C \in \mathscr{A}$. Then $G_T(x, C)$ is bounded for $x \in S$.*

PROOF. Use Proposition 3.2(ii) to pick j and $\delta < 1$ such that $\sup_{x \in C} P^x(T > T_C^{(ji)}) \leqq \delta^i$ for all $i \geqq 1$. If N is the number of visits to C before T occurs, one then has for $x \in C$ that

$$\begin{aligned} G_T(x, C) = E^x(N) &= \textstyle\sum_{k=1}^{\infty} P^x(N \geqq k) = \sum_{k=1}^{\infty} P^x(T > T_C^{(k)}) \\ &\leqq j - 1 + j \textstyle\sum_{i=1}^{\infty} P^x(T > T_C^{(ji)}) \leqq j - 1 + j \sum_{i=1}^{\infty} \delta^i , \end{aligned}$$

showing that $\sup_{x \in C} G_T(x, C) = M < \infty$. T's subterminality then guarantees that for any $x \in S$,

$$\begin{aligned} G_T(x, C) &= \textstyle\sum_{k=1}^{\infty} P^x(T > T_C^{(k)}) \leqq 1 + \sum_{k=2}^{\infty} P^x(T \circ \theta_{T_C} > T_C^{(k-1)} \circ \theta_{T_C}) \\ &= 1 + \textstyle\int_C P^x(X_{T_C} \in dy) G_T(y, C) \leqq 1 + M , \end{aligned}$$

establishing the proposition. □

We next investigate some closure properties of $\mathscr{U}$.

PROPOSITION 3.4. *If $R, S \in \mathscr{U}$ then*

(i) $\max(R, S) \in \mathscr{U}$,
(ii) $\min(R, S) \in \mathscr{U}$,

and

(iii) $R + S \circ \theta_R \in \mathscr{U}$.

PPOOF. It is evident that the stopping times defined in (i)—(iii) have bounded support. Subterminality is also easily checked in cases (i) and (ii). For case (iii), note first that for any i, j and k one has $X_i(\theta_j \theta_k \omega) = X_{i+j+k}(\omega) = X_i(\theta_{j+k} \omega)$ for all $\omega \in \Omega$. Therefore by the monotone class theorem, $U(\theta_j \theta_k) \doteq U(\theta_{j+k})$ a.s. for any $\mathscr{F}$-measurable random variable U. It follows then by S's subterminality that for any $i \leqq j$, $i + S(\theta_i) \leqq j + S(\theta_{j-i} \theta_i) = j + S(\theta_j)$ a.s. Now let $T = R + S \circ \theta_R$, let n be arbitrary, and suppose $R(\omega) = i \leqq n + R(\theta_n \omega) \equiv j$. One then obtains $T(\omega) = i + S(\theta_i \omega) \leqq j + S(\theta_j \omega) = n + R(\theta_n \omega) + S(\theta_{n+R(\theta_n\omega)}\omega) = n + R(\theta_n \omega) + S(\theta_{R(\theta_n\omega)}\theta_n \omega) = n + T(\theta_n \omega)$ a.s., showing that T is indeed subterminal.

We next verify the uniform lower bound property. Since $P^x(\min(R, S) \leqq n) \geqq P^x(R \leqq n)$ for any x and n, $\min(R, S)$ has the uniform lower bound property if R does. For (i), let C be a given compact set. For given $\varepsilon > 0$, use Proposition 3.2(iii) to pick N so large that $P^x(R > n) < \varepsilon/2$ and $P^x(S > n) < \varepsilon/2$ whenever $x \in C$ and $n \geqq N$. Then $P^x(\max(R, S) > n) < \varepsilon$ for all such n and x, so Proposition 3.2(iii) implies that $\max(R, S)$ has the uniform lower bound property. As for (iii), let A be compact and pick m so that $\inf_{x \in A} P^x(R \leqq m) = \delta > 0$. Let R have support in a bounded set B, and pick n so that $\inf_{y \in B} P^y(S \leqq n) = \varepsilon > 0$. Then for $x \in A$ one obtains $P^x(R + S \circ \theta_R \leqq m + n) \geqq P^x(R \leqq m; S \circ \theta_R \leqq n) = \int_B P^x(R \leqq m; X_R \in dy) P^y(S \leqq n) \geqq \delta\varepsilon$, so $R + S \circ \theta_R$ has the uniform lower bound property. □

We shall now give some examples of stopping times in $\mathscr{U}$. Our examples are intended to be indicative rather than exhaustive. As remarked earlier, $T_B \in \mathscr{U}$ for any $B \in \mathscr{B}$. By writing $T_B{}^{(k)} = T_B{}^{(k-1)} + T_B \circ \theta_{T_B{}^{(k-1)}}$ and using induction, it follows from Proposition 3.4(iii) that $T_B{}^{(k)} \in \mathscr{U}$ for any k. This may be generalized by considering a general additive functional $\{A_n\}$ defined by $A_n = \sum_{m=1}^n v(X_m)$ for some function $v \geqq 0$. Let c be a fixed positive number. If v has compact support and $v \geqq \delta \cdot 1_B$ for some $\delta > 0$ and $B \in \mathscr{B}$, then $T = \inf\{n > 0 : A_n \geqq c\} \in \mathscr{U}$. For $c = k$ and $v = 1_B$, T of course reduces to $T_B{}^{(k)}$. As another example, let B be bounded and let A be a (not necessarily bounded) set such that $A \supseteq C$ for some $C \in \mathscr{B}$ such that $\inf_{y \in C} P^y(X_1 \in B) > 0$. Then $T = \inf\{n > 1 : X_n \in B; X_{n-1} \in A\} \in \mathscr{U}$. One can of course generalize this to $T = \inf\{n > n_1 : X_{n-n_1} \in A; \cdots; X_{n-n_k} \in A_k; X_n \in B\}$ for appropriate sets A_i and fixed integers $n_1 > n_2 > \cdots > n_k > 0$. Next, suppose that $B_1, B_2, \cdots, B_k$ are all in $\mathscr{B}$. Then Proposition 3.4(iii) guarantees that $T_{B_1} + T_{B_2} \circ \theta_{T_{B_1}}$, the time of the first visit to B_2 after hitting B_1, is in $\mathscr{U}$. By the same reasoning, $\mathscr{U}$ also contains $T =$ first time that B_1, then $B_2, \cdots$, then B_k are visited. Finally, Proposition 3.4(i) shows that $T = \max\{T_{B_1}, \cdots, T_{B_k}\}$, the first time that all of the B_i are visited, is in $\mathscr{U}$.

4. The function L_T. In this section we define the function L_T which will arise as the limit in Theorem 1.1. Recall first that $\mathscr{B}_0 = \{B \in \mathscr{A} : \operatorname{int}(B) \neq \emptyset\} \subseteq \mathscr{B}$ and that Port and Stone in [4] have defined for each $C \in \mathscr{B}_0$ a function $L_C : S \to \mathbb{R}$ with the following properties:

(i) $L_C \geqq 0$ and $\int_C L_C(x)\,dx = 1$;

(ii) if $A \in \mathscr{A}$ and $|\partial A| = 0$, then $|A| L_C(x) = \operatorname{Lim}_{|y|\to\infty} G_C(x, A + y)$ uniformly for x in compacts; and

(iii) $L_C(x)$ is bounded for x in compacts.

(Here $\operatorname{Lim}_{|y|\to\infty} \equiv \lim_{|y|\to\infty}$ unless $d = 1$ and μ has finite variance, in which case $\operatorname{Lim}_{|y|\to\infty} \equiv \frac{1}{2}(\lim_{y\to+\infty} + \lim_{y\to-\infty})$.)

DEFINITION 4.1. Let $T \in \mathscr{U}$. Define

$$L_T(x) = L_C(x) + \int_C G_T(x, dy) L_C(y) \qquad x \in S,$$

where C is any member of $\mathscr{B}_0$ such that T has support in C.

PROPOSITION 4.2. *L_T is well defined and is bounded on compacts. Moreover, if $B \in \mathscr{B}_0$ then $L_{T_B} = L_B$.*

PROOF. Let $T \in \mathscr{U}$ and $x \in S$, and suppose C and F in $\mathscr{B}_0$ are such that T has support in both C and F. Let A be a ball such that $A \cap (C \cup F) = \emptyset$. For $n \geqq 1$ write

$$P^x(T > n; X_n \in A) = P^x(T_C > n; X_n \in A) + \sum_{m=1}^{n} P^x(\sigma_n(C) = m; T > n; X_n \in A),$$

where $\sigma_n(C)$ is the time of the last visit to B in the interval $[1, n]$. Since T cannot occur if the process is not in C, for each term in the summation the condition $T > n$ is equivalent to $T > m$. Conditioning on $\mathscr{F}_m$ then results in

$$P^x(T > n; X_n \in A) = P^x(T_C > n; X_n \in A) + \sum_{m=1}^{n} \int_C P^x(T > m; X_m \in dy) P^y(T_C > n - m; X_{n-m} \in A),$$

and summing this on n yields

$$G_T(x, A) = G_C(x, A) + \int_C G_T(x, dy) G_C(y, A)$$

since $A \cap C = \emptyset$. Clearly the same decomposition may be made according to T_F, and equating these expressions gives

$$G_C(x, A) + \int_C G_T(x, dy) G_C(y, A) = G_F(x, A) + \int_F G_F(x, dy) G_F(y, A).$$

Now replace A by $A + z$ in the above expression and let $|z| \to \infty$. It follows from the finiteness of $G_T(x, C)$ and $G_T(x, F)$ together with property (ii) of L_C that

$$L_C(x) + \int_C G_T(x, dy) L_C(y) = L_F(x) + \int_F G_T(x, dy) L_F(y),$$

proving that L_T is well defined. That L_T is bounded on compacts follows immediately from Proposition 3.3 and property (iii) of L_C. For the third assertion, let $B \in \mathscr{B}_0$ and use $C = B$ in Definition 4.1. Since $G_{T_B}(\cdot, B) \equiv 0$, it follows that $L_{T_B} = L_B$. □

Henceforth we shall denote L_{T_B} by L_B for $B \in \mathscr{B}$. We have just seen that property (iii) of L_C extends to L_T. The next result deals with properties (i) and (ii).

PROPOSITION 4.3. *Let $T \in \mathscr{U}$.*

(i) *If $A \in \mathscr{A}$ and $|\partial A| = 0$, then $|A| L_T(x) = \operatorname{Lim}_{|y| \to \infty} G_T(x, A + y)$ uniformly for x in compacts.*

(ii) *If $T = T_B$ for $B \in \mathscr{B}$, then $\int_B L_B(x)\, dx = 1$.*

PROOF. For (i), let $A \in \mathscr{A}$ be such that $|\partial A| = 0$ and pick $C \in \mathscr{B}_0$ such that T has support in C. If $|z|$ is large enough that $(A + z) \cap C = \emptyset$, one has as in the preceding proposition that

$$G_T(x, A + z) = G_C(x, A + z) + \int_C G_T(x, dy)G_C(y, A + z) .$$

Letting $|z| \to \infty$ and using property (ii) of L_C together with Proposition 3.3 then establishes (i). To prove the second assertion, pick $C \in \mathscr{B}_0$ containing B. By using the definition of L_B and reversing the process one obtains

$$\begin{aligned}\int_B L_B(x)\, dx &= \int_B L_C(x)\, dx + \int_B dx \int_{C\setminus B} \sum_{n=1}^{\infty} P^x(T_B \geqq n; X_n \in dy)L_C(y) \\ &= \int_B L_C(x)\, dx + \int_{C\setminus B} dy\, L_C(y) \int_B \sum_{n=1}^{\infty} \hat{P}^y(T_B \geqq n; X_n \in dx) \\ &= \int_B L_C(x)\, dx + \int_{C\setminus B} \sum_{n=1}^{\infty} \hat{P}^y(T_B = n)L_C(y)\, dy \\ &= \int_B L_C(x)\, dx + \int_{C\setminus B} \hat{P}^y(T_B < \infty)L_C(y)\, dy .\end{aligned}$$

Now Proposition 3.2 shows that $P^x(T_B < \infty) = 1$ for all $x \in S$, and hence results in Port and Stone ([6], page 216) guarantee that $\hat{P}^y(T_B < \infty) = 1$ for (Haar) a.e. $y \in S$. The right-hand side above then reduces to $\int_C L_C(x)\, dx = 1$. □

Part (i) of the last proposition says that $L_T(x)$ is in some sense the expected number of visits from x to ∞ before T occurs, which is interesting in view of the role $L_T(x)$ plays as $\lim_{n\to\infty} P^x(T > n)/R_n$ in Theorem 1.1. It is also of interest to know if L_T cannot be identically 0. Since by definition $L_T \geqq L_C$ for any $C \in \mathscr{B}_0$ in which T has support, $\int_C L_C = 1$ guarantees, as was noted in [5], that L_T is bounded away from 0 on a set of positive measure. The following proposition shows that in fact L_T is bounded away from 0 on an open set.

PROPOSITION 4.4. *If $C \in \mathscr{B}_0$ and $|\partial C| = 0$, then there exists an open set $U \subseteqq C$ such that*

$$\inf_{y \in U} L_C(y) > 0 .$$

PROOF. Let $I_n = \{x \in S: ||x|| < 1/n\}$ and let $C_n = C + I_n$. Since $L_{C_n} \leqq L_C$ for each n, property (iii) of L_C guarantees that $\sup_n \sup_{z \in C_n} L_{C_n}(z) < \infty$ and consequently that $\lim_{n\to\infty} \int_{C_n \setminus C} L_{C_n} = 0$. Since $\int_{C_n} L_{C_n} = 1$ for all n, it follows that $\lim_{n\to\infty} \int_C L_{C_n} = 1$. In particular, there is some $y \in \operatorname{int}(C)$ (recall that $|\partial C| = 0$) and some m such that $L_{C_m}(y) > 0$.

Now let A_0 and A be balls such that $A_0 + I_m \subseteqq A$. For any $z \in S$, $u \in I_m$, and n one has $P^{y+u}(T_C > n; X_n \in A + z) \geqq P^y(T_{C_m} > n; X_n \in A_0 + z)$ and hence $G_C(y + u, A + z) \geqq G_{C_m}(y, A_0 + z)$. Part (i) of Proposition 4.3 then shows that $L_C(y + u) \geqq (|A_0|/|A|)L_{C_m}(y)$, so setting $U = \operatorname{int}(C) \cap (y + I_m)$ completes the proof. □

5. Some properties of R_n. The main purpose of this section is to develop some analytical properties of R_n and $\mu^{(n)}$ which will be needed in the sequel. Several of these results use simple modifications of Kesten and Spitzer's lattice case arguments. As their proofs are rather long, only their statements are given here.

First note that since μ is not degenerate, Lemma 1 in [2] gives the following result:

LEMMA 5.1. *Let $C \in \mathscr{A}$. Then there exists $A < \infty$ such that for all $m \geqq 1$*

$$\sup_{x \in S} \mu^{(m)}(x + C) \leqq \frac{A}{m^{d/2}} .$$

The following result is easily proven by applying Lemma 5.1 and Ornstein's methods to the proof of the simpler half of Theorem 3 in [1]:

PROPOSITION 5.2. $(d = 2.)$ *If $B \in \mathscr{B}_0$, then*

$$\lim_{n\to\infty} \frac{r_n(B, B)}{r_{2n}(B, B)} = 1 .$$

In particular, $r_n(B, B)$ is slowly varying at infinity.

For $B \in \mathscr{A}$ define the following generating functions:

$$U_B(z) = \textstyle\sum_{n=0}^{\infty} \mu^{(n)}(B) z^n \qquad \text{and} \qquad R_B(z) = \sum_{n=0}^{\infty} r_n(B, B) z^n \qquad z \in (0, 1) .$$

LEMMA 5.3. *Let $B \in \mathscr{B}_0$ and suppose $B_2 \supseteqq B - B$. Then*

(i) $$\frac{|B|}{(1 - z)\hat{U}_{B_2}(z)} \leqq R_B(z) \leqq \frac{|B_2|}{(1 - z)U_B(z)} , \qquad z \in (0, 1) ;$$

and

(ii) *there exists $a > 0$ such that for n sufficiently large,*

$$\frac{1}{2e} \textstyle\sum_{k=0}^{n} \mu^{(k)}(B) \leqq U_B\left(1 - \frac{1}{n}\right) \leqq a \sum_{k=0}^{n} \mu^{(k)}(B_2) .$$

PROOF. (i) is of course a generalization of a well-known fact from renewal theory. To prove the first half of (i), write for any n

$$\begin{aligned} |B| &= \textstyle\int_B 1\, dx \\ &= \textstyle\int_B dx \left[\sum_{k=1}^{n} \int_B P^x(X_k \in dy) r_{n-k}(y, B) + r_n(x, B)\right] \\ &= \textstyle\sum_{k=0}^{n} \int_B \hat{P}^y(X_k \in B) r_{n-k}(y, B)\, dy \\ &\leqq \textstyle\sum_{k=0}^{n} \hat{\mu}^{(k)}(B_2) r_{n-k}(B, B) . \end{aligned}$$

Multiplying by z^n and summing on n then gives the desired result. The second half of (i) is proved in a similar manner. The proof of (ii) proceeds exactly as the proof of Lemma 3 in [1], except that estimate (4.8) there is replaced by

$$\begin{aligned} \textstyle\sum_{k=m+1}^{(r+1)n} \mu^{(k)}(B) &= \textstyle\sum_{j=rn+1}^{(r+1)n} \int_B P^0(T_B \circ \theta_{rn} = j;\, X_j \in dy) \sum_{k=0}^{(r+1)n-j} P^y(X_k \in B) \\ &\leqq 1 + \textstyle\sum_{k=1}^{n} \mu^{(k)}(B_2) . \qquad \square \end{aligned}$$

Using in part Lemma 5.3, the proof of Lemma 4 of Kesten and Spitzer [1] can be modified to give the following result:

LEMMA 5.4. $(d = 1.)$ *Let $A, B \in \mathscr{B}_0$. Then*

$$\lim_{s\to\infty} \frac{\sum_{k=0}^{ns} \mu^{(k)}(A)}{s \cdot \sum_{k=0}^{n} \mu^{(k)}(B)} = 0 \qquad \textit{uniformly in } n .$$

6. Proof of Theorems 1.1 and 1.3. We shall now prove Theorems 1.1 and 1.3. Recall first that the following is true:

THEOREM 6.1. *If $C \in \mathscr{B}_0$ is such that $|\partial C| = 0$, then*

(i) $$\lim_{n\to\infty} \frac{r_n(x, C)}{r_n(C, C)} = L_C(x) \qquad \text{uniformly for } x \text{ in compacts};$$

(ii) $$\lim_{n\to\infty} \frac{r_n(C, C)}{r_{n+1}(C, C)} = 1 ;$$

and

(iii) $$\sup_n \frac{r_n(C, C)}{r_{2n}(C, C)} < \infty .$$

We note that Lemmas 5.3 and 5.4 can be combined with arguments in [1] to show that (iii) holds even without $|\partial C| = 0$.

REMARKS. Theorem 6.1 was proved by Port and Stone ([4]) for one-dimensional random walk with finite variance, and by Ornstein ([3]) for one-dimensional random walk with infinite variance. Although Ornstein's results are only proved for C a bounded interval, his arguments may be extended to $C \in \mathscr{B}_0$ with $|\partial C| = 0$ (and to the two-dimensional case) in the following manner: Observe first that Lemmas 6 through 19 in [3] do not depend on the dimension of the random walk, and that they depend on C being an interval only in that an interval is a bounded set with nonempty interior with a boundary which allows it to be suitably approximated by other such sets. More precisely, the proofs of those lemmas may be easily modified to cover sets in $\mathscr{B}_0$ with boundary measure 0 because of the following result ($d(A, B)$ is the distance between two sets A and B):

LEMMA 6.2. *Let $C \in \mathscr{B}_0$ have $|\partial C| = 0$. Then for any $\varepsilon > 0$ there exist $\rho > 0$ and sets $A, F \in \mathscr{B}_0$ such that*

(i) $$|\partial A| = |\partial F| = 0 ;$$

(ii) $$d(A, C^c) \geqq \rho \qquad \textit{and} \qquad d(C, F^c) \geqq \rho ;$$

and

(iii) $$|F \backslash A| < \varepsilon .$$

PROOF. Simple compactness argument. □

Lemma 6.2 may be used in a number of instances to pick, for given $\varepsilon > 0$, ρ and sets A and F satisfying (i) and (iii) which are such that

$$P^{x+I_\rho}(T_F > n) \leqq |I_\rho| P^x(T_C > n) \leqq P^{x+I_\rho}(T_A > n)$$

for any $x \in S$, where $I_\rho = \{y \in S\colon ||y|| \leqq \rho\}$. With some other minor changes, Lemmas 6 through 19 in [3] may then be extended to our more general setting.

Theorem 6.1 will then be proved as stated once one proves an analogue to Ornstein's Lemma 5, viz., the following result:

LEMMA 6.3. *Let $B \in \mathscr{B}_0$ have $|\partial B| = 0$. Then there exist $\rho > 0$ and sets $A \subseteq B \subseteq F$ such that*

(i) *$A, F \in \mathscr{B}_0$ and $|\partial A| = |\partial F| = 0$;*

(ii) *$d(A, B^c) \geqq \rho$ and $d(B, F^c) \geqq \rho$;*

and

(iii) *there exist n_0 and $M < \infty$ such that*

$$\sup_{A \subseteq C \subseteq F} \sup_{n \geqq n_0} \frac{r_n(C, C)}{r_{2n}(C, C)} \leqq M .$$

PROOF. For the one-dimensional case, the conclusion follows from Lemmas 5.3 and 5.4 by the same arguments as in [1]. For $d = 2$, the uniformity in C which is required in (iii) may be gotten as follows: By reasoning as in Proposition 4.4, use Lemma 6.2 to pick A, F, and ρ satisfying conclusions (i) and (ii) of this lemma, and also such that $\int_A L_F > 0$. Corollary 5.2 and Theorem 5.4 in [4], together with our Proposition 5.2, then imply that

$$\lim_{n\to\infty} \frac{r_n(A, F)}{R_n} = \int_A L_F \qquad \text{and} \qquad \lim_{n\to\infty} \frac{r_n(F, A)}{R_n} = \int_F L_A .$$

Thus

$$\limsup_{n\to\infty} \frac{r_n(F, A)}{r_{2n}(A, F)} < \infty ,$$

from which (iii) follows. □

The following result is essentially Theorem 4.b in [1].

PROPOSITION 6.4. *Let $C \in \mathscr{B}_0$ have $|\partial C| = 0$. Then for each $k \geqq 1$*

$$\lim_{n\to\infty} \frac{P^x(T_C^{(k)} > n)}{r_n(C, C)} = \sum_{j=0}^{k-1} \int P^x(X_{T_C^{(j)}} \in dy) L_C(y)$$
$$= L_{T_C^{(k)}}(x) ,$$

uniformly for x in compacts.

PROOF. The equality of the two expressions for the limit is clear (recall that $P^x(X_{T_C^{(0)}} \in \cdot) = P^x(X_0 \in \cdot) = \delta_x(\cdot)$). To prove convergence, we proceed by induction on k. Let F be a compact set. For $k = 1$ the result is simply Theorem 6.1(i). Suppose the proposition true for $k = l$. For $k = l + 1$ a last entrance decomposition gives for $x \in F$

$$P^x(T_C^{(l+1)} > n) = P^x(T_C^{(l)} > n) + \sum_{j=1}^{n} \int_C P^x(T_C^{(l)} = j; X_j \in dy) r_{n-j}(y, C) .$$

By the induction hypothesis, it suffices to show that the sum above, when divided by $r_n(C, C)$, converges to $\int_C P^x(X_{T_C^{(l)}} \in dy) L_C(y)$ as $n \to \infty$. We also need the convergence to be uniform for $x \in F$. Let $\varepsilon > 0$ be given, and write the new sum as

$$\sum_{j=1}^{n} \int_C P^x(T_C^{(l)} = j; X_j \in dy) \frac{r_{n-j}(y, C)}{r_n(C, C)}$$
$$= \sum_{j=1}^{M} + \sum_{j=M+1}^{[n/2]} + \sum_{j=[n/2]+1}^{n-N} + \sum_{j=n-N+1}^{n}$$
$$= \quad \mathrm{I} \quad + \quad \mathrm{II} \quad + \quad \mathrm{III} \quad + \quad \mathrm{IV}$$

for M and N fixed but arbitrary. By Theorem 6.1

$$\lim_{n\to\infty} \mathrm{I} = \sum_{j=1}^{M} \int_C P^x(T_C^{(l)} = j;\ X_j \in dy) L_C(y)\,,$$

uniformly for $x \in S$. Since $T_C^{(l)} \in \mathscr{U}$, Proposition 3.2(iii) together with the boundedness of L_C on compacts shows that for sufficiently large M the difference between $\lim_{n\to\infty} \mathrm{I}$ and $\int_C P^x(X_{T_C^{(l)}} \in dy) L_C(y)$ is less than $\varepsilon/3$.

For the remaining parts of the sum, the following estimates may be made:

$$\mathrm{II} \leqq P^x(T_C^{(l)} \geqq M) \sup_{y\in C} \frac{r_{[n/2]}(y, C)}{r_n(C, C)}\,,$$

so Theorem 6.1 together with the boundedness of L_C on compacts and Proposition 3.2(iii) shows that $\limsup_{n\to\infty} \sup_{x\in F} \mathrm{II} < \varepsilon/3$ if M is sufficiently large. Also,

$$\mathrm{III} \leqq \frac{P^x(T_C^{(l)} > [n/2])}{r_n(C, C)} \sup_{y\in C} r_N(y, C)\,.$$

The induction hypothesis combined with Theorem 6.1 and Proposition 3.2(iii) then proves that $\limsup_{n\to\infty} \sup_{x\in F} \mathrm{III} < \varepsilon/3$ if N is large enough. Finally,

$$\begin{aligned}\mathrm{IV} &\leqq \frac{P^x(n - N < T_C^{(l)} \leqq n)}{r_n(C, C)} \\ &= \frac{P^x(T_C^{(l)} > n - N)}{r_n(C, C)} - \frac{P^x(T_C^{(l)} > n)}{r_n(C, C)}\,.\end{aligned}$$

The induction hypothesis combined with Theorem 6.1 then shows that for fixed N, $\lim_{n\to\infty} \sup_{x\in F} \mathrm{IV} = 0$. Combining these estimates completes the proof. □

The following result is crucial for the proof of Theorem 1.1:

LEMMA 6.5. *Let $C \in \mathscr{B}_0$. Then there exists $K < \infty$ such that*

$$\frac{1}{n} \sum_{m=1}^{n} r_m(C, C) \leqq K r_n(C, C)$$

for all $n \geqq 1$.

PROOF. Denote $r_n(C, C)$ by r_n. If $d = 2$, then by Proposition 5.2 r_n is slowly varying at ∞. Since r_n is also monotone, standard Tauberian theorems imply that in fact $(1/n) \sum_1^n r_m \sim r_n$ as $n \to \infty$.

If $d = 1$, we shall first show that if $\gamma > 0$ is sufficiently small then

$$\limsup_{n\to\infty} \frac{\sum_{m=1}^{[\gamma n]} r_m}{\sum_{m=1}^{n} r_m} < \tfrac{1}{2}\,. \tag{6.1}$$

To do this note first that, as in the proof of Lemma 5.3, for any $n \geqq 2$ one has

$$\frac{1}{2e} \sum_{m=1}^{n} r_m \leqq R_C\left(1 - \frac{1}{n}\right).$$

On the other hand, for any n,

$$R_C\left(1-\frac{1}{n}\right) \leqq \sum_{m=0}^{n} r_m + r_{n+1} \sum_{m=n+1}^{\infty}\left(1-\frac{1}{n}\right)^m$$
$$\leqq 2 \sum_{m=0}^{n} r_m$$

by the monotonicity of r_m. Therefore there is some n_0 such that

$$\frac{\sum_{m=1}^{n} r_m}{\sum_{m=1}^{jn} r_m} \leqq 5e \frac{R_C(1-1/n)}{R_C(1-1/jn)}$$

for all $j \geqq 1$ and $n \geqq n_0$. Let $C_2 = C - C$. Then Lemma 5.3 and the above estimate show that for some n_1

$$\frac{\sum_{m=1}^{n} r_m}{\sum_{m=1}^{jn} r_m} \leqq \frac{5e|C_2|}{|C|} \frac{(U_{C_2}(1-1/(jn))}{jU_C(1-1/n)}$$
$$\leqq c \frac{\sum_{m=0}^{jn} \mu^{(m)}(C_2)}{j \sum_{m=0}^{n} \mu^{(m)}(C)}$$

for all $j \geqq 1$ and all $n \geqq n_1$, where c is a finite constant. By Lemma 5.4 there exists J such that if $j \geqq J$ then the last expression above is less than $\frac{1}{4}$ for all n. Letting $\gamma = 1/J$ then establishes (6.1).

From (6.1) one obtains

$$\sum_{m=[\gamma n]+1}^{n} r_m > \tfrac{1}{2} \sum_{m=1}^{n} r_m$$

for all n sufficiently large, and hence by the montonicity of r_n

$$(n - \gamma n) r_{[\gamma n]} > \tfrac{1}{2} \sum_{m=1}^{n} r_m \tag{6.2}$$

for all such n. Since

$$\frac{r_{[\gamma n]}}{r_n} \leqq \frac{r_{[\gamma n]}}{r_{2[\gamma n]}} \cdot \frac{r_{2[\gamma n]}}{r_{4[\gamma n]}} \cdot \cdots \cdot \frac{r_{2^{j-1}[\gamma n]}}{r_{2^j[\gamma n]}}$$

if $j = [\log_2(1/\gamma)] + 2$ and n is large enough, it is clear from the comment following Theorem 6.1(iii) that (6.2) in fact proves the lemma. □

Lemma 6.5 is used in obtaining the following result:

LEMMA 6.6. *Let $T \in \mathscr{U}$ haves upport in $C \in \mathscr{B}_0$ with $|\partial C| = 0$. Then there exists a finite constant A such that*

$$\sup_{y \in C} P^y(T > m) \leqq A r_m(C, C)$$

for all m.

PROOF. As in the proof of Proposition 4.2, one has for any y that

$$P^y(T > k) = r_k(y, C) + \sum_{j=1}^{k} \int_C P^y(T > j; X_j \in dz) r_{k-j}(z, C)$$

and consequently

$$\begin{aligned} mP^y(T > m) \leqq \sum_{k=1}^{m} P^y(T > k) &\leqq \sum_{k=1}^{m} r_k(y, C) \\ &+ \sum_{j=1}^{m} \int_C P^y(T > j; X_j \in dz) \sum_{i=0}^{m-j} r_i(z, C) \end{aligned} \tag{6.3}$$

for any m. Now Theorem 6.1 (i) and the boundedness of L_C on compacts guarantee the existence of a finite constant A' such that

$$\sup_{u \in C} \sum_{k=1}^{m} r_k(u, C) \leqq A' \sum_{k=1}^{m} r_k(C, C)$$

for all m. Using this inequality in (6.3), together with the fact that $\sum_{k=1}^{\infty} r_k(C, C) = \infty$, then shows that for some A''

$$mP^y(T > m) \leqq A''[1 + G_T(y, C)] \sum_{k=1}^{m} r_k(C, C)$$

for all $y \in C$ and all m. Combining this with Lemma 6.5 and Proposition 3.3 then completes the proof. □

PROOF OF THEOREMS 1.1 AND 1.3. Let T have support in $C \in \mathscr{B}_0$ with $|\partial C| = 0$. We shall first prove that

$$\lim_{n\to\infty} \frac{P^x(T > n)}{r_n(C, C)} = L_T(x) \tag{6.4}$$

uniformly for x in compacts. For any x and n one has

$$\frac{P^x(T > n)}{r_n(C, C)} = \frac{r_n(x, C)}{r_n(C, C)} + \sum_{k=1}^{\infty} \frac{P^x(N_n(C) = k; T > n)}{r_n(C, C)}, \tag{6.5}$$

where $N_n(C)$ is the number of visits to C by time n. For each $k \leqq n$

$$P^x(N_n(C) = k; T > n) = \sum_{j=k}^{n} \int_C P^x(T > j; T_C^{(k)} = j; X_j \in dy) r_{n-j}(y, C)$$

since a.s. T can only occur when the process is in C. The same arguments as in Proposition 6.4 then show that

$$\begin{aligned} \lim_{n\to\infty} & \frac{P^x(N_n(C) = k; T > n)}{r_n(C, C)} \\ &= \sum_{j=k}^{\infty} \int_C P^x(T > j; T_C^{(k)} = j; X_j \in dy) L_C(y) \\ &= \int_C P^x(T > T_C^{(k)}; X_{T_C^{(k)}} \in dy) L_C(y) \end{aligned} \tag{6.6}$$

uniformly for x in compacts. Let F be a compact set. Since

$$\lim_{n\to\infty} \frac{r_n(x, C)}{r_n(C, C)} = L_C(x)$$

and

$$\lim_{l\to\infty} \sum_{k=1}^{l} \int_C P^x(T > T_C^{(k)}; X_{T_C^{(k)}} \in dy) L_C(y) = \int_C G_T(x, dy) L_C(y)$$

uniformly for $x \in F$, it follows from (6.5), (6.6), and the definition of L_T that (6.4) will be established once we show

$$\lim_{l\to\infty} \limsup_{n\to\infty} \sup_{x \in F} \frac{P^x(N_n(C) \geqq l; T > n)}{r_n(C, C)} = 0 . \tag{6.7}$$

To this end, fix l and observe that

$$\begin{aligned} P^x(N_n(C) \geqq l; T > n) &= P^x(T_C^{(l)} \leqq n; T > n) \\ &\leqq \sum_{j=1}^{n} P^x(T_C^{(l)} = j; T > j; j + T \circ \theta_j > n) , \end{aligned}$$

the second line following from T's subterminality. Since T is an $\{\mathscr{F}_n\}$ stopping time, we may condition on $\mathscr{F}_j$ in the summation above to obtain

$$P^x(N_n(C) \geqq l;\, T > n) \leqq \sum_{j=1}^n \int_C P^x(T_C^{(l)} = j;\, T > j;\, X_j \in dy) P^y(T > n - j)$$

for any x and n. Dividing by $r_n(C, C)$ and using Lemma 6.6 to bound $P^y(T > n - j)$ then shows that for some $A < \infty$ (which is independent of l),

$$\frac{P^x(N_n(C) \geqq l;\, T > n)}{r_n(C, C)} \leqq A \sum_{j=1}^n P^x(T > T_C^{(l)} = j) \frac{r_{n-j}(C, C)}{r_n(C, C)}$$

for any x and all n. By reasoning as in the proof of Proposition 6.4, one sees that the $\limsup_{n\to\infty} \sup_{x\in F}$ of the right-hand side above is

$$\leqq A \sup_{x\in F} P^x(T > T_C^{(l)}) ,$$

which together with Proposition 3.2(ii) proves (6.7) and hence (6.4).

We now prove Theorem 1.3. If $T = T_B$ for $B \in \mathscr{B}$ where $B \subsetneqq C \in \mathscr{B}_0$, then the uniformity (on compacts) of the convergence in (6.4) shows that

$$\begin{aligned} \lim_{n\to\infty} \frac{r_n(B, B)}{r_n(C, C)} &= \int_B \lim_{n\to\infty} \frac{r_n(x, B)}{r_n(C, C)}\, dx \\ &= \int_B L_B(x)\, dx \\ &= 1 \end{aligned}$$

by Proposition 4.3. Since any sets A and B in $\mathscr{B}$ may be inscribed in a common $C \in \mathscr{B}_0$, this proves Theorem 1.3. Theorem 1.3 combined with (6.4) then proves Theorem 1.1. □

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