

## WEAK CONVERGENCE OF THE RAO-BLACKWELL ESTIMATOR OF A DISTRIBUTION FUNCTION<sup>1</sup>

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Under the condition that the minimal sufficient statistics are transitive, the sequence of Rao-Blackwell estimators of distribution function has been shown to form a reverse martingale sequence. Weak convergence of the corresponding empirical process to a Gaussian process has been established by assuming that the sufficient statistics are  $U$ -statistics and utilizing certain results on the convergence of conditional expectations of functions of  $U$ -statistics along with the functional central limit theorems for (reverse) martingales by Loynes (1970) and Brown (1971).

**1. Introduction.** Let  $\{X_i, i \geq 1\}$  be a sequence of independent and identically distributed random vectors (i.i.d. rv) defined on a probability space  $(\Omega, \mathcal{A}, P)$  with each  $X_i$  having a continuous distribution function (df)  $F(\mathbf{x})$ ,  $\mathbf{x} \in R^p$ , the  $p$  ( $\geq 1$ ) dimensional Euclidean space. For every  $n$  ( $\geq 1$ ), the empirical df  $F_n$  is defined by

$$(1.1) \quad F_n(\mathbf{x}) = n^{-1} \sum_{i=1}^n c(\mathbf{x} - X_i), \quad \mathbf{x} \in R^p,$$

where for a  $p$ -vector  $\mathbf{u} = (u_1, \dots, u_p)'$ ,

$$(1.2) \quad c(\mathbf{u}) = 1 \text{ if } \mathbf{u} \geq \mathbf{0} \text{ and is } 0 \text{ otherwise,}$$

and by  $\mathbf{x} \geq \mathbf{y}$  we denote the coordinatewise inequalities  $x_j \geq y_j$ ,  $1 \leq j \leq p$ . We assume that the df  $F$  admits of the existence of a complete sufficient statistic (vector)  $T_n$ , where

$$(1.3) \quad T_n' = (T_{n1}, \dots, T_{nq}), \quad \text{for some } q \geq 1.$$

[When we encounter the asymptotic situation where  $n \rightarrow \infty$ , we assume that  $q$  remains fixed. Note that the sample order statistics (for  $p = 1$ ) or the collection matrix (for  $p \geq 1$ ) constitute a sufficient statistic where  $q = n$ , and in that case, our study is of no real interest.]

Let  $\mathcal{B}_n = \mathcal{B}(X_1, \dots, X_n)$  be the  $\sigma$ -field generated by  $X_1, \dots, X_n$ , and let  $\mathcal{B}_n^{(s)} = \mathcal{B}(T_n)$  be the  $\sigma$ -field generated by  $T_n$ , so that  $\mathcal{B}_n^{(s)} \subset \mathcal{B}_n$ . Also, let  $\mathcal{F}_n = \mathcal{F}(T_n, T_{n+1}, \dots)$  be the  $\sigma$ -field generated by  $\{T_k, k \geq n\}$ ,  $n \geq 1$ , so that  $\mathcal{F}_n$  is a monotone nonincreasing  $\sigma$ -field. Finally, let  $\mathcal{C}_n$  be the  $\sigma$ -field generated by the unordered collection  $\{X_1, \dots, X_n\}$  and  $X_{n+1}, X_{n+2}, \dots$ ,  $n \geq 1$ , so that  $\mathcal{C}_n$  is monotone nonincreasing and  $\mathcal{F}_n \subset \mathcal{C}_n$ .

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Consider the usual *empirical process*

$$(1.4) \quad V_n(\mathbf{x}) = n^{\frac{1}{2}}[F_n(\mathbf{x}) - F(\mathbf{x})], \quad \mathbf{x} \in R^p, n \geq 1.$$

Then, the Rao-Blackwell empirical process is defined by

$$(1.5) \quad V_n^*(\mathbf{x}) = E[V_n(\mathbf{x}) | \mathcal{B}_n^{(s)}] = n^{\frac{1}{2}}[\phi_n(\mathbf{x}) - F(\mathbf{x})], \quad \mathbf{x} \in R^p,$$

where for every  $\mathbf{x} \in R^p$ ,

$$(1.6) \quad \phi_n(\mathbf{x}) = E[F_n(\mathbf{x}) | \mathcal{B}_n^{(s)}] = E[c(\mathbf{x} - \mathbf{X}_1) | \mathcal{B}_n^{(s)}]$$

is the Rao-Blackwell *empirical df.* Note that when  $T_n$  is a complete sufficient statistic,  $\phi_n$  is the unique minimum variance unbiased estimator (UMVU) of  $F$ . Specific expressions for  $\phi_n$  for various  $F$  (mostly belonging to the exponential family) have been worked out by various workers; we may refer to Tate (1959), Ghurye and Olkin (1969) where additional references are cited. In the context of the UMVU estimation of the density function  $f(\mathbf{x})$  and its various functions, the recent works of Seheult and Quesenberry (1971) and O'Reilly and Quesenberry (1972) deserve mention.

In view of the fact that the empirical process in (1.4) weakly converges to an appropriate multiparameter Gaussian process [viz., Neuhaus (1971) for  $p \geq 1$  and Billingsley (1968) for  $p = 1$ ], our interest centers here on deriving similar weak convergence results for the process  $V_n^*$  in (1.5). That, in general, the weak convergence of  $V_n$  does not necessarily imply the same for  $V_n^*$  can easily be verified with the simple example of the uniform  $[0, \theta]$  df, where  $\phi_n(x) = (n-1)x/nX_{(n)}$ , if  $0 \leq x < X_{(n)}$  and is 1 for  $x \geq X_{(n)}$ ;  $X_{(n)} = \max\{X_i: 1 \leq i \leq n\}$ , and the distribution of  $\sup\{|\phi_n(x) - F(x)|: 0 \leq x \leq \theta\}$  is independent of  $\theta$ . Hence, assuming  $\theta = 1$ ,  $n^{\frac{1}{2}}\{\sup_{0 \leq x \leq 1} |\phi_n(x) - F(x)|\} \leq \max\{n^{\frac{1}{2}}|(n-1)/nX_{(n)} - 1|, n^{\frac{1}{2}}|1 - X_{(n)}|\} \rightarrow 0$ , in probability, as  $n \rightarrow \infty$ . We also note that by definition in (1.6),  $\phi_n(\mathbf{x})$  ceases to be an average of i.i.d. rv and  $(n+1)\phi_{n+1}(\mathbf{x}) - n\phi_n(\mathbf{x})$  is no longer stochastically independent of  $n\phi_n(\mathbf{x})$ . Thus, the basic technique of deriving the functional central limit theorems, displayed in detail in Billingsley (1968) and extended to the multiparameter case by others, is not readily applicable. Our task is accomplished by showing that because of transitivity of  $T_n$ , for every  $\mathbf{x} \in R^p$ ,  $\{\phi_n(\mathbf{x}), \mathcal{F}_n; n \geq 1\}$  is a reverse martingale on which the central limit theorems of Loynes (1970) and Brown (1971) can be applied under parallel regularity conditions.

The basic results on  $\phi_n$  and some properties of  $U$ -statistics are studied in Section 2. The main theorem is stated and proved in Section 3. The last section includes (by way of concluding remarks) certain additional results.

**2. Some basic lemmas.** Let us denote by

$$(2.1) \quad B(\mathbf{a}, \mathbf{b}) = \{\mathbf{x}: \mathbf{a} \leq \mathbf{x} \leq \mathbf{b}\} \quad \text{where } \mathbf{a} \leq \mathbf{b};$$

$$(2.2) \quad P_F(\mathbf{a}, \mathbf{b}) = P\{\mathbf{X}_1 \in B(\mathbf{a}, \mathbf{b})\} = P\{\mathbf{a} \leq \mathbf{X}_1 \leq \mathbf{b}\}.$$

Also, for every  $n \geq 1$ , let

$$(2.3) \quad Z_n^*(\mathbf{a}, \mathbf{b}) = P\{X_1 \in B(\mathbf{a}, \mathbf{b}) | \mathcal{B}_n^{(s)}\} \\ = \sum^* (-1)^{j'1} \phi_n(j_i a_i + (1 - j_i) b_i, 1 \leq i \leq p),$$

where the summation  $\sum^*$  extends over all  $\mathbf{j}' = (j_1, \dots, j_p)$ ,  $j_i = 0, 1$ ,  $i = 1, \dots, p$ .

LEMMA 2.1. *Let  $C$  and  $h$  be some positive numbers such that*

$$(2.4) \quad 1 \leq nh \leq n \quad \text{and} \quad P_F(\mathbf{a}, \mathbf{b}) \leq Ch.$$

*Then, for every positive integer  $s$ , there exists a constant  $K_s$  (independent of  $h$ ,  $n$  and  $P_F(\mathbf{a}, \mathbf{b})$ ), such that*

$$(2.5) \quad n^s E|Z_n^*(\mathbf{a}, \mathbf{b}) - P_F(\mathbf{a}, \mathbf{b})|^{2s} \leq K_s h^s.$$

PROOF. Let us define

$$(2.6) \quad Z_n(\mathbf{a}, \mathbf{b}) = \sum^* (-1)^{j'1} F_n(j_i a_i + (1 - j_i) b_i, 1 \leq i \leq p),$$

so that by (1.6) and (2.3),

$$(2.7) \quad Z_n^*(\mathbf{a}, \mathbf{b}) = E[Z_n(\mathbf{a}, \mathbf{b}) | \mathcal{B}_n^{(s)}].$$

Therefore, by the Jensen inequality,

$$(2.8) \quad E|Z_n^*(\mathbf{a}, \mathbf{b}) - P_F(\mathbf{a}, \mathbf{b})|^{2s} = E|E\{Z_n(\mathbf{a}, \mathbf{b}) - P_F(\mathbf{a}, \mathbf{b}) | \mathcal{B}_n^{(s)}\}|^{2s} \\ \leq E|Z_n(\mathbf{a}, \mathbf{b}) - P_F(\mathbf{a}, \mathbf{b})|^{2s}.$$

Now,  $nZ_n(\mathbf{a}, \mathbf{b})$  has the binomial distribution with the parameters  $(n, P_F(\mathbf{a}, \mathbf{b}))$ , so that by Lemma 5.2 of Neuhaus (1971), under (2.4),

$$(2.9) \quad E|Z_n(\mathbf{a}, \mathbf{b}) - P_F(\mathbf{a}, \mathbf{b})|^{2s} \leq K_s n^{-s} h^s.$$

The lemma follows directly from (2.8) and (2.9).  $\square$

LEMMA 2.2. *For every  $\mathbf{x} \in R^p$ ,  $\{\phi_n(\mathbf{x}), \mathcal{F}_n; n \geq 1\}$  is a reverse martingale.*

PROOF. Since  $\{X_n, n \geq 1\}$  are i.i.d. random variable and  $\{\mathcal{T}_n\}$  is complete and sufficient, it follows from Bahadur (1954, page 447) that it is transitive. Hence, by definition in (1.6)

$$(2.10) \quad \phi_{n+1}(\mathbf{x}) = E\{c(\mathbf{x} - X_1) | \mathcal{B}_{n+1}^{(s)}\} \\ = E\{E\{c(\mathbf{x} - X_1) | \sigma(\mathcal{B}_n^{(s)}, \mathcal{B}_{n+1}^{(s)})\} | \mathcal{B}_{n+1}^{(s)}\}.$$

Since  $\{\mathcal{T}_n\}$  is transitive, by the Wijsman theorem [viz., Zacks (1971, page 84)],  $\mathcal{B}_n$  and  $\mathcal{B}_{n+1}^{(s)}$  are conditionally independent given  $\mathcal{B}_n^{(s)}$ . Therefore,

$$(2.11) \quad E\{c(\mathbf{x} - X_1) | \sigma(\mathcal{B}_n^{(s)}, \mathcal{B}_{n+1}^{(s)})\} = E\{c(\mathbf{x} - X_1) | \mathcal{B}_n^{(s)}\} = \phi_n(\mathbf{x}) \quad \text{a.s.}$$

Hence, from (2.10) and (2.11), we have for every  $n \geq 1$ ,  $\mathbf{x} \in R^p$ ,

$$(2.12) \quad \phi_{n+1}(\mathbf{x}) = E\{\phi_n(\mathbf{x}) | \mathcal{B}_{n+1}^{(s)}\} = E\{\phi_n(\mathbf{x}) | \mathcal{F}_{n+1}\} \quad \text{a.s.},$$

and the lemma follows.  $\square$

For a large class of df's (including the exponential family),  $T_n$  can be equivalently expressed in terms of a set of  $U$ -statistics. Thus, we write

$$(2.13) \quad T_n' = [U_n^{(1)}, \dots, U_n^{(q)}], \quad \text{for } n \geq m \geq 1,$$

where  $m$  is the maximum of the individual degrees of the kernels for the  $U_n^{(j)}$ . From the results of Hoeffding (1948), it follows that if the kernels are all square integrable, then

$$(2.14) \quad E\{(T_n - ET_n)'(T_n - ET_n)\} = n^{-1}A_1 + n^{-2}A_2 + \dots,$$

where the  $A_k, k \geq 1$  are positive semidefinite matrices. For proving a few results on  $U$ -statistics, to follow in the Lemmas 2.3-2.6, we assume for simplicity that  $q = 1$ , and let  $\theta = E\phi(X_1, \dots, X_m)$  where  $\phi$  is the symmetric kernel for  $U_n$ . Let then

$$(2.15) \quad \phi_h(x_1, \dots, x_h) = E\phi(x_1, \dots, x_h, X_{h+1}, \dots, X_m),$$

$$(2.16) \quad \zeta_h = E\{\phi_h^2(X_1, \dots, X_h) - \theta^2\}, \quad h = 0, \dots, m,$$

so that  $\phi_0 = 0, \zeta_0 = 0$  and for  $q = 1, A_1 = m^2\zeta_1$ .

LEMMA 2.3. *If  $E\{\phi^2(X_1, \dots, X_m)\} < \infty$ , then*

$$(2.17) \quad n(n+1)E\{(T_n - T_{n+1})^2 | \mathcal{E}_{n+1}\} \rightarrow m^2\zeta_1 \text{ a.s.}, \text{ as } n \rightarrow \infty.$$

PROOF. Following Miller and Sen (1972), for all  $n \geq m$ ,

$$(2.18) \quad T_n = \binom{n}{m}^{-1} \sum_{C_{n,m}} \int_{R^{pm}} \dots \int \phi(x_1, \dots, x_m) \prod_{j=1}^m d[c(x_j - X_{i_j})] \\ = \theta + \sum_{h=1}^m \binom{m}{h} T_{n,h},$$

where

$$(2.19) \quad T_{n,1} = n^{-1} \sum_{i=1}^n [\phi_1(X_i) - \theta],$$

$$(2.20) \quad T_{n,h} = \binom{n}{h}^{-1} \sum_{C_{n,h}} \phi_h^*(X_{i_1}, \dots, X_{i_h});$$

$$(2.21) \quad \phi_h^*(x_1, \dots, x_h) \\ = \phi_h(x_1, \dots, x_h) - \sum_{j=1}^h \phi_{h-1}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_h) \\ + \sum_{1 \leq j < k \leq h} \phi_{h-2}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_h) \\ - \dots + (-1)^h \theta, \quad \text{for } h = 2, \dots, m.$$

To avoid notational complexities, we shall take the case of  $m = 2$ ; the extension to the case of  $m \geq 3$  can be made in a similar but more laborious way. For  $m = 2$ ,

$$(2.22) \quad T_n - T_{n+1} = 2(T_{n,1} - T_{n+1,1}) + (T_{n,2} - T_{n+1,2}).$$

Hence,

$$(2.23) \quad n(n+1)(T_n - T_{n+1})^2 \\ = 4n(n+1)(T_{n,1} - T_{n+1,1})^2 + n(n+1)(T_{n,2} - T_{n+1,2})^2 \\ + 4n(n+1)(T_{n,1} - T_{n+1,1})(T_{n,2} - T_{n+1,2}).$$

Now,

$$(2.24) \quad T_{n,1} - T_{n+1,1} = n^{-1}[T_{n+1,1} - \phi_1(\mathbf{X}_{n+1}) + \theta],$$

so that

$$(2.25) \quad n(n+1)E\{(T_{n,1} - T_{n+1,1})^2 | \mathcal{E}_{n+1}\} \\ = n^{-1}(n+1)[(n+1)^{-1} \sum_{i=1}^{n+1} \{\phi_1(\mathbf{X}_i) - \theta\}^2 - T_{n+1,1}^2].$$

By the Khintchine strong law of large numbers,  $(n+1)^{-1} \sum_{i=1}^{n+1} \{\phi_1(\mathbf{X}_i) - \theta\}^2 \rightarrow \zeta_1$  a.s., as  $n \rightarrow \infty$ , whereas  $\{T_{n+1,1}, \mathcal{E}_{n+1}\}$  is a reverse martingale sequence which converges a.s. to its expectation 0, and hence,  $T_{n+1,1}^2$  also a.s. converges to 0 as  $n \rightarrow \infty$ . Again

$$(2.26) \quad T_{n,2} - T_{n+1,2} = \frac{2}{n-1} T_{n+1,2} - \binom{n}{2}^{-1} \sum_{i=1}^n \phi_2^*(\mathbf{X}_i, \mathbf{X}_{n+1}).$$

Hence,

$$(2.27) \quad n(n+1)(T_{n,2} - T_{n+1,2})^2 \leq (n-1)^{-2} 8n(n+1)T_{n+1,2}^2 \\ + n(n+1)2\binom{n}{2}^{-2} \left\{ \sum_{i=1}^n \phi_2^*(\mathbf{X}_i, \mathbf{X}_{n+1}) \right\}^2.$$

By the Berk (1966) reverse martingale property of  $U$ -statistics,  $\{T_{n+1,2}, \mathcal{E}_{n+1}\}$  is a reverse martingale with expectation 0, so that  $E(T_{n+1,2}^2 | \mathcal{E}_{n+1}) = T_{n+1,2}^2$  a.s. converges to 0 as  $n \rightarrow \infty$ . Whereas

$$(2.28) \quad E\left\{ \left( \sum_{i=1}^n \phi_2^*(\mathbf{X}_i, \mathbf{X}_{n+1}) \right)^2 \middle| \mathcal{E}_{n+1} \right\} \\ = n\binom{n+1}{2}^{-1} \sum_{1 \leq i < j \leq n+1} \phi_2^{*2}(\mathbf{X}_i, \mathbf{X}_j) \\ + n(n-1)\binom{n+1}{3}^{-1} \sum_{1 \leq i < j < k \leq n+1} \phi_2^*(\mathbf{X}_i, \mathbf{X}_j)\phi_2^*(\mathbf{X}_i, \mathbf{X}_k).$$

By (2.28), the 2nd term on the rhs of (2.27) is given by

$$(2.29) \quad \frac{8(n+1)}{n-1} \left\{ \frac{1}{n-1} \binom{n+1}{2}^{-1} \sum_{1 \leq i < j \leq n+1} \phi_2^{*2}(\mathbf{X}_i, \mathbf{X}_j) \right. \\ \left. + \binom{n+1}{3}^{-1} \sum_{1 \leq i < j < k \leq n+1} \phi_2^*(\mathbf{X}_i, \mathbf{X}_j)\phi_2^*(\mathbf{X}_i, \mathbf{X}_k) \right\} \\ = 8(n-1)^{-1}(n+1)[(n-1)^{-1}\tilde{U}_n^{(1)} + \tilde{U}_n^{(2)}], \quad \text{say.}$$

Now,  $\tilde{U}_n^{(1)}$  is a  $U$ -statistic, and hence, converges a.s. to its expectation which is finite, while  $\tilde{U}_n^{(2)}$  is also a  $U$ -statistic with expectation  $E\phi_2^*(\mathbf{X}_1, \mathbf{X}_2)\phi_2^*(\mathbf{X}_1, \mathbf{X}_3) = 0$ , and hence, it a.s. converges to 0 as  $n \rightarrow \infty$ . Thus, (2.29) a.s. converges to 0 as  $n \rightarrow \infty$ . Finally,  $[E\{n(n+1)(T_{n,1} - T_{n+1,1})(T_{n,2} - T_{n+1,2}) | \mathcal{E}_{n+1}\}]^2$  converges a.s. to 0 [by the Schwarz inequality and (2.25)–(2.29)]. Hence, the proof is complete.  $\square$

LEMMA 2.4. Under the assumptions of the previous lemma,

$$(2.30) \quad n(n+1)E\{(T_n - T_{n+1})^2 | \mathcal{F}_{n+1}\} \rightarrow m^2\zeta_1 \quad \text{a.s.} \\ \text{and in } L_1, \quad \text{as } n \rightarrow \infty.$$

PROOF. Since  $\mathcal{F}_{n+1} \subset \mathcal{C}_{n+1}$ ,

$$(2.31) \quad n(n+1)E\{(T_n - T_{n+1})^2 \mid \mathcal{F}_{n+1}\} \\ = n(n+1)E\{E\{(T_n - T_{n+1})^2 \mid \mathcal{C}_{n+1}\} \mid \mathcal{F}_{n+1}\}.$$

Now, by Lemma 2.3,  $E\{n(n+1)(T_n - T_{n+1})^2 \mid \mathcal{C}_{n+1}\} \rightarrow m^2\zeta_1$  a.s., as  $n \rightarrow \infty$ ,  $\mathcal{F}_n$  is  $\downarrow$  in  $n$  and  $n(n+1)E\{(T_n - T_{n+1})^2 \mid \mathcal{C}_{n+1}\} \in L_1$ . Hence, by Loève (1963, page 409), the result follows.  $\square$

LEMMA 2.5. *If  $\phi^4(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_m) < \infty$ , then for every  $k \geq m$ ,*

$$(2.32) \quad \sup_{n \geq k} n(n+1)E\{(T_n - T_{n+1})^2 \mid \mathcal{C}_{n+1}\} \in L_1.$$

PROOF. Here also, for simplicity, we take  $m = 2$ . It suffices to show that for each  $j (= 1, 2)$ ,  $\sup_{n \geq k} n(n+1)E\{(T_{n,j} - T_{n+1,j})^2 \mid \mathcal{C}_{n+1}\}$  is integrable. Note that by (2.24),

$$(2.33) \quad \sup_{n \geq k} n(n+1)E\{(T_{n,1} - T_{n+1,1})^2 \mid \mathcal{C}_{n+1}\} \\ = \sup_{n \geq k} [n^{-1} \sum_{i=1}^{n+1} \{\phi_1(\mathbf{X}_i) - T_{n+1,1}\}^2] = \sup_{n \geq k} M_{n+1}, \text{ say.}$$

Now, by definition,  $\{M_n, \mathcal{C}_n; n \geq 2\}$  is a reverse martingale, so that by the Kolmogorov inequality, for every  $t \geq 0$ ,

$$(2.34) \quad P\{\sup_{n \geq k} M_n > t\} \leq t^{-2}E[M_k^2] = t^{-2}\zeta_1^*,$$

and this implies the integrability of  $\sup_{n \geq k} M_n$ , for every  $k \geq 1$ . A similar but more lengthy proof applies for  $j = 2$ .  $\square$

LEMMA 2.6. *Under the conditions of Lemma 2.5,*

$$(2.35) \quad n(n+1)E\{(T_n - T_{n+1})^2 \mid \mathcal{F}_{n+1}\} \rightarrow m^2\zeta_1 \text{ a.s., as } n \rightarrow \infty.$$

PROOF. Since  $E\{(T_n - T_{n+1})^2 \mid \mathcal{F}_{n+1}\} = E\{E\{(T_n - T_{n+1})^2 \mid \mathcal{C}_{n+1}\} \mid \mathcal{F}_{n+1}\}$  and since  $\mathcal{F}_n$  is  $\downarrow$  in  $n$ , the lemma follows directly from Lemma 2.5 and Loève (1963, page 409).  $\square$

**3. The main results.** To motivate the main theorem, we require to introduce certain assumptions. Note that by the Rao-Blackwellization, for every  $n \geq 1$ ,

$$(3.1) \quad nE\{\phi_n(\mathbf{x}) - F(\mathbf{x})\}^2 \leq nE\{F_n(\mathbf{x}) - F(\mathbf{x})\}^2 = F(\mathbf{x})[1 - F(\mathbf{x})], \quad \forall \mathbf{x} \in R^p.$$

But, as in the example of the uniform df, the lhs of (3.1) may tend to 0 as  $n \rightarrow \infty$ . So, in order to avoid this degeneracy, we assume that for every  $\mathbf{x}, \mathbf{y} \in R^p$ ,

$$(3.2) \quad \lim_{n \rightarrow \infty} nE\{[\phi_n(\mathbf{x}) - F(\mathbf{x})]\{\phi_n(\mathbf{y}) - F(\mathbf{y})\}\} = h(\mathbf{x}, \mathbf{y}),$$

where there exists a subset  $\mathcal{X} \subset R^p$  such that for every  $x_i \in \mathcal{X}, i = 1, \dots, m$ , the matrix  $(h(\mathbf{x}_i, \mathbf{x}_j))$  is positive definite for every  $m \geq 1$ . We also assume that

$$(3.3) \quad E\{n(n+1)[\phi_n(\mathbf{x}) - \phi_{n+1}(\mathbf{x})]^2 I(n|\phi_n(\mathbf{x}) - \phi_{n+1}(\mathbf{x})| > \lambda) \mid \mathcal{F}_{n+1}\}$$

converges to 0 as  $\lambda \rightarrow \infty$ , uniformly in  $n (\geq n_0)$ , and

$$(3.4) \quad n(n+1)E\{[\phi_n(\mathbf{x}_1) - \phi_{n+1}(\mathbf{x}_1)]\{\phi_n(\mathbf{x}_2) - \phi_{n+1}(\mathbf{x}_2)\} \mid \mathcal{F}_{n+1}\} \rightarrow h(\mathbf{x}_1, \mathbf{x}_2) \text{ a.s.,}$$

for every  $\mathbf{x}_1, \mathbf{x}_2 \in R^p$ , where  $I(A)$  stands for the indicator function of a set  $A$ .

We may note that (3.4) implies that

$$(3.5) \quad n \sum_{k=n}^{\infty} E[\{\phi_k(\mathbf{x}_1) - \phi_{k+1}(\mathbf{x}_1)\}\{\phi_k(\mathbf{x}_2) - \phi_{k+1}(\mathbf{x}_2)\} | \mathcal{F}_{k+1}] \rightarrow h(\mathbf{x}_1, \mathbf{x}_2) \quad \text{a.s.}$$

whereas the uniform integrability condition in (3.3) implies that whenever  $h(\mathbf{x}, \mathbf{x}) > 0$ , for every  $\varepsilon > 0$ ,

$$(3.6) \quad \begin{aligned} & [\sum_{k=n}^{\infty} E[\phi_k(\mathbf{x}) - \phi_{k+1}(\mathbf{x})]^2]^{-1} [\sum_{k=n}^{\infty} \{E[(\phi_k(\mathbf{x}) - \phi_{k+1}(\mathbf{x}))^2] \\ & \times I(n^{\frac{1}{2}}|\phi_k(\mathbf{x}) - \phi_{k+1}(\mathbf{x})| > \varepsilon h^{\frac{1}{2}}(\mathbf{x}, \mathbf{x})) | \mathcal{F}_{k+1}\}] \rightarrow 0 \quad \text{a.s.}, \end{aligned}$$

as  $n \rightarrow \infty$ .

First, we present the main theorem. Later on we shall show that (3.3) and (3.4) hold under general conditions on  $\phi_n$ .

**THEOREM 3.1.** *Under (3.2), (3.3) and (3.4),  $V_n^* = [V_n^*(\mathbf{x}), \mathbf{x} \in R^p]$  converges weakly to a multiparameter Gaussian function  $W = [W(\mathbf{x}), \mathbf{x} \in R^p]$  whose mean (function) is 0 and covariance function is  $h(\mathbf{x}, \mathbf{y})$ .*

**PROOF.** Let  $F_{[j]}$  be the marginal df of the  $j$ th component of  $X_1, j = 1, \dots, p$ . Let then  $Y_{ij} = F_{[j]}(X_{ij}), 1 \leq j \leq p, i \geq 1$ , and let  $\mathbf{Y}_i = (Y_{i1}, \dots, Y_{ip})$ ,

$$(3.7) \quad G_n(\mathbf{t}) = n^{-1} \sum_{i=1}^n c(\mathbf{t} - \mathbf{Y}_i), \quad \mathbf{t} \in E^p, n \geq 1,$$

where  $E^p$  denotes the unit  $p$ -cube  $\{\mathbf{t}: \mathbf{0} \leq \mathbf{t} \leq \mathbf{1}\}$ . Also, let

$$(3.8) \quad G(\mathbf{t}) = P\{\mathbf{Y}_i \leq \mathbf{t}\}, \quad \mathbf{t} \in E^p;$$

$$(3.9) \quad \tilde{V}_n(\mathbf{t}) = E\{n^{\frac{1}{2}}[G_n(\mathbf{t}) - G(\mathbf{t})] | \mathcal{B}_n^{(s)}\}, \quad \mathbf{t} \in E^p.$$

Then, it suffices to show that  $\tilde{V}_n$  converges weakly in the Skorokhod  $J_1$ -topology on  $D^p[0, 1]$  to a Gaussian function  $\tilde{W} = [\tilde{W}(\mathbf{t}), \mathbf{t} \in E^p]$ . For this, we require to show (a) that the process  $\tilde{V}_n$  is tight and (b) that the finite dimensional distributions (f.d.d.) of  $\tilde{V}_n$  converge to the corresponding ones of  $\tilde{W}$ , as  $n \rightarrow \infty$ . For the proof of (a), we principally follow the treatment of Section 5 of Neuhaus (1971) where the original empirical process has been considered. We only need to replace his Lemma 5.2 by our Lemma 2.1, and the rest follows on the same line.

The proof of (b) is a little more complicated. By virtue of our Lemma 2.2, for every  $\mathbf{t} \in E^p, \{\tilde{V}_n(\mathbf{t}), \mathcal{F}_n, n \geq 1\}$  is a reverse martingale. As such, for every  $m \geq 1, \mathbf{0} \leq \mathbf{t}_1 \neq \dots \neq \mathbf{t}_m \leq \mathbf{1}$  and  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \neq \mathbf{0}, \{\sum_{j=1}^m \lambda_j \tilde{V}_n(\mathbf{t}_j), \mathcal{F}_n, n \geq 1\}$  is a reverse martingale. This tempts us to use the recent functional central limit theorems of Loynes (1970) and Brown (1971). For this, we let  $s_n^2 = E[\sum_{k=n}^{\infty} Z_k]^2$ , where

$$(3.10) \quad Z_k = \sum_{j=1}^m \lambda_j \{E[G_k(\mathbf{t}_j) | \mathcal{B}_k^{(s)}] - E[G_{k+1}(\mathbf{t}_j) | \mathcal{B}_{k+1}^{(s)}]\}, \quad k \geq 1.$$

Then, by the Brown (1971) modifications of Loynes' (1970) results, we are only to show that as  $n \rightarrow \infty$ ,

$$(3.11) \quad [\sum_{k=n}^{\infty} E(Z_k^2 | \mathcal{F}_{k+1})] / s_n^2 \rightarrow 1, \quad \text{in probability,}$$

$$(3.12) \quad s_n^{-2} [\sum_{k=n}^{\infty} E\{Z_k^2 I(|Z_k| > \varepsilon s_n) | \mathcal{F}_{k+1}\}] \rightarrow 0, \quad \text{in probability,}$$

for every  $\varepsilon > 0$ . Now, (3.11) follows from (3.4) and (3.5) while (3.12) follows from (3.3), (3.6) and the inequality

$$(3.13) \quad \left\{ \sum_{i=1}^m u_i \right\}^2 I(\left| \sum_{i=1}^m u_i \right| > m\varepsilon) \leq m2^{m-1} \left\{ \sum_{i=1}^m u_i^2 I(|u_i| > \varepsilon) \right\}.$$

Let us now examine the conditions (3.3) and (3.4). Note that  $\phi_n(\mathbf{x}) = \phi_n(\mathbf{x}, \mathbf{T}_n)$  is a bounded function and for a broad class of df's (including the exponential family having a set of minimal complete sufficient statistics), we may by the usual Taylor's expansion write

$$(3.14) \quad \begin{aligned} \phi_n(\mathbf{x}) &= \phi_n(\mathbf{x}; \boldsymbol{\theta} + (\mathbf{T}_n - \boldsymbol{\theta})) \\ &= F(\mathbf{x}) + n^{-1}v_0(\mathbf{x}) + \sum_{j=1}^q v_j(\mathbf{x}, \boldsymbol{\theta})[T_{nj} - \theta_j] \\ &\quad + \sum_{j=1}^q \sum_{j'=1}^q v_{jj'}(\mathbf{x}, \boldsymbol{\theta})[T_{nj} - \theta_j][T_{nj'} - \theta_{j'}] + R_n \end{aligned}$$

where  $|R_n| < cn^{-2}$ ,  $0 < c < \infty$  and  $v_j$ ,  $j = 0, 1, \dots, q$ ,  $v_{jj'}$ ,  $j, j' = 1, \dots, q$  are all bounded and continuous functions of  $\mathbf{x}$  and  $\boldsymbol{\theta}$ . For example, consider the case of the univariate normal df with mean  $\mu$  and variance  $\sigma^2$ ; the corresponding  $\phi_n(x)$  [cf. Lieberman and Resnikoff (1955)] for  $n > 2$  is given by

$$(3.15) \quad \begin{aligned} \phi_n(x) &= 0, \quad \text{if } x < \bar{X}_n - (n-1)^{\frac{1}{2}}S_n, \\ &= 1, \quad \text{if } x > \bar{X}_n + (n-1)^{\frac{1}{2}}S_n, \\ &= G_{n-2}\{(n-2)^{\frac{1}{2}}(x - \bar{X}_n)/[(n-1)S_n^2 - (x - \bar{X}_n)^2]^{\frac{1}{2}}\}, \\ &\quad \text{if } |x - \bar{X}_n| < (n-1)^{\frac{1}{2}}S_n, \end{aligned}$$

where  $\mathbf{T}_n = (\bar{X}_n, S_n^2) = (n^{-1} \sum_{i=1}^n X_i, n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2)$  and  $G_{n-2}$  is the Student  $t$  distribution with  $n-2$  degrees of freedom. Now, from Johnson and Kotz (1970), we have

$$(3.16) \quad G_n(t) = \Phi(t) - Z(t)[\frac{1}{4}t^2 + 1]n^{-1} + O(n^{-2}),$$

so that (3.14) is valid in this case.

From (3.14), we have

$$(3.17) \quad \begin{aligned} \phi_n(\mathbf{x}) - \phi_{n+1}(\mathbf{x}) &= v_0(\mathbf{x})/n(n+1) + \sum_{j=1}^q v_j(\mathbf{x}, \boldsymbol{\theta})(T_{nj} - T_{n+1j}) \\ &\quad + \sum_{j=1}^q v_{jj}(\mathbf{x}, \boldsymbol{\theta})(T_{nj} - T_{n+1j})(T_{nj} + T_{n+1j} - 2\theta_j) \\ &\quad + \sum_{1 \leq j \neq j' \leq q} v_{jj'}(\mathbf{x}, \boldsymbol{\theta})[(T_{nj} - T_{n+1j})(T_{nj'} - \theta_{j'}) \\ &\quad + (T_{n+1j} - \theta_j)(T_{nj'} - T_{n+1j'})] + O(n^{-2}), \quad \text{a.s.} \end{aligned}$$

Thus, if the  $\{\mathbf{T}_n\}$  form a  $U$ -statistics sequence, by the a.s. convergence results of Berk (1966),  $\mathbf{T}_n \rightarrow \boldsymbol{\theta}$  a.s., as  $n \rightarrow \infty$ , and hence, by (3.17), we have

$$(3.18) \quad \begin{aligned} \phi_n(\mathbf{x}) - \phi_{n+1}(\mathbf{x}) \\ &= \sum_{j=1}^q (T_{nj} - T_{n+1j})[v_j(\mathbf{x}, \boldsymbol{\theta}) + 2 \sum_{j'=1}^q v_{jj'}(\mathbf{x}, \boldsymbol{\theta})\{o(1)\}] + O(n^{-2}) \end{aligned}$$

with probability one, as  $n \rightarrow \infty$ . Consequently, (3.3) and (3.4) follow from (3.18), Lemma 2.5 and Lemma 2.6.

REMARKS. Srinivasan (1970), Lilliefors (1967) and others have used the studentized Kolmogorov-Smirnov statistic (in the univariate case)

$$(3.19) \quad n^{\frac{1}{2}}\{\sup_x |\phi_n(x) - F_n(x)|\}$$



as a goodness of fit statistic when  $F$  is an exponential or normal df, and in those cases, the distribution of (3.19) is known to be independent of the population parameters. Condition for this statistic to be independent of nuisance parameters in the general case has been given by O'Reilly (1971). We should note that if  $\phi_n(x, \mathbf{T}_n)$  is absolutely continuous then  $\phi_n(X_i, \mathbf{T}_n)$ ,  $i = 1, \dots, n$  are ancillary statistics, each having marginally the uniform  $(0, 1)$  df [viz., O'Reilly and Quesenberry (1973)]. Let us now assume that the probability structure of the problem is such that Stein's theorem [cf. Zacks (1971, page 79)] is applicable, and let  $\mathcal{S}$  denote the group of transformations operating on the sample space. Then, if  $g \in \mathcal{S}$  is a monotone increasing transformation, it is easy to show that

$$(3.20) \quad \phi_n(gX_i, \mathbf{T}_n(g\mathbf{X})) = \phi_n(X_i, \mathbf{T}_n(\mathbf{X})); \quad \mathbf{X} = (X_1, \dots, X_n);$$

$$(3.21) \quad \phi_n(gX_i, \mathbf{T}_n(g\mathbf{X})) - F_n(gX_i) = \phi_n(X_i, \mathbf{T}_n(\mathbf{X})) - F_n(X_i).$$

Hence, noting that  $\sup_x |\phi_n(x, \mathbf{T}_n) - F_n(x)| = \max_{1 \leq i \leq n} \{|\phi_n(X_i, \mathbf{T}_n) - F_n(X_i)|, |\phi_n(X_i, \mathbf{T}_n) - F_n(X_i - 0)|\}$  if  $\phi_n$  is a continuous function, and by (3.21) we conclude that (3.19) is an invariant statistic and that the distribution of (3.19) depends on the maximal invariant function in the parameter space. Thus, if the maximal invariant function is constant, then (3.19) has a distribution independent of the parameters. Even when this distribution is not free of nuisance parameters, under fairly general conditions,  $n^{\frac{1}{2}}[\phi_n(x, \mathbf{T}_n) - F_n(x)]$ ,  $x \in R$  will converge to a Gaussian process and the percentage points of the original Kolmogorov-Smirnov statistic [viz.,  $n^{\frac{1}{2}} \sup_x |F_n(x) - F(x)|$ ] will give a crude but distribution free upper bound for the corresponding percentage points of the asymptotic distribution of (3.19).

**4. Some concluding remarks.** First, let us consider a real valued  $g(\mathbf{x})$ ,  $\mathbf{x} \in R^p$ , such that  $\int g(\mathbf{x}) dF(\mathbf{x}) = \mu$  and  $\int g^2(\mathbf{x}) dF(\mathbf{x}) < \infty$ . Let then

$$(4.1) \quad W_n = \int g(\mathbf{x}) d\phi_n(\mathbf{x}) \quad \text{so that} \quad W_n = E[g(\mathbf{X}_1) | \mathcal{B}_n^{(s)}], \quad n \geq 1.$$

By Lemma 2.2,  $\{W_n, \mathcal{F}_n, n \geq 1\}$  is a reverse martingale, and hence, under the conditions of Brown (1971) or Loynes (1970),  $n^{\frac{1}{2}}(W_n - \mu)$  is asymptotically normal. Consider then the general  $U$ -statistic

$$(4.2) \quad U_n = \binom{n}{m}^{-1} \sum_{C_{n,m}} g(\mathbf{X}_{i_1}, \dots, \mathbf{X}_{i_m}), \quad n \geq m.$$

By Berk (1966),  $\{U_n\}$  forms a reverse martingale sequence. Loynes (1970) cites this and conjectured that the functional central limit theorem is applicable to the  $U$ -statistics. He, however, fails to provide the necessary verification of the underlying conditions needed to apply his main theorem. By virtue of our Lemmas 2.2-2.6, we immediately conclude that the assumptions of Loynes' main theorem are all met for  $U_n$ , and hence, the results hold good. In fact if  $g(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_m)$  is a symmetric kernel of an estimable functional  $\theta(F)$ , then the Rao-Blackwell estimator

$$(4.3) \quad U_n^* = E[g(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_m) | \mathcal{B}_n^{(s)}], \quad n \geq m$$

is a reverse martingale and by using the results of Miller and Sen (1972), the asymptotic normality and tightness of the process based on  $U_n^*$  can be established under parallel regularity conditions. Similar results hold for the Rao-Blackwell version of von Mises' differentiable statistical function. It may be pointed out also that if  $\{T_n\}$  is neither complete nor minimal sufficient but is sufficient and transitive,  $\{U_n^*, n \geq m\}$  will be a reverse martingale sequence. Similar modifications can be made in the statements of Lemma 2.2 and Theorem 3.1.

Now if for  $n \geq n_0$  and for each  $T_n = t$ , the measure induced by the distribution function  $\phi_n(x, t)$  is absolutely continuous with respect to the Lebesgue measure and we choose that version of the Radon-Nikodym derivative which is jointly measurable with respect to the appropriate product  $\sigma$ -algebra, then the density function can be easily shown to be a reverse martingale by applying Fubini's theorem. Hence under conditions essentially similar to those in Theorem 3.1, the asymptotic normality of the density function, appropriately standardized, can be established.

It has been shown in Sen, Bhattacharyya and Suh (1973) that the bundle strength of filaments under suitable conditions may be represented as  $Z_n = \sup\{x[1 - F_n(x)]: x \geq 0\}$ . In view of the Rao-Blackwell estimator  $\phi_n(x)$  of  $F(x)$ , we consider

$$(4.4) \quad Z_n^* = \sup\{x[1 - \phi_n(x)]: x \geq 0\}.$$

Then, it is easily seen that  $\{Z_n^*\}$  is a reverse submartingale sequence. Also,

$$(4.5) \quad P\{\sup_x n^{1/2}|\phi_n(x) - F(x)| > K_\varepsilon\} < \varepsilon, \quad K_\varepsilon < \infty.$$

Hence on using our Theorem 3.1 and then proceeding as in the proof of the main theorem in Sen, Bhattacharyya and Suh (1973) or Sen and Bhattacharyya (1976) it follows that the asymptotic normality results apply to  $\{Z_n^*\}$  as well.

Finally, the weak convergence of  $V_n^*$  for random sample sizes can also be established by using our Lemmas 2.1-2.6 along with the results of Sen (1973) on the weak convergence of  $V_n$  for random sample sizes.

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