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A CORRELATION INEQUALITY FOR MARKOV PROCESSES IN PARTIALLY ORDERED STATE SPACES[1]

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Let E be a finite partially ordered set and M_p the set of probability measures in E giving a positive correlation to each pair of increasing functions on E. Given a Markov process with state space E whose transition operator (on functions) maps increasing functions into increasing functions, let U_t be the transition operator on measures. In order that $U_t M_p \subset M_p$ for each $t \geqq 0$, it is necessary and sufficient that every jump of the sample paths is up or down.

1. Introduction. Let E be a finite set with a partial ordering $\leqq$. A probability measure in E is determined by a density μ, $\sum_{x \in E} \mu(x) = 1$; if f is a real function on E, then $\mu(f)$ denotes $\sum f(x)\mu(x)$. Call f *increasing* if $x < y$ implies $f(x) \leqq f(y)$ and let C_i be the set of increasing functions. We say that μ has *positive correlations* if $\mu(fg) \geqq \mu(f)\mu(g)$ whenever $f, g \in C_i$. Let M_p be the set of μ with positive correlations.

Let $\{X_t, t \geqq 0\}$ be a Markov process with step-function paths in the state space E and a stationary transition density $p(t, x, y)$, and let $T_t f(x) = \sum_y p(t, x, y) f(y)$, $U_t \mu(y) = \sum_x \mu(x) p(t, x, y)$. We call $\{X_t\}$ or $\{T_t\}$ *monotone* if $T_t C_i \subset C_i$, $t \geqq 0$. Conditions for monotonicity of a process have been given in [5], Section 9. See [1] for some applications of monotonicity.

It is sometimes useful to know that U_t maps M_p into itself. For example if X_t is a random subset of a set Z, we may want to know that $P_x\{a \in X_t, b \in X_t\} \geqq P_x\{a \in X_t\} \cdot P_x\{b \in X_t\}$, $a, b \in Z$. There are criteria for determining whether a measure has positive correlations; see [2] and [3]. However, they are not readily applied to $U_t \mu$, which is usually not known explicitly. The criterion of the following theorem relates directly to the behavior of the process.

(1.1) THEOREM. *Let $\{X_t\}$ be a monotone process in a finite partially ordered state space E. In order that $U_t M_p \subset M_p$ for each $t > 0$ it is necessary and sufficient that each jump of $\{X_t\}$ is up or down.*

That is, if $\{X_t\}$ can jump from x to y, then $x < y$ or $x > y$.

(1.2) COROLLARY. *Let $E^n = E \times E \times \cdots \times E$ have the product partial ordering and let f and g be increasing functions on E^n. Then $f(X_{t_1}, \cdots, X_{t_n})$ and $g(X_{t_1}, \cdots, X_{t_n})$ are positively correlated under the conditions of the theorem, if the distribution of X_0 has positive correlations.*

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The same is true, taking limits, for indicators of events such as $\{X_t \geqq y, 0 \leqq t \leqq T\}$ where $y \in E$. Also we can sometimes deal with infinite sets E by taking limits.

For monotone processes in discrete time the up-down condition is neither necessary nor sufficient for positive correlations. For this and other variations see Section 3.

NOTE. For the "necessary" part of the theorem, monotonicity is not required.

2. Proof of Theorem 1.1.

Sufficiency. Assume all jumps are up or down. We first assume E has a least element O and a greatest element I and then remove this assumption.

Let the generator of $\{X_t\}$ have the matrix $\mathscr{A}(x, y)$, $x, y \in E$, where $\mathscr{A}(x, y)$ is the transition intensity $x \to y$ if $x \neq y$, and $\sum_y \mathscr{A}(x, y) = 0$. Let $E_x^+ = \{y : y > x, \mathscr{A}(x, y) > 0\}$, $E_x^- = \{y : y < x, \mathscr{A}(x, y) > 0\}$. Pick $\Delta > 0$ small enough so that the probabilities defined below are between 0 and 1. Let $X_0', X_1' \cdots$ be a Markov chain in the state space E whose law will be defined, supposing $X_0' = x$, by exhibiting X_1' as a function of x and some random quantities. For the moment x and Δ are fixed. We use $\mathscr{E}$ and $\mathscr{E}'$ for expectations with respect to $\{X_t\}$ and $\{X_n'\}$ respectively.

Let S be the set of functions s from $E_x^+ \cup E_x^-$ into $\{0, 1\}$; coordinates of s will be denoted by s_y^+ if $y \in E_x^+$, s_y^- if $y \in E_x^-$. Let ν be the product probability measure on S such that $\nu\{s_y^+ = 1\} = \Delta\mathscr{A}(x, y)$, $y \in E_x^+$, and $\nu\{s_y^- = 1\} = 1 - \Delta\mathscr{A}(x, y)$ $y \in E_x^-$. Define X_1' as follows:

(i) if two or more $s_z^- = 0$, $X_1' = O$;

(ii) if $s_y^- = 0$ and no other $s_z^- = 0$, $X_1' = y$;

(iii) if all $s_z^- = 1$, or if E_x^- is empty, then: $X_1' = x$ if all $s_z^+ = 0$ or if E_x^+ is empty; $X_1' = y$ if $s_y^+ = 1$ and all other $s_z^+ = 0$; $X_1' = I$ if two or more $s_z^+ = 1$.

From the construction we see that if S is given the product partial ordering then $s'' > s'$ implies $X_1'(s'') \geqq X_1'(s')$. It follows that if $f \in C_i$ then $f \circ X_1'$ is an increasing function from S into R_1. Since ν is a product measure we have $\nu(s' \vee s'') \cdot \nu(s' \wedge s'') = \nu(s') \cdot \nu(s'')$. It follows (see [3]) that if $f, g \in C_i$, then

$$\mathscr{E}_x' f(X_1')g(X_1') \geqq \mathscr{E}_x' f(X_1')\mathscr{E}_x' g(X_1') . \tag{2.1}$$

Moreover

$$\Pr\{X_1' = y \mid X_0 = x\} = \Delta\mathscr{A}(x, y) + \theta c_1 \Delta^2 , \quad x \neq y , \tag{2.2}$$

where $|\theta| \leqq 1$ and c_1 does not depend on x, y, or Δ. Also (supremum norm)

$$\mathscr{E}_x f(X_\Delta) = \Delta \textstyle\sum_{y \neq x} \mathscr{A}(x, y)f(y) + [1 + \Delta\mathscr{A}(x, x)]f(x) + c_2\theta\Delta^2\|f\| , \tag{2.3}$$

where c_2 and θ have the same properties as in (2.2). It follows that

$$|\mathscr{E}_x f(X_\Delta) - \mathscr{E}_x' f(X_1')| \leqq c\Delta^2\|f\| , \tag{2.4}$$

where c does not depend on f or Δ. The c which appears below is the same as in (2.4).

We show that if $f, g \in C_i$ then

$$\mathscr{E}_x' f(X_n')g(X_n') \geqq \mathscr{E}_x' f(X_n')\mathscr{E}_x' g(X_n') - (n-1)K\Delta^2 \|f\| \cdot \|g\| , \quad n = 1, 2, \cdots , \tag{2.5}$$

where the constant K will be determined. For $n = 1$, (2.5) is just (2.1). Suppose (2.5) is true for $n = 1, 2, \cdots, N$, for each $f, g \in C_i$. Using (2.1) and (2.4),

$$\begin{aligned} \mathscr{E}_x' f(X'_{N+1})g(X'_{N+1}) &= \mathscr{E}_x'\mathscr{E}'_{X'_N} f(X_1')g(X_1') \\ &\geqq \mathscr{E}_x'\{\mathscr{E}'_{X'_N} f(X_1')\mathscr{E}'_{X'_N} g(X_1')\} \\ &\geqq \mathscr{E}_x'\{\mathscr{E}_{X'_N} f(X_\Delta)\mathscr{E}_{X'_N} g(X_\Delta)\} - (2c + c^2)\Delta^2 \cdot \|f\| \cdot \|g\| . \end{aligned} \tag{2.6}$$

The function $x \to \mathscr{E}_x f(X_\Delta)$ and $x \to \mathscr{E}_x g(X_\Delta)$ have norms $\leqq \|f\|$ and $\|g\|$ respectively, and are increasing in x. Hence, from the inductive hypothesis and (2.4)

$$\begin{aligned} &\mathscr{E}_x'\{\mathscr{E}_{X'_N} f(X_\Delta)\mathscr{E}_{X'_N} g(X_\Delta)\} \\ &\geqq [\mathscr{E}_x'\mathscr{E}_{X'_N} f(X_\Delta)] \cdot [\mathscr{E}_x'\mathscr{E}_{X'_N} g(X_\Delta)] - (N-1)K\Delta^2 \|f\| \cdot \|g\| \\ &\geqq [\mathscr{E}_x'\mathscr{E}'_{X'_N} f(X_1')] \cdot [\mathscr{E}_x'\mathscr{E}'_{X'_N} g(X_1')] \\ &\qquad - (2c + c^2)\Delta^2 \|f\| \cdot \|g\| - (N-1)K\Delta^2 \|f\| \cdot \|g\| . \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\begin{aligned} \mathscr{E}_x' f(X'_{N+1})g(X'_{N+1}) &\geqq \mathscr{E}_x' f(X'_{N+1})\mathscr{E}_x' g(X'_{N+1}) \\ &\qquad - [(4c + 2c^2) + (N-1)K] \cdot \Delta^2 \|f\| \cdot \|g\| . \end{aligned}$$

If we take $K = 4c + 2c^2$, the inductive step is completed. Hence (2.5) is true.

Now fix t and let $n \to \infty$, taking $\Delta = t/n$. It follows from (2.5), (2.4), and a known result about approximations to continuous time chains by discrete time chains (see [6], Theorem 5.3) that

$$\mathscr{E}_x f(X_t)g(X_t) \geqq \mathscr{E}_x f(X_t)\mathscr{E}_x g(X_t) . \tag{2.8}$$

If E does not have a least or greatest element, augment E to E^* by adjoining new elements O and I that will be least and greatest. Extend $\{X_t\}$ to $\{X_t^*\}$ on E^* by making O and I absorbing states. Then $\{X_t^*\}$ is still monotone and still has the up-down property. If f and g are increasing on E, extend them to increasing f^* and g^* on E^*. If $x \in E$,

$$\begin{aligned} \mathscr{E}_x f(X_t)g(X_t) &= \mathscr{E}_x^* f^*(X_t^*)g^*(X_t^*) \\ &\geqq \mathscr{E}_x^* f^*(X_t^*) \cdot \mathscr{E}_x^* g^*(X_t^*) = \mathscr{E}_x f(X_t) \cdot \mathscr{E}_x g(X_t) . \end{aligned}$$

It is readily seen from (2.8) that $\mu \in M_p$ implies $U_t\mu \in M_p$. This completes the proof of sufficiency.

Necessity. If $w \in E$, the indicator of the set $\{z : z \in E, z \geqq w\}$ is in C_i. If $\mathscr{A}(x, y) > 0$ for some x and y that are not comparable then

$$\begin{gathered} P_x\{X_t \geqq x\} \geqq P_x\{X_t = x\} \to 1 , \quad t \downarrow 0 , \\ P_x\{X_t \geqq y\} = t\mathscr{A}(x, y) + t \textstyle\sum_{z>y} \mathscr{A}(x, z) + o(t) , \\ P_x\{X_t \geqq x, X_t \geqq y\} \leqq t \textstyle\sum_{z>y} \mathscr{A}(x, z) + o(t) , \end{gathered}$$

showing that $P_x\{X_t \geqq x, X_t \geqq y\} < P_x\{X_t \geqq x\} \cdot P_x\{X_t \geqq y\}$ for sufficiently small $t > 0$.

3. Change of conditions. The following two examples show that a non-monotonic process with the up-down condition may or may not have positive correlations. (a) If E is simply ordered, it is known that M_p contains every probability measure in E. (b) Let $E = \{a, b, c\}$, $a < c$, $b < c$, a and b not comparable. The transitions $c \to a$ and $c \to b$ each have intensity 1. The process is not monotone because $1 = P_a\{X_t \in \{a, c\}\} > P_c\{X_t \in \{a, c\}\}$ if $t > 0$. Also

$$P_c\{X_t \geqq a, X_t \geqq b\} = P_c\{X_t = c\} = e^{-2t},$$
$$P_c\{X_t \geqq a\} \cdot P_c\{X_t \geqq b\} \to \tfrac{1}{4} \qquad \text{as} \quad t \to \infty,$$

so we do not have positive correlations.

Theorem 1.1 is not true for processes in discrete time. For let μ be a probability measure not in M_p on a space E having a greatest element I. Adjoin a point z to E less than each point of E and let $\{X_n\}$ be a process on $\{z\} \cup E$ that jumps out of z with the distribution μ and that jumps from each $x \in E$ (including $x = I$) directly into I. Then $\{X_n\}$ is monotone with up or down transitions but does not have positive correlations. On the other hand there are monotone discrete-time processes in a space E that is not simply ordered, without the up-down property, but having positive correlations. An example is given in Lemma 1 of [4]. In fact this is true of any monotone process $\{X_n\}$ of subsets of a finite set Z if conditional to $X_n = x$ the events $\{a \in X_{n+1}\}$, $a \in Z$, are independent.

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