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A STRONG LAW FOR WEIGHTED AVERAGES OF INDEPENDENT, IDENTICALLY DISTRIBUTED RANDOM VARIABLES WITH ARBITRARILY HEAVY TAILS[1]

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Let $X_1, X_2, \cdots$ be independent, identically distributed, nondegenerate random variables, let w_k be a sequence of positive numbers and for $n = 1, 2, \cdots$ let $S_n = \sum_{k=1}^{n} w_k X_k$ and $W_n = \sum_{k=1}^{n} w_k$. The weak (strong) law is said to hold for $\{X_k, w_k\}$ if and only if S_n/W_n converges in probability (almost surely) to a constant. Jamison, Orey and Pruitt (1965) (*Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** 40–44) studied conditions related to these laws of large numbers. In considering the strong law, only distributions with finite first moments are discussed. However, Theorem 2 of this paper shows that a sequence of random variables and a sequence of weights can be chosen so that the strong law holds and so that the random variables have arbitrarily heavy tails. This result also answers some interesting questions concerning the weak law.

1. Introduction. Let $X, X_1, X_2, \cdots$ be a sequence of independent, identically distributed, nondegenerate random variables, let $\{w_k\}$ be a sequence of positive numbers and for $n = 1, 2, \cdots$ let $S_n = \sum_{k=1}^{n} w_k X_k$ and $W_n = \sum_{k=1}^{n} w_k$. Jamison, Orey and Pruitt (1965) considered weak and strong laws of large numbers for $\{X_k, w_k\}$. The weak (strong) law is said to hold for $\{X_k, w_k\}$ if and only if S_n/W_n converges in probability (almost surely) to a constant. They have shown that in order for the weak law to hold the weights must satisfy

$$W_n \to \infty \quad \text{and} \quad w_n/W_n \to 0 \quad \text{as} \quad n \to \infty \tag{1}$$

or equivalently, $\max_{1 \leq k \leq n} w_k/W_n \to 0$ as $n \to \infty$. Throughout this paper we will refer to a sequence which satisfies (1) as one which qualifies.

Restricting attention to whose X's for which $E|X| < \infty$, they proved that the strong law holds whenever the tail probabilities of X and the function

$$N(x) = \text{card.}\, \{n : W_n/w_n \leqq x\} \tag{2}$$

grow slowly enough so that a certain integral is finite. (See condition (4) of this note.) Because of this interaction between $N(x)$ and the tail probabilities

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of X, it is possible for the strong law to hold for $\{X_k, w_k\}$ but not for $\{X_k, w_k'\}$ even though $\{w_k'\}$ qualifies. While the classical strong law ($w_k \equiv 1$) is a first moment result, one might, in view of the last remark, wonder if there is a sequence $\{X_k, w_k\}$ for which the strong law holds even though the first moment of X does not exist. We construct an example to show that, given any function g defined on the nonnegative reals with $g(x) \to \infty$ as $x \to \infty$, there is a sequence of random variables $\{X_k\}$ and a sequence of weights $\{w_k\}$ for which the strong law holds and $Eg(X^+) = Eg(X^-) = \infty$. Corollaries 1 and 2 of Chow and Teicher (1971) also provide examples for which the strong law holds and $E|X| = \infty$ but in both cases $\lim_{x\to\infty} x^r P[|X| \geqq x] = 0$ for all $0 < r < 1$ which implies that $E|X|^r < \infty$ for all $0 < r < 1$. However, choosing $g(x) = (\log x)^+$ in the example presented here we see that the strong law holds for a sequence $\{X_k, w_k\}$ with $E|X|^r = \infty$ for all $r > 0$. It should be noted that there are sequences $\{X_k\}$ for which the strong law fails for every sequence of weights. One such example is obtained by letting the X_k have a Cauchy distribution.

In the case of the weak law, Jamison et al. have shown that if the weak law holds for $\{X_k, 1\}$ and if $\{w_k\}$ qualifies then the weak law also holds for $\{X_k, w_k\}$. One might speculate that if the weak law holds for some qualified sequence of weights then it holds for all such sequences. However, by choosing g properly, the example mentioned above also shows that it is possible for the weak law (and, in fact, the strong law) to hold for $\{X_k, w_k\}$ but not for $\{X_k, 1\}$. So the examples presented here demonstrate that the classical weights, $w_k \equiv 1$, play a special role in the weak law of large numbers.

Chapter 4 of Stout (1974) contains a more detailed discussion of the work of Jamison et al. and Chow and Teicher.

2. Examples. Denoting the distribution function of X by F, we establish the following result which is Theorem 2 of Jamison et al., modified slightly to allow for random variables which do not have a first moment.

THEOREM 1. *If* $\{w_k\}$ *qualifies, if*

$$\int_{|x|<T} x\, dF(x) \to \mu \qquad \textit{as} \quad T \to \infty \tag{3}$$

and if

$$\int x^2 \int_{y \geqq |x|} N(y) y^{-3}\, dy\, dF(x) < \infty\,, \tag{4}$$

then the strong law holds for $\{X_k, w_k\}$ *and in fact* $W_n^{-1} S_n$ *converges almost surely to* μ.

PROOF. The proof is like that given by Jamison et al. except in showing $W_n^{-1} E(T_n) \to \mu$. In our case, $E(Y_k) = E(X I_{[|X| < W_k / w_k]}) \to \mu$ by (1) and (3) and so $W_n^{-1} E(T_n) \to \mu$.

In constructing the examples of this section, it is convenient to obtain the weights, w_k, via the function $N(x)$. (However, it should be noted that the function $N(x)$ does not uniquely determine a sequence $\{w_k\}$.) Let $\{x_k\}$ be a strictly increasing unbounded sequence of real numbers with $x_1 = 1$, let $\{n_k\}$ be a strictly increasing sequence of integers with $n_1 = 1$ and let $N(x) = n_k$ for $x_k \leqq x < x_{k+1}$ and $k = 1, 2, \cdots$. It is possible to construct weights w_k so that they correspond to $N(x)$ according to definition (2) and so that $w_k/W_k \to 0$ as $k \to \infty$ (but it may be the case that $W_k \nrightarrow \infty$). One way to do this is to set $w_1 = 1$, choose w_j so that $W_j/w_j = x_2$ for $j = 2, \cdots, n_2$, then choose w_j so that $W_j/w_j = x_3$ for $j = n_2 + 1, \cdots, n_3$, etc.

We note for future reference that given a sequence of weights $\{w_k\}$ with $w_k/W_k \to 0$ as $k \to \infty$ there exist sequences $\{x_k\}$ and $\{n_k\}$ satisfying the conditions above such that the $N(x)$ corresponding to $\{w_k\}$ satisfies $N(x) = n_k$ for $x_k \leqq x < x_{k+1}$ and $k = 1, 2, \cdots$. Next we characterize those functions $N(x)$ which correspond to weights $\{w_k\}$ with $W_k \to \infty$.

PROPOSITION. *Let $\{w_k\}$ be a sequence of positive numbers satisfying $w_k/W_k \to 0$ and let $N(x)$ be defined by* (2), *then $W_k \to \infty$ if and only if $I = \int_1^\infty N(x)x^{-2}\,dx$ is infinite.*

PROOF. We first note that if $I < \infty$ then $N(x)/x \to 0$ as $x \to \infty$. This follows from the inequality $\int_M^\infty N(x)x^{-2}\,dx \geqq N(M)M^{-1}$.

With c chosen so that $1 < c < \inf_{k\geqq 2} W_k/w_k$, we see that

$$\begin{aligned}\lim_{n\to\infty} \log W_n - \log W_1 &= \textstyle\sum_{k=2}^\infty \log(W_k/W_{k-1}) \\ &= \textstyle\int_c^\infty \log(x/(x-1))\,dN(x)\,.\end{aligned}$$

Integrating $\int_c^d \log(x/(x-1))\,dN(x)$ by parts and observing that $\log(x/(x-1)) \leqq (x-1)^{-1}$ for $x > 1$, we see that W_n is bounded if $I < \infty$ and, on the other hand, $W_n \to \infty$ if $I = \infty$.

The following is a corollary to our Theorem 1 and contains Theorem 3 of Jamison et al. as a special case ($r = 1$). Its proof is like the one they have given and hence is omitted. We do comment, however, that the proof of the necessity is based on the following implication: if the strong law holds for $\{X_k, w_k\}$ then $EN(|X|) < \infty$.

COROLLARY. *Let $1 \leqq r < 2$ and let $\{w_k\}$ be a fixed sequence of weights which qualifies. The strong law holds for all X with $E|X|^r < \infty$ if and only if* $\limsup_{x\to\infty} N(x)/x^r < \infty$.

It is interesting to note that there are no qualified sequences of weights which satisfy $\limsup_{x\to\infty} N(x)/x^r < \infty$ with $0 < r < 1$. If there were such a sequence the corresponding integral I would be finite, contradicting the proposition.

We now show that sequences $\{X_k, w_k\}$ can be constructed so that the strong law holds and X has arbitrarily heavy tails; that is, we prove the following result.

THEOREM 2. *Let g be a nonnegative function defined for nonnegative real numbers with $g(x) \to \infty$ as $x \to \infty$. There exists a sequence $\{X_k, w_k\}$ for which the strong law holds and $Eg(X^+) = Eg(X^-) = \infty$.*

PROOF. We construct a function $N(x)$ and a random variable X such that $E[g(X^+)] = E[g(X^-)] = \infty$. We then obtain our weights from $N(x)$ as in the discussion preceding our proposition and show that they qualify by arguing that $I = \infty$. Finally we use Theorem 1 to obtain the strong law. Set $n_1 = 1$; choose $a_1 > n_1$ with $n_1/g(a_1) \leqq 1$ and then n_2 an integer with $n_2 \geqq a_1^2$; choose $a_2 > n_2$ with $n_2/g(a_2) \leqq 2^{-1}$ and then choose an integer $n_3 \geqq a_2^2$; continuing in this fashion we obtain sequences $\{n_k\}$ and $\{a_k\}$ which for $k = 1, 2, \cdots$ satisfy

$$(5) \qquad a_k > n_k, \qquad n_k/g(a_k) \leqq k^{-1} \qquad \text{and} \qquad n_{k+1} \geqq a_k^2 .$$

Since $n_{k+1} > n_k^2$ for $k = 1, 2, \cdots$, $n_k \to \infty$ and $n_k/n_{k+1} \to 0$ as $k \to \infty$. Now define $N(x)$ by setting $N(x) = n_k$ for $n_k \leqq x < n_{k+1}$ and $k = 1, 2, \cdots$. We obtain a sequence of weights $\{w_k\}$ from $N(x)$ using the techniques discussed in the paragraph preceding the proposition and then use the proposition to show that they qualify. Since $n_k/n_{k+1} \to 0$, $I = \sum_{k=1}^{\infty} n_k(n_k^{-1} - n_{k+1}^{-1}) = \sum_{k=1}^{\infty} (1 - n_k/n_{k+1}) = \infty$.

By (5), $g(a_k) \geqq kn_k \geqq k$, and so we can choose c a positive constant so that $P_k = c(kg(a_k))^{-1}$ satisfies

$$\sum_{k=1}^{\infty} P_k = 1 , \qquad \sum_{k=1}^{\infty} g(a_k)P_k = \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} g(a_k)k^{-1}P_k < \infty .$$

Let X be a symmetric random variable with $P[X = a_k] = P_k/2$. The proof is completed if we can show that (4) holds for this choice of X and $\{w_k\}$. Since $n_{v+1} > n_v^2$ we note that for $k = 1, 2, \cdots$,

$$\sum_{v=k+1}^{\infty} n_v(n_v^{-2} - n_{v+1}^{-2}) < \sum_{v=k+1}^{\infty} n_v^{-1} < \sum_{v=k+1}^{\infty} (n_{k+1})^{-2^{(v-k-1)}} < \sum_{v=1}^{\infty} n_{k+1}^{-v}$$
$$= (n_{k+1} - 1)^{-1} \leqq n_2(n_2 - 1)^{-1} n_{k+1}^{-1} ,$$

and so

$$\int x^2 \int_{y \geqq |x|} N(y)y^{-3}\, dy\, dF(x)$$
$$= 2^{-1} \sum_{k=1}^{\infty} a_k^2 P_k \{n_k(a_k^{-2} - n_{k+1}^{-2}) + \sum_{v=k+1}^{\infty} n_v(n_v^{-2} - n_{v+1}^{-2})\}$$
$$< \sum_{k=1}^{\infty} n_k P_k + n_2(n_2 - 1)^{-1} \sum_{k=1}^{\infty} a_k^2 P_k n_{k+1}^{-1}$$
$$\leqq \sum_{k=1}^{\infty} g(a_k)k^{-1}P_k + n_2(n_2 - 1)^{-1} \sum_{k=1}^{\infty} P_k < \infty .$$

It has already been noted that if X has a Cauchy distribution then the strong law does not hold for any sequence of weights. This is, in fact, true for any X for which $\liminf_{x\to\infty} xP[|X| \geqq x] > 0$ (see Theorem 1 of Chow and Teicher (1971)). With $g(x) = (\log x)^+$, Theorem 2 shows that there exists a sequence $\{X_k, w_k\}$ for which the strong law holds and $E(\log |X|)^+ = \infty$. So in this example, $\limsup_{x\to\infty} x^r P[|X| \geqq x] > 0$ for each $r > 0$ but $\liminf_{x\to\infty} xP[|X| \geqq x]$ must be zero.

Furthermore, Theorem 2 shows that the choice of weights also has an effect when considering the weak law. Theorem 1 of Jamison et al. states that the weak law holds for all qualified sequences of weights if and only if (3) and

$$(6) \qquad \lim_{T\to\infty} TP[|X| \geqq T] = 0$$

hold. This result and Theorem A on page 278 of Loève (1963) show that with the distribution of X fixed the weak law holds for all qualified sequences of weights if and only if the classical weak law ($w_k \equiv 1$) holds. Using Theorem 2 with $g(x) = x^r$ with $r < 1$, one can obtain a sequence $\{X_k, w_k\}$ for which the weak law (and in fact, the strong law) holds but (6) does not (it is well known that (6) implies $E|X|^r < \infty$ for all $r < 1$) and so the weak law does not hold for $\{X_k, 1\}$.

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