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LÉVY SYSTEMS FOR TIME-CHANGED PROCESSES

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After a study of the process Y, obtained from a right process X by time-changing it with respect to a continuous additive functional A, we relate the jumps of Y in $\Phi = \text{supp}\, A$ to the excursions of X out of Φ and to the jumps of X inside Φ.

1. Introduction. Let $X = (\Omega, \mathbf{F}, \mathbf{F}_t, X_t, \Theta_t, P^x)$ be a right process (see [4]) with state space $(E_\Delta, \mathbf{E}_\Delta)$, where $E_\Delta = E \cup \{\Delta\}$, E being a universally measurable subset of a compact metric space, and Δ is isolated in E_Δ. $\mathbf{E}_\Delta$ denotes the Borel subsets of E_Δ. $\mathbf{E}_\Delta{}^*$ will denote the universally measurable subsets of E_Δ.

Let $A = (A_t)_{t \geqq 0}$ be a continuous additive functional of X, and let $(\tau_t)_{t \geqq 0}$ be its right continuous inverse. We shall assume that $A_t < \infty$ for all t. We shall put $\Phi = \text{supp}\, A = \{x \in E : P^x(\tau_0 = 0) = 1\}$. From the hypotheses on X it follows that Φ is a universally measurable subset of E.

If we put $Y_t = X_{\tau_t}$, $\mathbf{G}_t = \mathbf{F}_{\tau_t}$, $\hat{\Theta}_t = \Theta_{\tau_t}$, and if $f \in b\mathbf{\Phi}^*$ ($\mathbf{\Phi}^* = \mathbf{E}^*|_\Phi$), then $Q_t f(x) = E^x f(Y_t)$ provides us with a semigroup of kernels on $(\Phi, \mathbf{\Phi}^*)$ which makes of $(Y_t)_{t \geqq 0}$ a right process in a sense that we explain below.

The main result of this work concerns the study of the jumps of the process Y. If $f \in (\mathbf{\Phi}^* \times \mathbf{\Phi}^*)_+$, $x \in \Phi$, then Theorem 3.3 below asserts that $E^x\{\sum_{0<s\leqq t} f(Y_{s-}, Y_s) 1_R(s)\}$, where $R = \{(t, \omega) : Y_{t-}(\omega)$ exists, $Y_{t-}(\omega) \neq Y_t(\omega)\}$, can be expressed in terms of the excursions of X out of Φ and in terms of the jumps of X inside Φ.

Since the family $(\mathbf{F}_t)$ may not satisfy the usual hypotheses (see [3]), we will say that a process Z is previsible (resp. well-measurable, progressively measurable) with respect to $(\mathbf{F}_t)$, if for every law μ on E, there exists a process Z^μ previsible (resp. well-measurable, progressively measurable) with respect to $(\Omega, \mathbf{F}^\mu, \mathbf{F}_t{}^\mu, P^\mu)$ which is P^μ-indistinguishable from Z.

For later use we state without proof the following lemmas whose proofs are easy.

LEMMA 1.1. *Let μ be a probability on E. If T is a $\{\mathbf{G}_t{}^\mu\}$ stopping time, then τ_t is an $\{\mathbf{F}_t{}^\mu\}$ stopping time.*

LEMMA 1.2. *Let μ be a probability on E. If Z is previsible with respect to $\{\mathbf{G}_t{}^\mu\}$, then Z_A is previsible with respect to $\{\mathbf{F}_t{}^\mu\}$.*

Also, to the given continuous additive functional A we will associate a Markov set (see [5]) $M = \{(t, \omega) : t > 0, A_{t+\varepsilon}(\omega) - A_{t-\varepsilon}(\omega) > 0, \forall\, \varepsilon > 0\}$. It is easy to

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see that the contiguous intervals to M are the random sets of the form $((\tau_{s-}, \tau_s)) = \{(t, \omega): \tau_{s-}(\omega) < t < \tau_s(\omega)\}$ and therefore that the set of left endpoints of the contiguous intervals is given by $\vec{M} = \{(\tau_{s-}(\omega), \omega): s > 0, \omega \in \Omega, \tau_{s-}(\omega) \neq \tau_s(\omega)\}$.

The description of the jumps of a right process X can be given through the Ray–Knight compactification method. The method consists of introducing a new state space $\bar{E}$ and a metric ρ on $\bar{E}$ which makes it a compact space, such that E sits in $\bar{E}$ as a ρ-dense, universally measurable subset. $\bar{\mathbf{E}}$ will denote the Borel subsets of $\bar{E}$.

Also, there is a unique Ray resolvent (see [4]) $(\bar{U}^\alpha)_{\alpha>0}$ on $(\bar{E}, \bar{\mathbf{E}})$, such that for any $f \in \mathbf{C}(\bar{E})$, $\bar{U}^\alpha f|_E = U^\alpha(f|_E)$. We also have $\bar{U}^\alpha: \mathbf{C}(\bar{E}) \to \mathbf{C}(\bar{E})$ and there exists a unique semigroup $(\bar{P}_t)_{t\geqq 0}$ on $(\bar{E}, \bar{\mathbf{E}})$ whose resolvent is $(\bar{U}^\alpha)$. The relationship between the process X and the process constructed with the aid of $(\bar{P}_t)$ as the details of the Ray–Knight compactification method are well explained in [4].

Let d denote the original metric on E. Let Ω represent the set of all maps $\omega: \mathbb{R}_+ \to E$ which are right continuous in the original metric d and which are right continuous in ρ and have left limits in ρ. Now one can construct all the associated σ-algebras $\mathbf{F}_t^0$, $\mathbf{F}_t^\mu$, $\mathbf{F}_t$, etc. in the usual way; then Theorem 11.8 and Theorem 13.4 of [4] apply to give us the main results that we shall need below.

2. Study of the process Y. It is easy to verify that τ_s is an $\{\mathbf{F}_t\}$ stopping time for every s. From Proposition 6.7 ([2], Chapter 1) it follows that $Y_t \in \mathbf{G}_t$, $t \geqq 0$, and that $\{\mathbf{G}_t\}$ is a right continuous family of σ-algebras. It is also easy to see that $t \to Y_t$ is right continuous.

We will now prove that a.s. $(P^x)u \to Y_u \in \Phi$ for u in $[0, A_\zeta)$. From Proposition 2.3 ([2], Chapter 5) it follows that $Y_{u+v} = Y_u \circ \hat{\theta}_v$ on $\{\tau_u < \zeta\}$ and that $\tau_0 \circ \theta_{\tau_u} = 0$ on $\{\tau_u < \zeta\} = \{u < A_\zeta\}$. This implies that $P^x\{\tau_u < \zeta\} = P^x\{\tau_0 \circ \theta_{\tau_u} = 0, \tau_u < \zeta\} = E^x\{P^{X(\tau_u)}(\tau_0 = 0); \tau_u < \zeta\}$ which implies that a.s. (P^x), $P^{X(\tau_u)}(\tau_0 = 0) = 1$ if $u < A_\zeta$. Using the right continuity of $u \to Y_u$ and the fact that X is a right process, one can verify that for every x there exists a set N_x with $P^x(N_x) = 0$ and such that $Y_u(\omega) \in \Phi\ \forall\ \omega \notin N_x\ \forall\ u < A_\zeta$.

We will write Φ_Δ instead of $\Phi\ U\{\Delta\}$ and we will have $\mathbf{\Phi}^* = \mathbf{E}^*|_\Phi$ and $\mathbf{\Phi}_\Delta{}^* = \mathbf{E}_\Delta{}^*|_\Phi$. When we have $f \in \mathbf{\Phi}^*$ we will extend it to $\mathbf{\Phi}_\Delta{}^*$ by putting $f(\Delta) = 0$. If we put $Q_t f(x) = E^x f(Y_t)$ for $f \in \mathbf{\Phi}_+$ and $x \in \Phi$, it can be seen easily that $(Q_t)_{t\geqq 0}$ defines a semigroup of transition kernels on $(\Phi, \mathbf{\Phi}^*)$, which can be made Markov by extending the Q_t's to $(\Phi_\Delta, \mathbf{\Phi}_\Delta{}^*)$ in the usual way.

If μ is carried by Φ, then $P^\mu(Y_0 \in A) = \mu(A)$ for all $A \in \mathbf{\Phi}^*$. We can also define $W^\alpha f(x) = \int_0^\infty e^{-\alpha t} Q_t f(x)\, dt$ for $f \in \mathbf{\Phi}^*$ and $x \in \Phi$. Then certainly $W^\alpha: b\mathbf{\Phi}^* \to b\mathbf{\Phi}^*$ and $W^\alpha - W^\beta = (\beta - \alpha) W^\alpha W^\beta$; $\alpha, \beta > 0$. Since $E^x f(X_{\tau_t})$ makes sense for any $f \in b\mathbf{E}^*$ and $x \in E$, it follows that $W^\alpha f(x) = \int_0^\infty e^{-\alpha t} E^x f(X_{\tau_t})\, dt$ makes sense for $f \in b\mathbf{E}^*$ and $x \in E$.

We will say that $f \in \mathbf{\Phi}_+{}^*$ is $\alpha - Y$-excessive if $\beta W^{\alpha+\beta} f(x) \uparrow f(x)$ for all $x \in \Phi$ as $\beta \to \infty$, or equivalently if $e^{-\alpha t} Q_t f(x) \uparrow f(x)$ for all $x \in \Phi$ as $t \downarrow 0$. Given $f \in \mathbf{\Phi}_+{}^*$ we can extend it to E by putting $\bar{f}(x) = E^x f(X_{\tau_0})$. It can be easily verified that $\beta W^{\alpha+\beta} \bar{f}(x) \uparrow \bar{f}(x)$ for all $x \in E$ as $\beta \to \infty$.

Let us now prove

PROPOSITION 2.1. *Let f be $\alpha - Y$-excessive. Then for all $x \in \Phi$, $t \to f(Y_t)$ is* a.s. *(P^x) right continuous.*

PROOF. There exists (Proposition 2.9, [4]) a sequence $\{f_n\} \in b\Phi^*$ such that $W^\alpha f_n \uparrow f_n$ on Φ as $n \to \infty$. Extending f_n and f as indicated above, it follows that $W^\alpha \bar{f}_n \uparrow \bar{f}$ as $n \to \infty$.

Put $A_t^\alpha = \int_0^t e^{-A_s} dA_s$. Then $(A_t^\alpha)_{t \geq 0}$ is a continuous additive functional of (X, N^α) (see Definition 1.1, [2], Chapter 4) where $N_t^\alpha = e^{-\alpha A_t}$ is a multiplicative functional of X. Rewrite $W^\alpha \bar{f}_n(x)$ as $U_A \alpha \bar{f}_n(x) = E^x \int_0^\infty \bar{f}(X_t)\, dA_t^\alpha$, which is then (X, N^α)-excessive and therefore is $\beta - (X, N^\alpha)$-excessive for all $\beta > 0$.

If we put $V^\beta g(x) = E^x \int_0^\infty e^{-\beta t} g(X_t) N_t^\alpha\, dt$, then there is a sequence $\{g_n\} \in b\mathbf{E}_+$ such that $V^\beta g_n(x) \uparrow \bar{f}(x)$. It is easy to see that $h_n(x) = U^\beta g_n(x) - V^\beta g_n(x)$ is β-excessive. Since X is a right process, $t \to h_n(X_t)$ and $t \to U^\beta g_n(X_t)$ are right continuous and a fortiori so is $t \to V^\beta g_n(X_t)$.

Since $V^\beta g_n$ is $\beta - (X, N^\alpha)$-excessive, it follows that $\{e^{-\beta t} N_t^\alpha V^\beta g_n(X_t), \mathbf{F}_t, P^x\}$ is a right continuous supermartingale for all $x \in E$. It follows from Theorem 4.1 of [4] that $\{e^{-\alpha t} N_t^\alpha \bar{f}(X_t), \mathbf{F}_t, P^x\}$ is a right continuous supermartingale. From our assumptions on A it follows that $t \to f(Y_t)$ is right continuous a.s. (P^x) for all $x \in \Phi$.

From these facts we see that the process Y satisfies the following versions of the Hypothéses Droites.

HD 1′. *For any probability μ on Φ_Δ, the process Y with state space $(\Phi_\Delta, \Phi_\Delta{}^*)$ is Markov with transition semigroup $(Q_t)_{t \geq 0}$, and has right continuous trajectories and μ as initial measure.*

HD 2′. *If f is $\alpha - Y$-excessive, then for any probability μ on Φ, $t \to f(Y_t)$ is almost surely right continuous.*

3. Lévy system of the time-changed process. Let us begin by describing the jumps of X. We will say that ω has a jump at t if $X_{t-}^*(\omega) = d - \lim_{s \uparrow t} X_s$ exists and $X_{t-}^*(\omega) \neq X_t(\omega)$ ($d - \lim$ means limit with respect to the d metric on E). By $X_{t-}(\omega)$ we shall mean $\rho - \lim_{s \uparrow t} X_s(\omega)$ (which always exists in $\bar{E}$).

With very simple modifications, Theorem 3.1 in [1] can be restated as follows.

THEOREM 3.1. *If X is a Ray process with state space $E_\Delta \subset \bar{E}$, there exists a continuous additive functional H with* supp *$H \subset E$, a kernel N on $(\bar{E}, \bar{\mathbf{E}}^*)$ which can be taken to satisfy $N(x, \cdot) = 0$ for $x \notin E_\Delta$ and $N(x, \{x\}) = 0$ for all $x \in E_\Delta$, such that for all $f \in (\bar{\mathbf{E}} \times \bar{\mathbf{E}})_+$, all positive Z, previsible with respect to $\{\mathbf{F}_t\}$, one has*

$$E^x\{\textstyle\sum_t Z_t f(X_{t-}, X_t) 1_{\{X_{t-} \neq X_t, X_{t-} \in E_\Delta\}}\} = E^x\{\int_0^\infty Z_s\, dH_s \int_{E_\Delta} N(X_s, dy) f(X_s, y)\}\,. \tag{3.1}$$

Let us define the following sets: $S = \{(t, \omega) : X_{t-}(\omega) \in E_\Delta, X_{t-}(\omega) \neq X_t(\omega)\}$, $J = \{(t, \omega) : X_{t-}^*(\omega)$ exists, $X_{t-}^*(\omega) \neq X_t(\omega)\}$ and $\Gamma = \{(t, \omega) : X_{t-}^*(\omega)$ does not exist or $X^*_{t-}(\omega) \neq X_t(\omega)\}$. With the aid of Proposition 13.4, [4], one can see that for any law μ on E, $S = J \cap \Gamma^c$ a.s. (P^μ).

This gives us Theorem 1.1 of [1] if we take $f \in b(\mathbf{E}_\Delta \times \mathbf{E}_\Delta)_+$ and extend it to $\bar{E}_\Delta \times \bar{E}_\Delta$, putting $f = 0$ off $\bar{E}_\Delta \times \bar{E}_\Delta$, since the left-hand side of 3.1 can now be written as

$$E^x\{\textstyle\sum_t Z_t f(X_{t-}, X_t)1_S(t)\} = E^x\{\sum_t Z_t f(X^*_{t-}, X_t)1_S(t)\}\,.$$

Let us now restate Theorem 1.1 of [1], with obvious modifications, for the sake of later reference.

THEOREM 3.2. *If X is a right process, there exists a Lévy system (N, H), consisting of a positive kernel N on $(E_\Delta, \mathbf{E}_\Delta{}^*)$ such that $N(x, \{x\}) = 0$ for all $x \in E_\Delta$, and a continuous additive functional H such that $E^x(H_t) < \infty$ for all $x \in E_\Delta$, for all $t \geqq 0$, with the property that for all $f \in (\mathbf{E}_\Delta \times \mathbf{E}_\Delta)_+$ and all positive previsible Z one has*

$$E^x\{\textstyle\sum_t Z_t f(X^*_{t-}, X_t)1_S(t)\} = E^x \int_0^\infty Z_s\, dH_s \int_{E_\Delta} N(X_s, dy) f(X_s, y)\,. \tag{3.2}$$

Note that (3.2) does not describe all the jumps of X, for $J = J \cap \Gamma + J \cap \Gamma^c$, which for all x is P^x-indistinguishable from $J \cap \Gamma + S$. In the course of the proof of Proposition 13.4 of [4], it is proved that $J \cap \Gamma$ is accessible and from Theorem 13.1 of [4] it follows that S is the totally inaccessible part of the jumps. With the help of Proposition 13.8 of [4], if f vanishes off $E \times E$ it follows that $E^x\{\sum_t f(X^*_{t-}, X_t)1_{J\cap\Gamma}(t)\}$ vanishes when X is standard.

We will work under the following assumption.

ASSUMPTION. For any probability μ on E and any $f \in (\mathbf{E} \times \mathbf{E})_+$ $E^\mu\{\sum_t f(X_{t-}, X_t)1_{J\cap\Gamma}(t)\} = 0$.

Now, let us say a few words about the jumps at time $\zeta = \inf\{t > 0 : X_t = \Delta\}$. When using the Ray–Knight compactification method, it may happen that Δ is not isolated in E_Δ in the metric ρ, and the jump at ζ may be lost when taken in the ρ-metric. Some information about what happens at ζ can be obtained from 3.2 as follows. Let $f \in (\mathbf{E}_\Delta \times \mathbf{E}_\Delta)_+$ such that $f(x, y) = 0$ if $y \neq \Delta$ and $Z = e^{-\alpha s}$. Since $X^*_{\zeta-} \neq \Delta$ if it exists and since Δ is isolated in E_Δ,

$$\begin{aligned} E^x\{e^{-\alpha\zeta} f(X^*_{\zeta-}, \Delta);\, X^*_{\zeta-} \text{ exists}\} &= E^x\{e^{-\alpha\zeta_A} f(X^*_{\zeta_A-}, A);\, X^*_{\zeta-A} \text{ exists}\} \\ &\quad + E^x \int_0^\infty e^{-\alpha s}\, dH_s N(X_s, \Delta) f(X_s, \Delta) \end{aligned}$$

where ζ_A is the accessible part of ζ.

Let us now proceed to the study of the jumps of Y. We will say that ω has a jump at t if $Y_{t-}(\omega) = d - \lim_\uparrow Y_s(\omega) \neq Y_t(\omega)$. In order to get rid of an excess of minus signs we will put $\bar{X}_t = X^*_{t-}$ and $\tau_{t-} = \bar{\tau}_t$. With this $Y_{t-} = \bar{X}_{\bar{\tau}_t} = \bar{X}_{\tau_{t-}}$. Let us also put $R = \{(t, \omega) : Y_{t-}(\omega)$ exists, $Y_{t-}(\omega) \neq Y_t(\omega)\}$. Let us recall that the set $\vec{M}$ introduced in Section 1 can be written as $\vec{M} = \vec{M}_w \cup \vec{M}_\pi$, where $\vec{M}$ is a well-measurable, homogeneous, closed set, which can be written as a countable union of graphs of stopping times; and $\vec{M}_\pi$ is a progressively measurable homogeneous, closed set, which contains the graph of no stopping time (see [5]).

It can be proved (Theorem 4.1, [6]) that there exists a couple $(K, \hat{P})$ (the exit system for $\vec{M}_\pi$), K is a continuous additive functional of X and $\hat{P}$ is a kernel from $(\Omega, \mathbf{F})$ to $(E, \mathbf{E}^*)$ such that $\operatorname{supp} K \subset \Phi$ and $\hat{P}^x(\tau_0 = 0) = 0$ for all $x \in E$. Also, for any positive $\mathbf{F}^\circ$ or $\mathbf{F}^*$-measurable function h and any positive $\{\mathbf{F}_t\}$-well-measurable process ξ one has

$$E^x\{\sum_{s\in\vec{M}_\pi} \xi_s h \circ \theta_s\} = E^x\{\int_0^\infty \xi_s \hat{P}^{X_s}(h)\, dK_s\}\,.$$

The result we are after is contained in the following theorem.

THEOREM 3.3. *Let μ be a measure carried by Φ, Z a bounded $\{\mathbf{G}_t\}$-previsible process and $f \in (\mathbf{\Phi} \times \mathbf{\Phi})_+$. Then*

$$\begin{aligned} E^\mu\{\sum_t Z_t f(Y_{t-}, Y_t) 1_R(t)\} = {} & \sum_n E^\mu\{Z_{A_{T_n}} \hat{f}(X_{T_n}, X_{\tau_0} \circ \theta_{T_n})\} \\ & + E^\mu\{\int_0^\infty Z_{A_s} \hat{P}^{X_s} \hat{f}(X_0, X_{\tau_0})\, dK_s\} \\ & + E^\mu\{\int_0^\infty Z_{A_s}\, dH_s \int_{E_\Delta} \bar{N}(X_s, dy) f(X_s, y)\}\,. \end{aligned} \tag{3.3}$$

Here $\hat{f}(x, y) = f(x, y)$ if $x \neq y$ and 0 if $x = y$. $\{T_n\}_{n=1}^\infty$ is a collection of stopping times with disjoint graphs such that $\vec{M}_w = \bigcup_n [T_n]$, and $(K, \hat{P})$ is an exit system for $\vec{M}_\pi$ as mentioned above, $\bar{N}(x, dy) = b(x, y)N(x, dy)$ for some appropriate $b \in (\mathbf{E}^* \times \mathbf{E}^*)_+$, and (N, H) is a Lévy system for the original process.

PROOF. By writing $\mathbb{R}_+ \times \Omega = B_1 \cup B_2$ with $B_2 = B_1^c$ and $B_1 = \{(t, \omega) : \tau_{t-}(\omega) \neq \tau_t(\omega)\}$, the left-hand side of (3.3) can be written as $E_1 + E_2$ where $E_1 = E^\mu\{\sum_t Z_t f(Y_{t-}, Y_t) 1_R(t) 1_{B_1}(t)\}$ and E_2 is the obvious complement. Taking into account the description of $\vec{M}$ we gave in Section 1, the fact that $A_{\tau_t} = t$ on $\{\tau_t < \zeta\}$, and the definitions of Y and $\bar{X}$, E_1 can be written as follows:

$$E_1 = E^\mu\{\sum_{s\in\vec{M}} Z_{A_s} \hat{f}(\bar{X}_s, X_{\tau_0} \circ \theta_s)\}\,.$$

By writing $\vec{M} = \vec{M}_w \cup \vec{M}_\pi$, and using the fact about $\vec{M}_w$ and $\vec{M}_\pi$ mentioned above together with Maisonneuve's result (Theorem 4.1, [6]), it can be seen rather easily that

$$E_1 = \sum_n E^\mu\{Z_{A_{T_n}} \hat{f}(X_{T_n}, X_{\tau_0} \circ \theta_{T_n})\} + E^\mu \int_0^\infty Z_{A_s}\, dK_s \hat{P}^{X_s} f(X_0, X_{\tau_0})\}\,. \tag{3.4}$$

Let us turn our attention to

$$E_2 = E^\mu\{\sum_t Z_t f(Y_{t-}, Y_t) 1_R(t) 1_{B_2}(t)\}\,.$$

From the definitions it is easy to see that

$$E_2 = E^\mu\{\sum_s Z_{A_{\tau_s}} f(\bar{X}_{\tau_s}, X_{\tau_s}) 1_{\{\bar{X}_{\tau_s} \text{exits},\ X_{\tau_s} \neq X_{\tau_s}\}} 1_{\{\tau_s = \tau_s\}}\,.$$

When s ranges over $\mathbb{R}_+$, τ_s ranges over $\{t < \infty : \exists\, s, \tau_s = t\}$, which coincides with the set $\{t : A_{t+\varepsilon} - A_t > 0;\ \forall\, \varepsilon > 0\}$. Making use of our assumption to replace J by S and putting $G = \{(t, \omega) : A_{t+\varepsilon}(\omega) - A_t(\omega) > 0,\ \forall\, \varepsilon > 0\}$, we can write

$$E_2 = E^\mu\{\sum_t Z_{A_t} f(\bar{X}_t, X_t) 1_S(t) 1_{B_2}(A_t) 1_G(t)\}\,. \tag{3.5}$$

In [1] it is shown how meaning can be given to expressions of the form $C_t = \sum_{0<s\leqq t} 1_S(s)1_{B_2}(A_s)1_G(s)$. We can quote [1] provided we can prove that each summand is homogeneous and adapted. It is easy to see that $1_S(s)1_{B_2}(A_s)1_G(s) \circ \theta_t = 1_S(s+t)1_{B_2}(A_{s+t})1_G(s+t)$ for $s > 0$, $t \geqq 0^{\sim}$.

Let us now prove that $1_{B_2}(A_t)1_G(t) \in \mathbf{F}_t$ for $t \geqq 0$. This will make $C_t \in \mathbf{F}_t$. Put $W = 1_{B_2}$. Then for $s \geqq 0$, $W_s \in \mathbf{G}_s{}^\mu$ and therefore $W_{A_t} \in \mathbf{G}^\mu_{A_t}$. From Lemma 1.1 it follows that $\tau_{A_t} = t + \tau_0 \circ \theta_t$ is an $\mathbf{F}_t{}^\mu$-stopping time and then $W_{A_t} \in \mathbf{F}^\mu_{t+\tau_0\circ\theta_t}$. By definition this means that for any $s \geqq 0$ $\{1_{B_2}(A_t) = 1\} \in \mathbf{F}^\mu_{t+\tau_0\circ\theta_t}$ or $\{1_{B_2}(A_t) = 1\} \cap \{t + \tau_0 \circ \theta_{t-}s\} \in \mathbf{F}_s{}^\mu$. Therefore

$$\begin{aligned}\{1_{B_2}(A_t)1_G(t) = 1\} &= \{1_{B_2}(A_t) = 1\} \cap \{1_G(t) = 1\}\\ &= \{1_B(A_t) = 1\} \cap \{\omega : t \in G(\omega)\}\\ &= \{1_{B_2}(A_t) = 1\} \cap \{\tau_0 \circ \theta_t = 0\}\\ &= \{1_{B_2}(A_t) = 1\} \cap \{t + \tau_0 \circ \theta_t \leqq t\} \qquad \text{which is in} \quad \mathbf{F}_t{}^\mu .\end{aligned}$$

Now, from Theorem 1.2 in [1] it follows that there exists a $b \in (\mathrm{E}^* \times \mathrm{E}^*)_+$ such that C_t is indistinguishable from $\sum_{0<s\leqq t} b(X^*_{s-}, X_s)1_S(s)$, which allows us to rewrite (3.5) as

$$E^\mu\{\textstyle\sum_t Z_{A_t} f(X^*_{t-}, X_t) b(X^*_{t-}, X_t) 1_S(t)\} .$$

From this (3.3) drops out if we take Theorem 3.2 into account.

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