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UNIFORMITY IN STONE'S DECOMPOSITION OF THE RENEWAL MEASURE

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Stone has decomposed the renewal measure of a probability distribution F into two parts: a finite component with the same tail behaviour as that of F and an absolutely continuous one which is nearly stationary at infinity. Our theorem asserts the uniformity of this decomposition.

1. Introduction. The solution of the two lift problem [3] is based on the uniformity of Stone's decomposition of the renewal measure. The renewal measure determined by a probability distribution F has been decomposed by Stone [2] into two parts: a finite component with the same tail behaviour as that of F and an absolutely continuous one which is nearly stationary at infinity. Since the uniformity obtained is interesting in itself, its proof is given here, separated from the considerations of [3].

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2. The theorem. Let $\xi_1, \xi_2, \cdots, \eta_1^\varepsilon, \eta_2^\varepsilon, \cdots$ be independent nonnegative random variables with $\mathscr{L}(\xi_k) = F$ and $\mathscr{L}(\eta_k^\varepsilon) = G^\varepsilon$ $(k \geqq 1)$, where $\varepsilon \geqq 0$ is a small parameter. Define $\zeta_k^\varepsilon = \xi_k + \eta_k^\varepsilon$, and denote by H^ε the renewal measure of the process determined by the intervals ζ_k^ε:

$$H^\varepsilon(S) = E \sum_{k \geqq 1, \zeta_1^\varepsilon + \cdots + \zeta_k^\varepsilon \in S} 1 .$$

The characteristic function of a probability measure A will be denoted by $\hat{A}(t) = \int e^{itx}\, dA(x)$.

THEOREM. *Suppose that*

(i) $\int_0^\infty x\, dF(x) = \lambda$ $(0 < \lambda < \infty)$ *and* $\hat{F}^{n_0}$ *is integrable for some* $n_0 \geqq 0$;

(ii) $\int_0^\infty x\, dG^\varepsilon(x) = \varepsilon$ *and the family of measures* $\{\tilde{G}^\varepsilon(dx) = xG^\varepsilon(dx) : \varepsilon \leqq \varepsilon_0\}$ *is relatively compact in the weak topology.*

Then for a suitable $\varepsilon_0 > 0$ *all the measures* H^ε $(\varepsilon \leqq \varepsilon_0)$ *possess a decomposition* $H^\varepsilon = H_1^\varepsilon + H_2^\varepsilon$, *where*

(α) *the* H_1^ε *are finite and the family of measures* $\{\tilde{H}_1^\varepsilon(dx) = xH_1^\varepsilon(dx) : \varepsilon \leqq \varepsilon_0\}$ *is relatively compact in the weak topology*;

(β) *the* H_2^ε *are absolutely continuous with uniformly bounded densities* h_2^ε, i.e., $\sup_{\varepsilon \leqq \varepsilon_0} \sup_x h_2^\varepsilon(x) < \infty$, *and finally*

$$\lim_{\varepsilon' \to 0, x' \to \infty} \sup_{\varepsilon \leqq \varepsilon'} \sup_{x \geqq x'} \left| h_2^\varepsilon(x) - \frac{1}{\lambda} \right| = 0 . \tag{2.1}$$

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It is easy to see that the last assertion can also be formulated as follows:

$$\lim_{\varepsilon'\to 0, x'\to\infty} \sup_{\varepsilon\leqq\varepsilon'} \sup_{x\geqq x'} \left| h_2^\varepsilon(x) - \frac{1}{\lambda+\varepsilon} \right| = 0 .$$

An important special case is where $\eta^\varepsilon = \varepsilon\eta$ and η has a given distribution with expectation 1. In this case (ii) holds automatically.

3. **Proof.** Let $A^{(n)}$ denote the n-fold convolution of the probability measure A with itself. By the virtue of (i), $F^{(n_0)}$ can be written in the form $F^{(n_0)} = pF_1 + qF_2$, where $p > 0$, $p + q = 1$ and the density of F_1 is continuous and vanishes outside a finite interval. Denote $F^\varepsilon = F * G^\varepsilon$ and let $(F^\varepsilon)^{(n_0)} = pF_1 * (G^\varepsilon)^{(n_0)} + qF_2 * (G^\varepsilon)^{(n_0)} \doteq pF_1^\varepsilon + qF_2^\varepsilon$. Following Stone's arithmetic, let

$$H_1^\varepsilon = [1 + F^\varepsilon + \cdots + (F^2)^{(n_0-1)}] * \textstyle\sum_{k=0}^\infty q^k (F_2^\varepsilon)^{(k)}$$

and

$$H_2^\varepsilon = [pH_1^\varepsilon * (G^\varepsilon)^{(n_0)}] * F_1 * \textstyle\sum_{k=0}^\infty (F^\varepsilon)^{(n_0 k)} . \tag{3.1}$$

Then $H^\varepsilon = H_1^\varepsilon + H_2^\varepsilon$ (cf. [2]). Now (α) follows easily because $H_1^\varepsilon(R^1) = n_0 p^{-1}$ and (ii) imply the relatively compactness of the $\tilde{F}^\varepsilon$ ($\tilde{F}^\varepsilon(dx) = xF^\varepsilon(dx)$) and consequently that of the F_2^ε ($\tilde{F}_2^\varepsilon(dx) = xF_2^\varepsilon(dx)$) as well.

The core of the proof is the following lemma. Denote $L^\varepsilon = \sum_{k=0}^\infty (F^\varepsilon)^{(n_0 k)}$.

LEMMA 1. *For every finite interval I,*

$$\lim_{\varepsilon'\to 0, x'\to\infty} \sup_{\varepsilon\leqq\varepsilon'} \sup_{x\geqq x'} \left| L^\varepsilon(I + x) - \frac{|I|}{n_0\lambda} \right| = 0 .$$

PROOF. For simpler notations suppose in what follows that $n_0 = 1$. Assume that u and $\hat{u}$ are integrable and $u(x) = \int e^{itx}\hat{u}(t)\,dt$. It is easy to see that the inversion formula

$$\int u(y)L^\varepsilon(dy + x) = 2\pi \frac{\hat{u}(0)}{\lambda+\varepsilon} + 2\int e^{-itx}\hat{u}(t)\,\mathrm{Re}\,\frac{1}{1 - \hat{F}^\varepsilon(t)}\,dt$$

is valid if $u \geqq 0$ (cf. [1], page 221). Let

$$\begin{aligned} u_{a,\rho}(y) &= 1 && \text{if } |y| \leqq a \\ &= \rho^{-1}(a + \rho - |y|) && \text{if } a \leqq |y| \leqq a + \rho \\ &= 0 && \text{if } |y| \geqq a + \rho \qquad 0 < \rho < a . \end{aligned}$$

Putting $u^-(y) = u_{a-\rho,\rho}(y)$ and $u^+(y) = u_{a,\rho}(y)$ we have

$$\int u^-(y)L^\varepsilon(dy + x) \leqq L^\varepsilon(x - a, x + a) \leqq \int u^+(y)L^\varepsilon(dy + x) . \tag{3.2}$$

Since $\hat{u}_{a,\rho}(t) = (\pi\rho t^2)^{-1}(\cos at - \cos (a + \rho)t)$, the inversion formula can be applied to u^+ (and to u^- as well). Thus

$$\int u^+(y)L^\varepsilon(dy + x) = \frac{2a+\rho}{\lambda+\varepsilon} + 2\int \cos(tx)\hat{u}^+(t)\,\mathrm{Re}\,\frac{1}{1 - \hat{F}^\varepsilon(t)}\,dt$$

where $\hat{u}^+$ is integrable. We use this relation in the more convenient form

$$\int u^+(y)L^\varepsilon(dy + x) - \frac{2a}{\lambda}$$

$$(3.3) \qquad = \frac{\rho\lambda - 2\varepsilon a}{\lambda(\lambda + \varepsilon)} + 2 \int \cos(tx)\hat{u}^+(t) \operatorname{Re} \frac{1}{1 - \hat{F}^0(t)} dt$$

$$+ 2 \cos(tx)\hat{u}^+(t) \operatorname{Re} \left(\frac{1}{1 - \hat{F}^\varepsilon(t)} - \frac{1}{1 - \hat{F}^0(t)} \right) dt .$$

(Clearly $F^0(t) = F(t)$.) The first term on the right-hand side will be arbitrarily small if we choose ρ and ε small enough. Let us fix a small ρ. Then by the Riemann–Lebesgue lemma the second term can be made as small as we want if x is large enough. Finally observe that $\hat{u}^+$ is bounded and so the desired negligibility of the third term will follow if we prove

LEMMA 2.

$$\lim_{\varepsilon \to 0} \int \left| \operatorname{Re} \left(\frac{1}{1 - \hat{F}^\varepsilon(t)} - \frac{1}{1 - \hat{F}(t)} \right) \right| dt = 0 .$$

Accept the lemma for the moment. Our argument can also be applied to u^- and then by the inequality (3.2) we obtain the assertion of Lemma 1.

PROOF OF LEMMA 2. We use the arithmetic identity

$$\frac{1}{1 - \hat{F}^\varepsilon} - \frac{1}{1 - \hat{F}} = \frac{\hat{F}(\hat{G}^\varepsilon - 1)}{(1 - \hat{F}^\varepsilon)(1 - \hat{F})} = J^\varepsilon .$$

By the assumptions, $\lim_{t \to \infty} \hat{F}^\varepsilon(t) = 0$ uniformly in $\varepsilon \geqq 0$. Thus for $|t|$ large, $|\operatorname{Re} J^\varepsilon| \leqq 4|\hat{F}|$ (say) which—together with the integrability of $\hat{F}$—gives (for a suitable t_1) $\int_{|t| \geqq t_1} |\operatorname{Re} J^\varepsilon| \leqq \beta$, where $\beta > 0$ is arbitrary but fixed. Now we prove that there exist $t_2 > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon \leqq \varepsilon_0$

$$(3.4) \qquad \int_{|t| \leqq t_2} \left| \operatorname{Re} \frac{1}{1 - \hat{F}^\varepsilon(t)} \right| dt < \beta .$$

To do this we need a lemma that will be proved later.

LEMMA 3. *There exist ε^*, $t^* > 0$ such that for every $\varepsilon \leqq \varepsilon^*$, $|t| \leqq t^*$*

$$|1 - \hat{F}^\varepsilon(t)|^2 \geqq \frac{\lambda^2 t^2}{2} .$$

Suppose that ε^* and t^* are chosen according to this lemma. Then for $|t| \leqq t^*$ and $\varepsilon \leqq \varepsilon^*$

$$\operatorname{Re} \frac{1}{1 - \hat{F}^\varepsilon(t)} = \frac{\operatorname{Re}(1 - \hat{F}^\varepsilon(t))}{|1 - \hat{F}^\varepsilon(t)|^2} \leqq \frac{2}{\lambda^2} \frac{\operatorname{Re}(1 - \hat{F}^\varepsilon(t))}{t^2}$$

and for $t_0 \leqq t^*$ and $\varepsilon \leqq \varepsilon^*$

$$\int_{|t| \leqq t_0} t^{-2} \operatorname{Re}(1 - \hat{F}^\varepsilon(t))\, dt \leqq \int\int_{|t| \leqq t_0} t^{-2} |e^{ity} - 1 - i \sin ty|\, dt\, dF^\varepsilon(y)$$

$$\leqq \int y \int_{|s| \leqq t_0 y} s^{-2} |e^{is} - 1 - i \sin s|\, ds\, dF^\varepsilon(y) .$$

Since the measures $yF^\varepsilon(dy)$ are relatively compact and $\int s^{-2}|e^{is} - 1 - i \sin s|\, ds < \infty$, we can apply the Lebesgue theorem and obtain (3.4).

To finish the proof of Lemma 2 let us fix t_1 and t_2 as above and observe that $\hat{F}(t) \neq 1$ if $t \neq 0$. Then

$$|1 - \hat{F}^\varepsilon| \geqq |1 - \hat{F}| - |\hat{F}||1 - \hat{G}^\varepsilon| ,$$

and since the second term on the right-hand side tends to 0 uniformly for $|t| \in [t_2, t_1]$, we can conclude that $\hat{F}^\varepsilon(t)$ is uniformly bounded away from 1 for $t_2 \leqq |t| \leqq t_1$, if ε is small enough. Consequently $\lim_{\varepsilon\to 0} J^\varepsilon(t) = 0$ uniformly for $t_2 \leqq |t| \leqq t_1$. Hence Lemma 2.

PROOF OF LEMMA 3. We start from the elementary inequality

$$(3.5) \qquad |\hat{F}^\varepsilon(t) - 1 - i(\lambda + \varepsilon)t| \leqq \int |e^{itx} - 1 - itx|\, dF^\varepsilon(x)$$
$$\leqq \frac{t^2}{2} \int_{|x| \leqq 2|t|^{-1}} x^2\, dF^\varepsilon(x) + 2|t| \int_{|x| \geqq 2|t|^{-1}} |x|\, dF^\varepsilon(x) .$$

Note that the conditions of the theorem imply that the measures $\tilde{F}^\varepsilon(dx) = xF^\varepsilon(dx)$ are relatively compact in the weak topology. Consequently if t^* is small enough, then because of

$$\int_{|x| \geqq 2|t|^{-1}} |x|\, dF^\varepsilon(x) \leqq \int_{|x| \geqq 2(t^*)^{-1}} |x|\, dF^\varepsilon(x) ,$$

the second term on the very right-hand side of (3.5) will be less than $\delta|t|$ for all $|t| \leqq t^*$ and $\varepsilon \leqq \varepsilon_0$, where δ is any prescribed positive number. Moreover,

$$\frac{t^2}{2} \int_{|x| \leqq 2|t|^{-1}} x^2\, dF^\varepsilon(x) \leqq |t| \frac{c}{2} \tilde{F}^\varepsilon[0, \infty) + |t|\tfrac{1}{2}\tilde{F}^\varepsilon[c|t|^{-1}, \infty)$$

whenever $0 < c \leqq 2$. If c is sufficiently small the first term on the right-hand side will be less than $\delta|t|$ for any $\varepsilon \leqq \varepsilon_0$ and the same will be true for the second term if c is already fixed and $|t| \leqq t^*$, where t^* is small enough. Now Lemma 3 follows by taking the second power of (3.5).

Return to the proof of the theorem. Using Lemma 1 and the nice properties of F_1 one can see first that $F_1 * L^\varepsilon$ is absolutely continuous with a density satisfying

$$\lim_{\varepsilon'\to 0, x'\to\infty} \sup_{\varepsilon \leqq \varepsilon'} \sup_{x \geqq x'} \left| \frac{d}{dx} (F_1 * L^\varepsilon)(x) - \frac{1}{n_0 \lambda} \right| = 0 .$$

This easily leads to (2.1) if we observe that $\{pH_1^\varepsilon * (G^\varepsilon)^{(n_0)} : \varepsilon \leqq \varepsilon_0\}$ is a relatively compact family of measures each of total mass n.

Finally we prove the uniform boundedness of the h_2^ε's. But it is a simple consequence of the fact that for every finite I we have $\limsup_{\varepsilon\to 0} \sup_x L^\varepsilon(I + x) < \infty$, which comes from (3.2) and (3.3).

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