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ON COMPLETE CONVERGENCE OF DISTRIBUTIONS AND EXPECTED VALUES OF ORDER STATISTICS

BY K. AIYAPPAN NAIR

Edinboro State College

Let $\{F_m\}$ be a sequence of distribution functions and $X(r, n, m)$ be the rth order statistic in a sample of size n from F_m. In this note we establish a relationship between the convergences of F_m and $EX(r, n, m)$ as $m \to \infty$.

Let $X(1, n) \leqq \cdots \leqq X(n, n)$ be an ordered sample of size n from a distribution function F. Chan (1967) has shown that the sequence of expectations $\{EX(n, n)\}$ completely determines F. Konheim (1971) has provided an alternate proof for the same result. A more general result is given by Pollak (1973). He has shown that for any sequence $\{k(n)\}$ $(1 \leqq k(n) \leqq n)$ of integers, $\{EX(k(n), n)\}$ determines F. Gupta (1974) has derived the result of Chan (1967) and Konheim (1971) assuming that F is discrete.

In this note we prove a result concerning the convergence of distribution functions and of expected values of order statistics.

Let $X(r, n, m)$ and $X(r, n)$ be the rth order statistics in samples of size n from the distribution functions F_m and F respectively. For brevity we denote $X(1, 1, m)$ by $X(m)$ and $X(1, 1)$ by X. Expectations are denoted by E as in $E_F X$, the suffix F is used only when F is not clear from the context. In the following F_m and F may be continuous or discrete.

THEOREM 1. *Suppose $F_m(x) \to F(x)$ and $E|X(m)| \to E|X|$ as $m \to \infty$. Then $EX(r, n, m) \to EX(r, n)$ as $m \to \infty$ for all $r (1 \leqq r \leqq n)$ and n.*

PROOF. The distribution function G_m of $X(r, n, m)$ is given by

$$\begin{aligned} G_m(x) &= \sum_{i=r}^{n} \binom{n}{i} F_m{}^i(x)[1 - F_m(x)]^{n-i} \\ &= \sum_{i=r}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n}{i}\binom{n-i}{j} F_m{}^{i+j}(x) \end{aligned}$$

(see David (1970), page 7). Hence

$$EX(r, n, m) = \sum_{i=r}^{n} \sum_{j=0}^{n-i} (-1)^j \binom{n}{i}\binom{n-i}{j} \int x \, dF_m{}^{i+j}(x).$$

Since $|X|$ is uniformly integrable in F_m, it follows that $|X|$ is uniformly integrable in $F_m{}^{i+j}$, and since furthermore $F_m{}^{i+j} \to F^{i+j}$ we find that $EX(r, n, m) \to EX(r, n)$. (See Loève (1963), page 183.)

THEOREM 2. *Suppose for every n there exists a $k(n)$ $(1 \leqq k(n) \leqq n)$ such that $EX(k(n), n, m) \to \mu(k(n), n)$ as $m \to \infty$, and $E|X(m)|^p < c$ for some $p > 1$ and c finite, then there exists a* rv *X with distribution function F such that $F_m \to F$ completely as $m \to \infty$, and $\mu(k(n), n) = EX(k(n), n)$.*

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PROOF. Let $\{F_{m'}\}$ be a subsequence converging to some distribution function G. Then by a result on page 184 of Loève (1963), $F_{m'}$ converges to G completely and $E|X(m')| \to E_G|X|$ as $m' \to \infty$. Hence by Theorem 1, $EX(r, n, m') \to E_G X(r, n)$ as $m' \to \infty$ for all $1 \leqq r \leqq n$. Therefore $E_G X(k(n), n) = E_F X(k(n), n)$. This implies that $G = F$. (See Pollak (1973).) This proves the theorem.

As an application consider the arithmetic mean of a sample of size m from a population with mean zero and standard deviation unity. If n samples of size m are taken, the expected value of the rth smallest mean, for large m, is approximately equal to $m^{-\frac{1}{2}} E_F X(r, n)$ where $F(x)$ is the standard normal distribution.

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DEPARTMENT OF MATHEMATICS
EDINBORO STATE COLLEGE
EDINBORO, PENNSYLVANIA 16444