TWO APPLICATIONS OF A POISSON APPROXIMATION FOR DEPENDENT EVENTS[1]

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Recent results have estimated the error when sums of dependent non-negative integer-valued random variables are approximated in distribution by a Poisson variable. Two problems are considered where these results can be used to provide simple solutions. The first problem studies the asymptotic behavior, as $\alpha \to 0$, of the number of independent random arcs of length α needed to cover a circle of unit circumference at least m times ($m \geq 1$). The second problem deals with urn schemes.

1. Introduction. Freedman [2] has studied the problem of approximating in distribution a sum of dependent random variables by a Poisson variable. We first introduce some notation. For $\theta > 0$, let $N(\theta)$ denote a Poisson variable with parameter θ. If X_1 and X_2 are any two nonnegative integer-valued random variables, then define

$$d(X_1, X_2) = \sup_A |P(X_1 \in A) - P(X_2 \in A)|$$

where A ranges over all subsets of nonnegative integers. Also for any set B, I_B is its indicator function.

THEOREM A (Freedman [2]). *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let A_1, $A_2, \cdots$ be events, i.e., elements of $\mathscr{F}$. Let $\{\mathscr{F}_i\}$ be a collection of sub-σ-algebras of $\mathscr{F}$ and assume that*

$$A_i \in \mathscr{F}_i, \quad i \geq 1, \quad \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots.$$

Let $p_i = P(A_i | \mathscr{F}_{i-1})$. Let N equal the number of A_i that occur with $i \leq n$:

$$N = \sum_{i=1}^n I_{A_i}.$$

Let $a \leq b$ be nonnegative real numbers. Let ε, δ be nonnegative real numbers less than $\frac{1}{2}$. Suppose

$$(1) \qquad P(a \leq \sum_{i=1}^n p_i \leq b, \sum_{i=1}^n p_i^2 \leq \varepsilon) \geq 1 - \delta.$$

Then there is a positive constant C such that

$$d(N, N(a)) \leq C\varepsilon + (b - a) + 2\delta.$$

The difficulties in using Theorem A are to compute the $\{p_i\}$ and to establish (1). The next result considers a simple situation, where sufficient conditions are given which imply (1).

COROLLARY 1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $\{A_{ij}\}$ be a collection of events and $\{\mathscr{F}_{ij}\}$ a collection of sub-σ-algebras of $\mathscr{F}$. Assume that*

$$A_{ij} \in \mathscr{F}_{ij} \qquad \text{and} \qquad \mathscr{F}_{0j} \subset \mathscr{F}_{1j} \subset \cdots \quad \text{for all} \quad i, j.$$

Let $p_{ij} = P(A_{ij} \,|\, \mathscr{F}_{ij})$ and $\{n_j\}$ a sequence of integers tending to ∞. Let

$$N_j = \sum_{i=1}^{n_j} I_{A_{ij}}.$$

Suppose that as $j \to \infty$

(2) $\sum_{i=1}^{n_j} E(p_{ij}) \to \theta$ *a positive finite constant,*

(3) $E(|\sum_{i=1}^{n_j} (p_{ij} - E(p_{ij}))|) = o(1)$,

(4) $\sum_{i=1}^{n_j} E(p_{ij}^2) = o(1)$.

Then

$$d(N_j, N(\theta)) = o(1).$$

PROOF. Let $0 \leqq \varepsilon \leqq \frac{1}{2}$, $0 \leqq \delta \leqq \frac{1}{2}$. In view of (3) and (4), there exists a J_0 such that $j > J_0$ implies

$$P(|\sum_{i=1}^{n_j} (p_{ij} - E(p_{ij}))| < \varepsilon, \; \sum_{i=1}^{n_j} p_{ij}^2 < \varepsilon) > 1 - \delta.$$

Hence by Theorem A, for all $j > J_0$,

(5) $d(N_j, N(\sum_{i=1}^{n_j} E(p_{ij}) - \varepsilon)) \leqq \varepsilon(2 + C) + 2\delta.$

The result now follows from (2) and (5) and the observation ([2], Lemma 8) that

$$d(N(\sum_{i=1}^{n_j} E(p_{ij}) - \varepsilon), N(\theta)) \leqq |\sum_{i=1}^{n_j} E(p_{ij}) - \varepsilon - \theta|.$$

The purpose of this note is to give two applications where the computation of the $\{p_{ij}\}$ is possible and the conditions of Corollary 1 are verifiable.

The first application is to a problem studied by Flatto [1]. Let $\mathscr{C}$ be a circle with unit circumference and suppose that arcs of length α are thrown independently and uniformly on $\mathscr{C}$. Set $N_\alpha(m)$ equal to the number of arcs necessary to cover $\mathscr{C}$ at least m times ($m \geqq 1$). We then have

THEOREM 1 (Flatto [1]). *Let $-\infty < x < \infty$, and $m \geqq 1$. Then,*

$$\lim_{\alpha \to 0} P\left(N_\alpha(m) \leqq \frac{1}{\alpha}\left(\log\frac{1}{\alpha} + m\log\log\frac{1}{\alpha} + x\right)\right) = e^{-e^{-x}/(m-1)!}.$$

Using Corollary 1 and some properties of the Poisson process, we will give a simple proof of Theorem 1. The details are carried out in Section 2.

Our second problem was considered by Sevast'yanov [4]. Suppose we perform a sequence of experiments, where, for the lth experiment, we independently toss n_l balls into N_l urns according to some distribution $\{b_l(k)\}_{k=1}^{N_l}$. For ease of notation we drop the subscript l. We assume that on the set of integers $\{1, 2, \cdots, N\}$, there is a metric $\rho(i, j)$ possibly depending on l. We write Z_i for the number of the urn picked on the ith throw and let Z denote the number of pairs of tosses (i, j), $i < j$, with outcomes Z_i and Z_j such that $\rho(Z_i, Z_j) \leqq a$, where $a \geqq 0$ is a

certain fixed number. Let $\pi_k = \sum_{\rho(k,j) \leq a} b(j)$ be the probability of picking an urn whose number is in the sphere of radius a with center at k for one trial. We then have:

THEOREM 2. *Assume*

(6) $$n^3 \sum_{k=1}^{N} b(k)\pi_k^2 = o(1)$$

and

(7) $$\binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k \to \lambda < \infty .$$

Then,

$$d(Z, N(\lambda)) = o(1) .$$

(It should be noted that all the above limits are taken as $l \to \infty$.)

A proof of Theorem 2 is given in Section 3.

2. Proof of Theorem 1. For the remainder of this section we assume x and m are fixed and set

$$n = n(\alpha, m, x) = \left[\frac{1}{\alpha} \left(\log \frac{1}{\alpha} + m \log \log \frac{1}{\alpha} + x \right) \right],$$

where for any positive y, $[y]$ is the greatest integer in y. Our starting point is a result of Flatto [1].

LEMMA 1. *Let $n - 1$ $(n > m)$ points be chosen uniformly and independently on $[0, 1]$. Let $(Z_{(1)}, Z_{(2)}, \cdots, Z_{(n-1)})$ denote the order statistics. For convenience set $Z_{(0)} \equiv 0$ and $Z_{(n)} \equiv 1$. Let*

$$L_i = Z_{(i)} - Z_{(i-1)} \qquad 1 \leq i \leq n$$

and define L_i for all i by putting $L_i = L_{i+n}$. Set

$$S_i = \sum_{j=0}^{m-1} L_{i-j} \qquad m \leq i \leq n + m - 1 .$$

Then,

(8) $$P(N_\alpha(m) \leq n) = P(S_i \leq \alpha, m \leq i \leq n + m - 1) .$$

Lemma 1 is just a restatement of Theorem 2.1 of [1].

The problem is to determine the asymptotic behavior of the right-hand side of (8). Since the $\{L_i\}$ are exchangeable random variables ([3], Chapter 8), all the $\{S_i\}$ have the same distribution. Furthermore, Flatto [1] showed that $P(S_m > \alpha) = o(1)$. Thus there is no loss of generality in studying

$$A_\alpha(n) = P(S_i \leq \alpha, m \leq i \leq n - 1) .$$

The analysis depends on Corollary 1 and some properties of the Poisson process. Let $\{X(t)\}_{0 \leq t \leq 1}$ be a Poisson process with parameter equal to n. Denote the interarrival times by $\{T_i\}_{i \geq 1}$, and recall that they are i.i.d., having a negative exponential distribution with parameter n. Let

$$W_i = \sum_{j=i-m+1}^{i} T_j \qquad i \geq m .$$

We then have

(9) $\qquad P(W_i \leq \alpha, m \leq i \leq X(1); X(1) \geq m) = \sum_{k=m}^{\infty} \Lambda(k) \dfrac{e^{-n}n^k}{k!}$

where

$$\Lambda(k) = P(W_i \leq \alpha; m \leq i \leq k \mid X(1) = k).$$

It is well known ([3], Chapter 8) that conditioned on $X(1) = k$, $\{\sum_{j=1}^{i} T_j\}_{1 \leq i \leq k}$ have the same distribution as a set of order statistics of size k taken from a uniform distribution on $[0, 1]$. Thus $\Lambda(n - 1) = A_n(\alpha)$.

The idea now is to show, using Corollary 1, that the left-hand side of (9) is asymptotic to $\exp(-e^{-x}/(m - 1)!)$, and that the asymptotic difference between the right-hand side of (9) and $A_n(\alpha)$ is negligible; to do this we use the central limit theorem (CLT).

We first consider the left-hand side of (9). Let $\varepsilon > 0$. It follows from the law of large numbers that

$$P(W_i \leq \alpha; m \leq i \leq X(1); X(1) > m)$$
$$= P\left(W_i \leq \alpha; m \leq i \leq X(1); \left|\dfrac{X(1)}{n} - 1\right| < \varepsilon\right) + o(1).$$

Let $n_0 = n(1 - \varepsilon)$ and $n_1 = n(1 + \varepsilon)$. Using monotonicity we then have

$$P(W_i \leq \alpha; m \leq i \leq n_1) + o(1)$$

(10)
$$\leq P\left(W_i \leq \alpha; m \leq i \leq X(1); \left|\dfrac{X(1)}{n} - 1\right| < \varepsilon\right)$$

$$\leq P(W_i \leq \alpha; m \leq i \leq n_0) + o(1).$$

Consider the right-hand side of (10). The argument for the left-hand side is analogous. Define

$$A_i = (W_i > \alpha) \qquad\qquad m \leq i \leq n_0.$$

Then,

$$P(W_i \leq \alpha; m \leq i \leq n_0) = P(\sum_{i=m}^{n_0} I_{(W_i > \alpha)} = 0).$$

We now verify the conditions of Corollary 1. First we compute $\theta = \sum_{i=m}^{n_0} P(A_i)$. Since all the $\{A_i\}$ have the same probability

$$\theta = (n_0 - m)P(A_m).$$

But

$$P(A_m) = P(\sum_{i=1}^{m} T_i > \alpha)$$
$$= \int_{\alpha}^{\infty} \dfrac{n}{(m - 1)!} (nx)^{m-1} e^{-nx}\, dx$$
$$= \dfrac{(n\alpha)^{m-1}}{(m - 1)!} e^{-n\alpha} + o\left(\dfrac{1}{n}\right)$$
$$= \dfrac{\alpha e^{-x}}{(m - 1)!\, \log 1/\alpha} + o\left(\dfrac{1}{n}\right).$$

(The last equality follows from the definition of n.)

Hence

$$\theta = \frac{(1 - \varepsilon)e^{-x}}{(m - 1)!} + o(1) \,.$$

This proves (2).

Let $\mathscr{F}_i = \sigma(T_1, \cdots, T_i)$, $i \geqq m$. Then

$$\begin{aligned}
p_i &= P(A_j \mid \mathscr{F}_{i-1}) \\
&= P(T_i > \alpha - \textstyle\sum_{j=i-m+1}^{i-1} T_j \mid \mathscr{F}_{i-1}) \\
&= I_{(\sum_{j=i-m+1}^{i-1} T_j > \alpha)} + \exp[-n(\alpha - \textstyle\sum_{j=i-m+1}^{i-1} T_j)] I_{(\sum_{j=i-m+1}^{i-1} T_j \leq \alpha)} \\
&= W_{1i} + W_{2i} \,.
\end{aligned}$$

To prove (3) it suffices to verify

(11) $$B_j = E(|\textstyle\sum_{i=m}^{n_0} (W_{ji} - E(W_{ji}))|) = o(1) \qquad j = 1, 2 \,.$$

A straightforward computation shows that

(12) $$\textstyle\sum_{i=m}^{n_0} E(W_{1i}) = o(1)$$

which in turn implies that $B_1 = o(1)$. To handle B_2 we argue as follows. Observe

$$\begin{aligned}
E([\textstyle\sum_{i=m}^{n_0} (W_{2i} - E(W_{2i}))]^2) \\
\leqq \textstyle\sum_{i=m}^{n_0} E(W_{2i}^2) + \sum_{i=m}^{n_0} [\sum_{j \neq i} E((W_{2i} - EW_{2i})(W_{2j} - EW_{2j}))] \,.
\end{aligned}$$

By the independence of the $\{T_i\}$, the sum in the brackets has at most $2m$ nonzero terms and each of these, by Schwarz's inequality, is bounded by $E(W_{2i}^2)$. Thus

$$\begin{aligned}
E([\textstyle\sum_{i=m}^{n_0} (W_{2i} - E(W_{2i}))]^2) &\leqq 3m \textstyle\sum_{i=m}^{n_0} E(W_{2i}^2) \\
&= 3mn_0 E(W_{2m}^2) \quad \text{by exchangeability.}
\end{aligned}$$

But

(13) $$\begin{aligned}
E(W_{2m}^2) &= \textstyle\int_0^\alpha e^{-2n(\alpha - z)} \frac{n}{(m - 2)!} (nz)^{m-2} e^{-nz} \, dz \\
&= \frac{e^{-2n\alpha}}{(m - 2)!} \textstyle\int_0^{n\alpha} z^{m-2} e^z \, dz \\
&= \frac{e^{-2n\alpha}}{(m - 2)!} (n\alpha)^{m-2} e^{n\alpha} [1 + o(1)] \\
&= o\left(\frac{1}{n}\right) \,.
\end{aligned}$$

This completes the proof of (3). The proof of (4) follows from (12), (13) and the observation that

$$\textstyle\sum_{i=m}^{n_0} E(p_i^2) \leqq 2 \sum_{i=m}^{n_0} (E(W_{1i}) + E(W_{2i}^2)) \,.$$

We now conclude from Corollary 1 that as $\alpha \to 0$

(14) $$|P(W_i \leqq \alpha; \, m \leqq i \leqq n_0) - e^{-(1-\varepsilon)e^{-x}/(m-1)!}| = o(1) \,.$$

Similarly

(15) $$|P(W_i \leqq \alpha; \, m \leqq i \leqq n_1) - e^{-(1+\varepsilon)e^{-x}/(m-1)!}| = o(1) \,.$$

Since ε is arbitrary we obtain from (10), (14), (15) that

$$|P(W_i \leq \alpha; m \leq i \leq X(1); X(1) \geq m) - e^{-e^{-x}/(m-1)!}| = o(1) .$$

We now turn to the right-hand side of (9). It follows from the CLT that for any $\varepsilon > 0$, there exists a $D > 0$ such that for all n sufficiently large (i.e., for all α sufficiently small),

$$P\left(\left|\frac{X(1) - n}{n^{\frac{1}{2}}}\right| > D\right) < \varepsilon .$$

It therefore suffices to show for any $D > 0$,

(16) $\qquad\qquad \eta = \sup_{|k-n|\leq Dn^{\frac{1}{2}}} |\Lambda(k) - \Lambda(n - 1)| = o(1) .$

To prove (16) it is convenient to modify the definition of $\Lambda(k)$. Let

$$\tilde{\Lambda}(k) = P(W_i \leq \alpha; m \leq i \leq X(1); 1 - \textstyle\sum_{j=1}^{X(1)-m+1} T_j \leq \alpha \mid X(1) = k) .$$

Then for $|k - n| \leq Dn^{\frac{1}{2}}$

(17) $\qquad |\Lambda(k) - \tilde{\Lambda}(k)| \leq P(1 - \textstyle\sum_{j=1}^{X(1)-m+1} T_j > \alpha \mid X(1) = k)$

$$= O\left(\frac{1}{n}\right) \quad \text{(uniform in } k) .$$

The last equality is a straightforward computation which is omitted.

We next estimate

$$\beta_k = |\tilde{\Lambda}(k) - \tilde{\Lambda}(k - 1)| , \qquad\qquad |k - n| \leq Dn^{\frac{1}{2}} .$$

Let $Z_{(0)}, Z_{(1)}, \cdots, Z_{(k)}, Z_{(k+1)}$ be a set of order statistics of size k taken from the uniform distribution on $[0, 1]$, and set $S_i = Z_{(i)} - Z_{(i-m)}$ ($Z_{(0)} \equiv 0, Z_{(k+1)} = 1$). Let X be one more observation from uniform $[0, 1]$, $\{Z'_{(0)}, Z'_{(1)}, \cdots, Z'_{(k+1)}, Z'_{(k+2)}\}$ the new set of order statistics, and $S_i' = Z'_{(i)} - Z'_{(i-m)}$. Then,

$$\beta_k = |\tilde{\Lambda}(k) - \tilde{\Lambda}(k - 1)| \leq \textstyle\sum_{i=m}^{k} P(S_i > \alpha; S'_{(j)} \leq \alpha; m \leq j \leq k + 1) .$$

But

$$P(S_i > \alpha, S_j' \leq \alpha; m \leq j \leq k + 1)$$

$$\leq P(\alpha < S_i \leq m\alpha; Z_{(i-m)} < X < Z_{(i)})$$

(18) $\qquad\quad \leq m\alpha P(\alpha < S_i \leq m\alpha)$

$$= m\alpha P(\alpha < S_m \leq m\alpha) \quad \text{(by exchangeability)}$$

$$= m\alpha \frac{(k - 1)!}{(m - 1)! (k - 1 - m)!} \int_{\alpha}^{m\alpha} x^{m-1}(1 - x)^{k-1-m} dx$$

$$= O\left(\frac{\alpha^2}{(\log 1/\alpha)^m}\right) \quad \text{uniform in } k .$$

Hence

(19) $\qquad\qquad\qquad \beta_k = O(\alpha) \quad \text{uniform in } k ,$

and so using (17) and (19) we obtain

$$\eta \leq \textstyle\sum_{|k-n|<Dn^{\frac{1}{2}}} \beta_k + 2Dn^{\frac{1}{2}} \sup_{|k-n|\leq Dn^{\frac{1}{2}}} |\Lambda(k) - \tilde{\Lambda}(k)| = o(1) .$$

This establishes (16) and completes the proof of Theorem 1.

REMARK. Flatto's method for proving Theorem 1 was to estimate $P(\bigcup_{i=m}^{n+m-1}(S_i > \alpha))$. Our use of Corollary 1 allows us to avoid many of the analytical difficulties which he encountered.

3. Proof of Theorem 2. Define

$$\zeta_{ij} = 1 \qquad \text{if} \quad \rho(Z_i, Z_j) \leqq a$$
$$= 0 \qquad \text{otherwise} \qquad\qquad 1 \leqq i < j \leqq n \,,$$

and

$$X_j = \sum_{i<j} \zeta_{ij} \qquad\qquad\qquad 2 \leqq j \leqq n \,.$$

Note that

$$Z = \sum_{j=2}^{n} X_j \,.$$

Since $E(\zeta_{ij}) = \sum_{i=1}^{N} b(i)\pi_i$, for all i, j,

(20)
$$E(Z) = [\sum_{j=2}^{n} (j - 1)] \sum_{i=1}^{N} b(i)\pi(i) \,.$$

It follows then from (7) that $E(Z) \to \lambda$. Let $A_j = \{X_j = 1\}$, $\tilde{N} = \sum_{j=2}^{n} I_{(X_j=1)}$, and $p_j = P(A_j \mid Z_1, Z_2, \cdots, Z_{j-1}), j \geqq 2$.

Using the independence of the $\{Z_i\}$, it is not difficult to show that

(21)
$$P(X_j = 1) = E(P(X_j = 1 \mid Z_j))$$
$$= (j - 1) \sum_{i=1}^{N} b(i)\pi_i(1 - \pi_i)^{j-2} \,.$$

Thus

(22)
$$E(|Z - \tilde{N}|) \leqq \sum_{j=2}^{n} E(|X_j - I_{(X_j=1)}|)$$
$$= \sum_{j=2}^{n} (j - 1)[\sum_{i=1}^{N} b(i)\pi_i(1 - (1 - \pi_i)^{j-2})] \,.$$

The last equality follows from (21) and the fact that $X_j \geqq I_{(X_j=1)}$ with probability 1. Thus

(23)
$$E(|Z - \tilde{N}|) \leqq \{\sum_{j=2}^{n} (j - 1)(j - 2)\}\{\sum_{i=1}^{N} b(i)\pi_i^2\}$$
$$= o(1) \,.$$

The last equality follows from (6) and the observation that $\sum_{j=2}^{n} (j - 1)(j - 2) = O(n^3)$. To complete the proof of the theorem, we therefore need only show

(24)
$$d(\tilde{N}, N(\lambda)) = o(1) \,.$$

To do this we use Corollary 1, and so we need to verify its conditions. (2) follows from (23) and the already noted fact that $E(Z) \to \lambda$. We next verify (3). Adding and subtracting the appropriate quantities leads to:

$$E(|\sum_{j=2}^{n} (p_j - E(p_j))|) \leqq E(|\sum_{j=2}^{n} p_j - \sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1})|)$$
$$+ E(|\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \sum_{j=2}^{n} E(X_j)|)$$
$$+ E(|\sum_{j=2}^{n} E(X_j) - \sum_{j=2}^{n} E(p_j)|)$$
$$= I_1 + I_2 + I_3 \,.$$

Since for all i, $p_i \leqq E(X_i \mid \mathscr{F}_{i-1})$ with probability 1,

$$I_1 = \sum_{j=2}^{n} E(X_j) - \sum_{j=2}^{n} E(p_j) = I_3 \,.$$

Thus to establish (3) it suffices to prove $I_2 = o(1)$ and $I_3 = o(1)$. It follows from (22) and (7) that $I_3 = o(1)$. To show $I_2 = o(1)$, we argue as follows. In view of (21) it is enough to prove

$$E(|\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k|) = o(1).$$

However,

$$\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k = \sum_{j=1}^{n-1} (n-j)(\pi_{Z_j} - E(\pi_{Z_j})).$$

Squaring, taking expectations and using independence yields

$$E\{(\sum_{j=2}^{n} E(X_j \mid \mathscr{F}_{j-1}) - \binom{n}{2} \sum_{k=1}^{N} b(k)\pi_k)^2\} \leqq \{\sum_{j=1}^{n-1} (n-j)^2\} E(\pi_{Z_1}^2)$$
$$\leqq Cn^3 \sum_{k=1}^{N} b(k)\pi_k^2$$
$$= o(1).$$

Thus $I_3 = o(1)$ and the proof of (3) is complete.

Finally we need to establish (4). Observe

$$p_j \leqq E(X_j \mid Z_1, \cdots, Z_{j-1}) = \sum_{i=1}^{j-1} \pi_{Z_i}.$$

Thus

$$E(\sum_{j=2}^{n} p_j^2) \leqq E(\sum_{j=2}^{n} (\sum_{i=1}^{j-1} \pi_{Z_i})^2)$$
$$= \sum_{j=2}^{n} E(\sum_{i=1}^{j-1} \pi_{Z_i}^2 + \sum_{\substack{k=1 \\ i \neq k}}^{j-1} \sum_{i=1}^{j-1} \pi_{Z_i}\pi_{Z_k})$$
$$\leqq \sum_{j=2}^{n} ((j-1) + 2\binom{j-1}{2})(\sum_{i=1}^{N} b(i)\pi_i^2)$$
$$\leqq Cn^3 \sum_{i=1}^{N} b(i)\pi_i^2$$
$$= o(1).$$

This proves (4). Hence we can apply Corollary 1 and conclude that (24) is true, completing the proof of the theorem.

REMARK. Theorem 2 is slightly stronger than Sevast'yanov's result since he needed to assume that $\max_{1 \leqq k \leqq N} \pi_k = o(1/n)$.

REFERENCES

[1] FLATTO, L. (1973). A limit theorem for random coverings of a circle. *Israel J. Math.* **15** 167–184.
[2] FREEDMAN, DAVID (1974). The Poisson approximation for dependent events. *Ann. Probability* **2** 256–269.
[3] KARLIN, S. (1969). *A First Course in Stochastic Processes.* Academic Press, New York.
[4] SEVAST'YANOV, B. A. (1972). Poisson limit law for a scheme of sums of dependent random variables. *Theor. Probability Appl.* **17** 695–699.

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