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AN INEQUALITY FOR SUMS OF INDEPENDENT RANDOM VARIABLES INDEXED BY FINITE DIMENSIONAL FILTERING SETS AND ITS APPLICATIONS TO THE CONVERGENCE OF SERIES[1]

BY JEAN-PIERRE GABRIEL

Courant Institute of Mathematical Sciences

R. Pyke raised the question of the convergence of series indexed by filtering sets. This paper contains a generalization of an inequality of Marcinkiewicz-Zygmund for a certain class of filtering sets, which gives rise to the theory of series for this type of set.

1. Notation. We first recall some definitions. A set I with a partial order $\leqq$ is filtering to the right if for each α, β in I, there exists γ in I such that $\alpha \leqq \gamma$ and $\beta \leqq \gamma$; γ is called an upper bound of α and β. The notation $\alpha \leqq \beta$ (respectively $\alpha \geqq \beta$) means that α is less than or equal to β (respectively greater than or equal to β). If $\alpha \leqq \beta$ (resp. $\alpha \geqq \beta$) and $\alpha \neq \beta$, then we write $\alpha \ll \beta$ (resp. $\alpha \gg \beta$). Let $(a_\alpha)_{\alpha \in I}$ be a family of real numbers indexed by a set I filtering to the right. The limit superior and inferior of $(a_\alpha)_{\alpha \in I}$ are defined in the following way ($\pm\infty$ included):

$$\lim_{\alpha \to} \sup a_\alpha = \inf_{\alpha \in I} \sup_{\beta \geqq \alpha} a_\beta\,, \qquad \lim_{\alpha \to} \inf a_\alpha = \sup_{\alpha \in I} \inf_{\beta \geqq \alpha} a_\alpha\,.$$

If these two numbers are equal and finite then we say that $(a_\alpha)_{\alpha \in I}$, or more simply, a_α converges. The number

$$\lim_{\alpha \to} a_\alpha = \lim_{\alpha \to} \sup a_\alpha = \lim_{\alpha \to} \inf a_\alpha$$

is called the limit of a_α. In a set I filtering to the right, the symbol I_α designates the subset $I_\alpha = \{\beta \in I \mid \beta \leqq \alpha\}$, and $|\alpha|$ is the cardinality of I_α. A set I filtering to the right is said to be locally finite if $|\alpha|$ is finite for each α in I. For locally finite sets, one can introduce the notions of convergence of series and of products. Let $(a_\alpha)_{\alpha \in I}$ be a family of real numbers indexed by such a set. We say that the series $\sum_{\alpha \in I} a_\alpha$ converges if $S_\alpha = \sum_{\beta \in I_\alpha} a_\beta$ (partial sums) converges. In this case we write $S = \sum_{\alpha \in I} a_\alpha$, where $S = \lim_{\alpha \to} S_\alpha$. In the same way, we say that the product $\prod_{\alpha \in I} a_\alpha$ converges, if there exists γ in I, such that $a_\alpha \neq 0$ for each α belonging to $I_\gamma^{\,c}$, and if $\prod_{\beta \in I_\alpha \setminus I_\gamma} a_\beta$ converges to a nonzero real number. In this case we write $P = \prod_{\alpha \in I} a_\alpha$, where $P = \prod_{\beta \in I_\gamma} a_\beta \cdot \lim_{\alpha \to} \prod_{\beta \in I_\alpha \setminus I_\gamma} a_\beta$. The number P does not depend on γ.

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We now give some examples of filtering sets we will use in the following.

EXAMPLES.

(1) The symbol K_d, for each d in the set of positive integers N, designates the set of d-tuples of positive integers with the partial order induced by the coordinates. This relation is defined as follows:

$$\alpha = (r_1, r_2, \cdots, r_d) \lll \beta = (s_1, s_2, \cdots, s_d) \quad \text{iff} \quad r_1 \leqq s_1, r_2 \leqq s_2, \cdots, r_d \leqq s_d .$$

(2) The symbol K_∞ designates the set of sequences of positive integers with only finitely many elements different from 1, and partially ordered with the relation induced by the coordinates.

(3) $\tilde{Z}_2$ designates the set of pairs of integers (m, n), with the partial order induced by the coordinates, and such that $m + n \geqq 0$.

All these sets are countable, filtering to the right and locally finite.

2. Partially ordered sets and filtering sets. This section is concerned with a tentative classification of the filtering sets which could give rise to a theory of the summation of series. R. Smythe in [12] introduced the notion of local lattice in connection with the law of large numbers. We now define a dimension for filtering sets which are very close to local lattices.

LEMMA 1. *Every partially ordered set containing $d \geqq 4$ elements is isomorphic to a subset of K_{d-2}.*

PROOF. It is clear that every 4-element partially ordered set is isomorphic to a subset of K_2. Let us now suppose that the proposition is true for d, and let us consider a partially ordered set $E = \{a_1, a_2, \cdots, a_d, a_{d+1}\}$. This set contains a maximal element which we suppose to be a_{d+1}. According to our hypothesis, $F = \{a_1, a_2, \cdots, a_d\}$ is isomorphic to a subset of K_{d-2}. Let $(r_1^i, r_2^i, \cdots, r_{d-2}^i)$, $i = 1, 2, \cdots, d$, be the image of a_i through this isomorphism, and let us write $F_1 = \{a_i \in F \mid a_i \lll a_{d+1}\}$, and $F_2 = F \backslash F_1$. The mapping for which a_i is taken into $(r_1^i, r_2^i, \cdots, r_{d-2}^i, 1)$ or $(r_1^i, r_2^i, \cdots, r_{d-2}^i, 3)$ according as a_i is in F_1 or in F_2, and a_{d+1} into $(\max_{1\leqq i\leqq d} r_1^i, \max_{1\leqq i\leqq d} r_2^i, \cdots, \max_{1\leqq i\leqq d} r_{d-2}^i, 2)$ is an isomorphism from E to a subset of K_{d-1}.

To each α in a set I, filtering to the right, locally finite, we can now associate the number

$$r(\alpha) = \inf\{d \in N \mid I_\alpha \text{ is isomorphic to a subset of } K_d\} .$$

It is clear that $\alpha_1 \lll \alpha_2$ implies $r(\alpha_1) \leqq r(\alpha_2)$. We call the *dimension of I* the number

$$\dim I = \sup_{\alpha \in I} r(\alpha) ,$$

and we say that I has infinite dimension if $\dim I = \infty$. It is obvious that $\dim K_d = d$.

An increasing sequence $(\alpha_n)_{n\in N}$ in a partially ordered set I (i.e., $\alpha_n \lll \alpha_{n+1}$ for each n in N) is called a *generating sequence* (of I) if $I = \bigcup_{n=1}^\infty I_{\alpha_n}$.

PROPOSITION 2. *In a partially ordered countable set I, the two following propositions are equivalent*:

(a) *I is filtering to the right*;
(b) *I contains a generating sequence.*

PROOF. The nonobvious implication is (a) $\Rightarrow$ (b). Let $I = \{\beta_1, \beta_2, \cdots, \beta_n, \cdots\}$ be an enumeration of I, and let us define $\alpha_1 = \beta_1$. The set $\{\alpha_1, \beta_2\}$ has an upper bound α_2 in I; the set $\{\alpha_2, \beta_3\}$ has an upper bound α_3 in I; by indefinitely following this procedure, a sequence $(\alpha_n)_{n \in N}$ can be obtained and it is easy to see that it is a generating sequence.

We note that a set, filtering to the right and locally finite, contains a generating sequence iff it is countable. One might think that such a set is always countable, but this is not so. Let us choose an arbitrary noncountable set, and let I be the family of its finite subsets, ordered by inclusion. The set I is filtering to the right, locally finite but noncountable. This example shows that a noncountable filtering set does not always contain a generating sequence. The dimension of a set is completely determined by a generating sequence $(\alpha_n)_{n \in N}$ because of the fact that $\dim I = \lim_{n \to \infty} r(\alpha_n)$, the limit being independent of the particular sequence.

The purpose of the next theorem is to give a geometric characterization of the sets which are filtering to the right, locally finite and countable. We first give some lemmas.

LEMMA 3. *Let T be a totally ordered set and $(A_t)_{t \in T}$, $(B_t)_{t \in T}$ two increasing families of sets. Let us suppose that $A = \bigcup_{t \in T} A_t$ and $B = \bigcup_{t \in T} B_t$ are both ordered and that for each t in T, A_t and B_t are ordered by the respective relative order of A and B, and are isomorphic. If the following condition is satisfied then A and B are isomorphic.*

If ϕ_t designates the isomorphism between A_t and B_t, then for each s in T with $s \geqq t$, the restriction of ϕ_s to A_t coincides with ϕ_t.

PROOF. Let ϕ be the application from A into B defined by $\phi(\alpha) = \phi_{t(\alpha)}(\alpha)$, where $t(\alpha)$ is chosen in such a way that α is in $A_{t(\alpha)}$. According to our hypothesis, this application is well defined and is unique. It is easy to show that ϕ is an isomorphism.

LEMMA 4. *Let E_1 and E_2 be two finite subsets of a partially ordered set, both containing an upper bound denoted respectively by α_1 and α_2, and such that $\alpha_1 \ll \alpha_2$, $\{\alpha \in E_2 \mid \alpha \leqq \alpha_1\} = E_1$. If ϕ_1 is an isomorphism between E_1 and $A_1 \subset K_\infty$, then there exists a subset A_2 of K_∞ and an isomorphism ϕ_2 between E_2 and A_2, such that the restriction of ϕ_2 to E_1 coincides with ϕ_1.*

PROOF. The proof will be done with an induction on the number of elements of $E_2 \backslash E_1$. The lemma is certainly true when card $(E_2 \backslash E_1) = 1$. Let us now suppose that it is true for card $(E_2 \backslash E_1) = n$, and let us choose a maximal element δ in the set $E_2 \backslash (E_1 \cup \{\alpha_2\})$. According to the hypothesis of induction there exists an isomorphism $\tilde{\phi}_2$ between $E_2\backslash\{\delta\}$ and $A_2 \subset K_\infty$, such that the restriction to E_1 is identical to ϕ_1. Let us denote by $\beta_1, \beta_2, \cdots, \beta_p$ the elements of E_2. The set $E_2\backslash\{\delta\}$ being finite, there exists an integer m such that $\tilde{\phi}_2(\beta_i) = (r_1{}^i, r_2{}^i, \cdots, r_m{}^i, 1, 1, \cdots)$, $i = 1, 2, \cdots, p-1$. Let us define ϕ_2 in the following way:

$$\begin{aligned}\phi_2(\beta) &= (r_1{}^i, r_2{}^i, \cdots, r_m{}^i, 1, 1, \cdots) && \text{if} \quad \beta = \beta_i \in E_1 \cup \{\alpha \in E_2 \mid \alpha \ll \delta\}\,; \\ &= (r_1{}^i, r_2{}^i, \cdots, r_m{}^i, 3, 1, 1, \cdots) && \text{if} \quad \beta = \beta_i \in E_2\backslash(E_1 \cup \{\alpha \in E_2 \mid \alpha \lneqq \delta\})\,; \\ &= (\max_{\{i \mid \beta_i \in \{\alpha \in E_2 \mid \alpha \ll \delta\}\}} r_1{}^i, \cdots, \max_{\{i \mid \beta_i \in \{\alpha \in E_2 \mid \alpha \ll \delta\}\}} r_m{}^i, 2, 1, 1, \cdots) \\ & && \text{if} \quad \beta = \delta\,.\end{aligned}$$

It is easy to see that ϕ_2 is an isomorphism between E_2 and a subset of K_∞, such that the restriction to E_1 is ϕ_1.

THEOREM 5. *Every set I, filtering to the right, locally finite and countable is isomorphic to a subset of K_∞.*

PROOF. Let $(\alpha_n)_{n \in N}$ be a generating sequence of I. According to Lemma 1, there exists an isomorphism ϕ_1 between I_{α_1} and $A_1 \subset K_\infty$. The sets I_{α_1} and I_{α_2} satisfy the conditions given in Lemma 4, and therefore there exists an isomorphism ϕ_2 between I_{α_2} and $A_2 \subset K_\infty$, whose restriction to I_{α_1} is identical to ϕ_1. By indefinitely following this procedure we get two increasing families of sets, $(I_{\alpha_n})_{n \in N}$ and $(A_n)_{n \in N}$, which verify the conditions of Lemma 3. The latter assures that $I = \bigcup_{n=1}^\infty I_{\alpha_n}$ is isomorphic to $\bigcup_{n=1}^\infty A_n \subset K_\infty$.

3. Sums of independent variables. In the following, the random variables will always be real and are supposed to be defined on a probability space $(\Omega, \mathscr{F}, P)$. We will say that a set is of *finite type* if it is filtering to the right, locally finite, countable and its dimension is finite. The set $\tilde{Z}_2$ given in the example (3) is of finite type but is not a local lattice. In this section, I will always denote a set of finite type. Let $(Y_\alpha)_{\alpha \in I}$ be a family of independent random variables. Let us denote by $S_\alpha = \sum_{\beta \in I_\alpha} Y_\beta$ the family of partial sums associated with the series $\sum_{\alpha \in I} Y_\alpha$. The following theorem generalizes an inequality of Marcinkiewicz–Zygmund [6], [8]. The method of the proof is inspired by [6], [9].

THEOREM 6. *Let $(Y_\alpha)_{\alpha \in I}$ be a family of independent and integrable random variables, centered on their means and indexed by a set I of finite type. For each $p \geqq 1$ we have*

$$||\sup_{\alpha \in I} |S_\alpha|||_p \leqq A_{d,p} \sup_{\alpha \in I} ||S_\alpha||_p\,,$$

where $d = \dim I$ and $A_{d,p}$ is a number depending on d and p only.

PROOF. Let us first suppose that the inequality is true for random variables indexed by K_d, and let us consider a set I with $\dim I = d$ and a generating sequence of I denoted by $(\alpha_n)_{n \in N}$. For each n in N, the set I_{α_n} is isomorphic to a subset of K_d. By extending (if necessary) the family $(Y_\alpha)_{\alpha \in I_{\alpha_n}}$ with zero random variables, one can obtain a family indexed by a subset I_α of K_d, and then for each $p \geqq 1$ and each n in N, we have

$$||\sup_{\alpha \in I_{\alpha_n}} |S_\alpha|||_p \leqq A_{d,p} \sup_{\alpha \in I} ||S_\alpha||_p .$$

Using the monotone convergence theorem, we get

$$\begin{aligned}\lim_{n\to\infty} ||\sup_{\alpha \in I_{\alpha_n}} |S_\alpha|||_p &= ||\sup_{\alpha \in I} |S_\alpha|||_p \\ &\leqq A_{d,p} \sup_{\alpha \in I} ||S_\alpha||_p .\end{aligned}$$

Let us prove now the inequality for the case $I = K_d$. If the random variables are symmetric and indexed by K_d, then the following inequality holds [9]:

$$P\{\sup_{\beta \leqq \alpha} S_\beta \geqq \lambda\} \leqq 2^d P\{S_\alpha \geqq \lambda\} ,$$

for every $\lambda > 0$ and for every α in K_d. From this inequality one can prove that:

$$||\sup_{\beta \leqq \alpha} |S_\beta|||_p \leqq (2^{d+1})^{1/p} ||S_\alpha||_p$$

is verified for every α in K_d and every $p \geqq 1$ [5]. The general case is obtained by the standard procedure of "desymmetrization" [6]. One finally gets:

$$||\sup_{\beta \leqq \alpha} |S_\beta|||_p \leqq (2^{d+p} + 2^{p-1})^{1/p} ||S_\alpha||_p ,$$

which is true for every α in K_d and every $p \geqq 1$. The proof is now complete if we define $A_{d,p} = (2^{p+d} + 2^{p-1})^{1/p}$.

COROLLARY 7. *The partial sums $(S_\alpha)_{\alpha \in I}$ associated with a family of independent centered random variables indexed by a set of finite type converge almost everywhere if* $\sup_{\alpha \in I} ||S_\alpha||_1 < \infty$.

PROOF. The family $(S_\alpha)_{\alpha \in I}$ is a uniformly integrable martingale and the inequality of Theorem 6 implies its almost everywhere convergence [2], [11].

The central point for the almost eveywhere convergence of sums of independent random variables is the theorem which assures that quadratic mean convergence implies almost everywhere convergence. The latter is an easy consequence of Corollary 7 for the case of finite type index sets. If this theorem is true for a given filtering set, then all the theorems of the following list are also true. The most general index sets for which we can prove this theorem are the sets of finite type. We do not know if it is only a consequence of the fact that the preceding inequality is related to the finite dimensionality of the index set or if there are deeper reasons. We are not giving the proofs of the following theorems, because they are quite analogous to those given for the case $I = N$ [3]. The three series theorem is the only exception and we will do it later.

Let $(Y_\alpha)_{\alpha \in I}$ be a family of independent random variables indexed by a set I of finite type, and ϕ_α be the characterisic function of Y_α.

(a) If $(Y_\alpha)_{\alpha \in I}$ is uniformly bounded, then the almost everywhere convergence of $\sum_{\alpha \in I} Y_\alpha$ implies its quadratic mean convergence.

(b) $\sum_{\alpha \in I} Y_\alpha$ converges with probability 1 when centered if and only if $\prod_{\alpha \in I} |\phi_\alpha(t)|$ converges for each t in a set of positive measure.

(c) $\sum_{\alpha\in I} Y_\alpha$ converges almost everywhere if and only if $\prod_{\alpha\in I}\phi_\alpha(t)$ converges for each t in a set of positive measure.

(d) $\sum_{\alpha\in I} Y_\alpha$ converges almost everywhere if and only if $\prod_{\alpha\in I}\phi_\alpha(t)$ converges in a neighborhood of $t = 0$ to a function which is continuous at $t = 0$.

(e) The convergence of $\sum_{\alpha\in I} Y_\alpha$, almost everywhere, in probability and in distribution are equivalent.

Let us consider now the three series theorem. Let $(Y_{ij};\, i \geqq 3, j \geqq 1)$ be a family of independent random variables such that $\sum_{i\geqq 3}\sum_{j\geqq 1} Y_{ij}$ converges almost everywhere, and let us define $Y_{1j} = (-1)^j$, $Y_{2j} = (-1)^{j+1}$ for $j \geqq 1$. The family $(Y_{ij})_{(i,j)\in K_2}$ is still independent and $\sum_{(i,j)\in K_2} Y_{ij}$ converges almost everywhere. But $\sum_{(i,j)\in K_2} P\{|Y_{ij}| > \frac{1}{2}\} = \infty$ which is in contradiction with one of the conclusions of the classical three series theorem. The latter can be modified in the following way:

THEOREM 8 (three series theorem). *Let $(Y_\alpha)_{\alpha\in I}$ be a family of independent variables indexed by a finite type set I, and let us define*

$$\begin{aligned} \tilde{Y}_\alpha &= Y_\alpha \qquad \text{if} \quad |Y_\alpha - m_\alpha| \leqq C\,, \\ &= m_\alpha \qquad \text{if} \quad |Y_\alpha - m_\alpha| > C\,, \end{aligned}$$

where C is a positive number and m_α a median of Y_α. The series $\sum_{\alpha\in I} Y_\alpha$ converges almost everywhere iff *the three series $\sum_{\alpha\in I} P\{Y_\alpha \neq \tilde{Y}_\alpha\}$, $\sum_{\alpha\in I} E(\tilde{Y}_\alpha)$, $\sum_{\alpha\in I} \operatorname{Var}(\tilde{Y}_\alpha)$ converge.*

PROOF. The proof of the sufficiency is easy. To prove the necessity, we will use the inequalities of Doob–Wintner [3, page 41]. The almost everywhere convergence of $\sum_{\alpha\in I} Y_\alpha$ implies the existence of a set $A \subset [0, a]$, with positive measure ρ, such that

$$\begin{aligned} \textstyle\sum_{\alpha\in I} P\{Y_\alpha \neq \tilde{Y}_\alpha\} &\leqq -4L_1(C, \rho, a) \textstyle\int_A \log \prod_{\alpha\in I} |\phi_\alpha(t)|\, dt < \infty\,, \\ \textstyle\sum_{\alpha\in I} \operatorname{Var}(\tilde{Y}_\alpha) &\leqq -2L_4(C, \rho, a) \textstyle\int_A \log \prod_{\alpha\in I} |\phi_\alpha(t)|\, dt < \infty\,, \end{aligned}$$

where ϕ_α is the characteristic function of Y_α and $L_1(C, \rho, a)$, $L_4(C, \rho, a)$ are the constants involved in the Doob–Wintner inequalities. From this follows the almost everywhere convergence of $\sum_{\alpha\in I} (Y_\alpha - E(\tilde{Y}_\alpha))$, and consequently of $\sum_{\alpha\in I} E(\tilde{Y}_\alpha)$.

We will now show that the L_1-bounded condition implies more than the almost everywhere convergence of the series. Let $(a_\alpha)_{\alpha\in I}$ be a family of real numbers and $\gamma \to \alpha_\gamma$ an I to I bijection. We will call $(a_{\alpha_\gamma})_{\gamma\in I}$ a rearrangement of $(a_\alpha)_{\alpha\in I}$. Let $(Y_\alpha)_{\alpha\in I}$ be a family of random variables. We will say that the series $\sum_{\alpha\in I} Y_\alpha$ converges unconditionally almost everywhere if for each rearrangement $(Y_{\alpha_\gamma})_{\gamma\in I}$ of the family $(Y_\alpha)_{\alpha\in I}$, the series $\sum_{\gamma\in I} Y_{\alpha_\gamma}$ converges almost everywhere.

THEOREM 9. *Let $(Y_\alpha)_{\alpha\in I}$ be a family of independent centered random variables indexed by a finite type set I. If $\sup_{\alpha\in I} \|\sum_{\beta\in I_\alpha} Y_\beta\|_1 < \infty$, then the series $\sum_{\alpha\in I} Y_\alpha$ converges unconditionally almost everywhere.*

PROOF. According to Corollary 7, it is enough to prove that

$$\sup_{\alpha \in I} \|\sum_{\beta \in I_\alpha} Y_\beta\|_1 = \sup_{\alpha \in I} \|\sum_{\gamma \in I_\alpha} Y_{\alpha_\gamma}\|_1 ,$$

for each rearrangement $(Y_{\alpha_\gamma})_{\gamma \in I}$ of $(Y_\alpha)_{\alpha \in I}$. It is easy to do it by using a generating sequence and the inequality $E|X| \leqq E|X + Y|$, which is valid for each pair of independent and centered random variables [7, page 263].

In fact, it is possible to connect this kind of convergence with the theory of Burkholder's transforms [5].

THEOREM 10. *Let $(Y_\alpha)_{\alpha \in I}$ be a family of independent random variables indexed by a finite type set I. Let us define*

$$\begin{aligned} \tilde{Y}_\alpha &= Y_\alpha \qquad \text{if } |Y_\alpha| \leqq C, \\ &= 0 \qquad \text{if } |Y_\alpha| > C, \end{aligned}$$

where C is a positive number. The series $\sum_{\alpha \in I} Y_\alpha$ converges unconditionally almost everywhere iff *the three series $\sum_{\alpha \in I} P\{Y_\alpha \neq \tilde{Y}_\alpha\}$, $\sum_{\alpha \in I} |E(\tilde{Y}_\alpha)|$, $\sum_{\alpha \in I} \operatorname{Var}(\tilde{Y}_\alpha)$ converge.*

PROOF. According to the three series theorem, $\sum_{\alpha \in I} Y_\alpha$ converges unconditionally almost everywhere iff the three series $\sum_{\alpha \in I} P\{Y_\alpha \neq \tilde{Y}_\alpha\}$, $\sum_{\alpha \in I} |E(\tilde{Y}_\alpha)|$, $\sum_{\alpha \in I} \operatorname{Var}(\tilde{Y}_\alpha)$ converge. This is equivalent to saying that $\sum_{n=1}^\infty P\{Y_{\alpha_n} \neq \tilde{Y}_{\alpha_n}\}$, $\sum_{n=1}^\infty |E(\tilde{Y}_{\alpha_n})|$, $\sum_{n=1}^\infty \operatorname{Var}(\tilde{Y}_{\alpha_n})$ converge, where $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots\}$ is an arbitrary enumeration of I. But now the series are ordered by N and we can use the classical three series theorem. The proof is now complete [13].

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COURANT INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NEW YORK 10012