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QUADRATIC VARIATION OF FUNCTIONALS OF BROWNIAN MOTION

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The quadratic variation of functionals $F(t)$ of n-dimensional Brownian motion is investigated. Let $\Pi_n = \{t_1^n, t_2^n, \cdots, t_{l(n)}^n\}$ with $a = t_1^n < t_2^n < \cdots < t_{l(n)}^n = b$ be a family of partitions of the interval $[a, b]$. The limiting behavior of $Q^2(F, \Pi_n) = \sum_{k=1}^{l(n)-1} (F(t_{k+1}^n) - F(t_k^n))^2$ as $n \to \infty$, assuming $||\Pi_n|| \to 0$, is studied. And the existence of this limit is obtained for a fairly general class of functionals of Brownian motion.

0. Introduction. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $F = \{F(t), a \leqq t \leqq b\}$ be a stochastic process defined on Ω. Let Π_n be a sequence of partitions of $[a, b]$, given by $a = t_1^n < t_2^n < \cdots < t_{l(n)}^n = b$. Let $Q^2(F, \Pi_n) = \sum_{i=1}^{l(n)-1} (F(t_{i+1}^n) - F(t_i^n))^2$. If the limit of $Q^2(F, \Pi_n)$ exists in some sense, when $||\Pi_n|| = \max_i (t_{i+1}^n - t_i^n) \to 0$, it is called the quadratic variation of F.

The quadratic variation of a process was first studied by P. Lévy in the case of Brownian motion, and later on was pursued in the case of Gaussian processes and martingales by many other people. In [2], Brosamler studied the quadratic variation of $F(t) = p(X(t))$, where p is a Green potential on a Green domain $D \subseteqq R^n$ with $n \geqq 2$ and $X(t)$ is a Brownian motion on R^n stopped at the Martin boundary of D.

We study the quadratic variation of $F(t) = f(X(t))$ and $F(t) = f(t, X(t))$ for a class of functions f. Our results are stated in Theorem 1.3, 2.1 and 2.3. We also derive a result on the quadratic variation of potentials $p(X(t))$, originally due to Brosamler [2], as an application of our theorems. Our proof of Theorem 1.3 relies on Corollary 1.1, which is a generalization of one identity given in Berman [1].

Tanaka [6] proved that a continuous (homogeneous) additive functional $A(t)$ of one-dimensional Brownian motion $X(t)$ has the following representation:

$$A(t) = f(X(t)) - f(X(0)) + \int_0^t g(X(u))\, dX(u) ,$$

where f is a continuous function on R and $g \in L_2^{\text{loc}}(R)$. In Ventcel [7], a similar representation was obtained for continuous additive functionals of higher dimensional Brownian motion. We study the quadratic variation of a class of continuous additive functionals of Brownian motion and our conclusions are Theorems 1.4 and 2.2, which include Theorems 1.3 and 2.1 as special cases.

1. One-dimensional case. Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let $\{X(t), \mathscr{F}_t\}$, $0 \leqq t < \infty$, be a standard one-dimensional Brownian motion with

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$P(X(0)=0)=1$ and $(X(t)-X(s))$ independent of $\mathscr{F}_s$ whenever $s<t$. Let $\Pi_n=\{t_1^n, t_2^n, t_3^n, \cdots, t_{l(n)}^n\}$ be a family of partitions of interval $[a, b]$ where $a>0$, $a=t_1^n<t_2^n<\cdots<t_{l(n)}^n=b$ and $\lim_{n\to\infty}\|\Pi_n\|=0$ with $\|\Pi_n\|=\max_i(t_{i+1}^n-t_i^n)$. We put

$$\eta_i^n = X(t_{i+1}^n) - X(t_i^n)\,, \tag{1.1}$$

$$\tau_i^n = t_{i+1}^n - t_i^n\,. \tag{1.2}$$

Let $F(t, \omega)$ be a stochastic process adapted to $\mathscr{F}_t$ and defined on $[a, b]\times\Omega$. We let

$$B(F, \Pi_n) = \textstyle\sum_{k=1}^{l(n)-1} F(t_k^n)(\eta_k^n)^2\,, \tag{1.3}$$

$$R(F, \Pi_n) = \textstyle\sum_{k=1}^{l(n)-1} F(t_k^n)(\tau_k^n)\,. \tag{1.4}$$

When f is a real valued function defined on R,

$$B(f(X), \Pi_n) = \textstyle\sum_{k=1}^{l(n)-1} f(X(t_k^n))(\eta_k^n)^2\,, \tag{1.5}$$

$$R(f(X), \Pi_n) = \textstyle\sum_{k=1}^{l(n)-1} f(X(t_k^n))(\tau_k^n)\,. \tag{1.6}$$

We let

$$\Delta f_k^n = f(X(t_{k+1}^n)) - f(X(t_k^n))\,, \tag{1.7}$$

$$Q^2(f(X), \Pi_n) = \textstyle\sum_{k=1}^{l(n)-1} (\Delta f_k^n)^2\,, \tag{1.8}$$

$$Mf(x) = \sup_r \frac{1}{|r|}\int_x^{x+r} |f(y)|\,dy\,. \tag{1.9}$$

We use I_A to denote the indicator function of set A. We will use $Q^2(f(X))$ to denote $\lim_{n\to\infty} Q^2(f(X), \Pi_n)$, when the latter exists in a certain sense.

Since $f=g$ a.e. implies $B(f(X), \Pi_n)=B(g(X), \Pi_n)$, $R(f(X), \Pi_n)=R(g(X), \Pi_n)$ and $Q^2(f(X), \Pi_n)=Q^2(g(X), \Pi_n)$ a.s. and in $L_p(\Omega)$ (when one side is in $L_p(\Omega)$), we will replace a Lebesgue measurable function by its Borel measurable version in discussing these summations.

LEMMA 1.1 (Wong and Zakai). *Let $E\{(F(t))^2\}\leqq M$ for all $t\in[a, b]$, where M is a constant. Then*

$$\lim_{n\to\infty} E\{[B(F, \Pi_n) - R(F, \Pi_n)]^2\} = 0\,.$$

PROOF. From (1.3) and (1.4),

$$\begin{aligned} E\{[B(F, \Pi_n) - R(F, \Pi_n)]^2\} &= E\{[\textstyle\sum_{k=1}^{l(n)-1} F(t_k^n)((\eta_k^n)^2 - \tau_k^n)]^2\} \\ &= E[\textstyle\sum_{k=1}^{l(n)-1} (F(t_k^n))^2((\eta_k^n)^2 - \tau_k^n)^2] \\ &\quad + E[\textstyle\sum\sum_{k\neq j} F(t_k^n)F(t_j^n)((\eta_k^n)^2 - \tau_k^n)((\eta_j^n)^2 - \tau_j^n)]\,. \end{aligned}$$

Since $[(\eta_k^n)^2 - \tau_k^n]$ is independent of $\mathscr{F}_{t_k^n}$ the second summation in the right vanishes. Further,

$$\begin{aligned} &E[\textstyle\sum_{k=1}^{l(n)-1} (F(t_k^n))^2((\eta_k^n)^2 - \tau_k^n)^2] \\ &\quad = \textstyle\sum_{k=1}^{l(n)-1} E(F(t_k^n))^2 E((\eta_k^n)^2 - \tau_k^n)^2 \\ &\quad \leqq 2M \textstyle\sum_{k=1}^{l(n)-1} (\tau_k^n)^2 \leqq 2M(b-a)\|\Pi_n\| \to 0 \qquad \text{as} \quad n\to\infty\,. \end{aligned} \tag{1.10}$$

Hence $\lim_{n\to\infty} E\{[B(F, \Pi_n) - R(F, \Pi_n)]^2\} = 0$.

THEOREM 1.1. *Let $F(t, \omega)$, $t \in [a, b]$, be uniformly integrable. Then*

$$\lim_{n\to\infty} E|B(F, \Pi_n) - R(F, \Pi_n)| = 0 .$$

PROOF. Let

$$\begin{aligned} F_N(t, \omega) &= F(t, \omega) \qquad \text{if} \quad |F(t, \omega)| \leqq N \\ &= 0 \qquad \text{othewise.} \end{aligned}$$

Let $F^N(t, \omega) = F(t, \omega) - F_N(t, \omega)$.

$$\begin{aligned} E|B(F^N, \Pi_n) - R(F^N, \Pi_n)| &\leqq \sum_{k=1}^{l(n)-1} E|F^N(t_k^n)((\eta_k^n)^2 - \tau_k^n)| \\ &= \sum_{k=1}^{l(n)-1} E|F^N(t_k^n)|E|(\eta_k^n)^2 - \tau_k^n| \\ &\leqq 2(b - a) \sup_{1\leqq k\leqq l(n)-1} E|F^N(t_k^n)| . \end{aligned}$$

Given $\varepsilon > 0$, we choose N big enough to make

$$E|F^N(t)| < \varepsilon/4(b - a) \qquad \text{for all} \quad t \in [a, b] .$$

For this fixed N, we take $\delta < \varepsilon/4N(b - a)$. Then $||\Pi_n|| < \delta$, by (1.10), so

$$E|B(F_N, \Pi_n) - R(F_N, \Pi_n)| < 2N(b - a)\delta < \varepsilon/2 .$$

THEOREM 1.2 (Wong and Zakai). *Let $F(t)$ be continuous in $L_1(\Omega, P)$ for $t \in [a, b]$,* i.e., $\lim_{s\to t} E|F(t) - F(s)| = 0$ *for all $t \in [a, b]$. Then*

$$\lim_{n\to\infty} E|R(F, \Pi_n) - \int_a^b F(t)\, dt| = 0 .$$

PROOF. Define $F_{(n)}(t) = F(t_k^n)$ for $t_k^n \leqq t < t_{k+1}^n$. Then

$$\begin{aligned} E|R(F, \Pi_n) - \int_a^b F(t)\, dt| &= E|\int_a^b (F_{(n)}(t) - F(t))\, dt| \\ &\leqq \int_a^b E|F_{(n)}(t) - F(t)|\, dt \to 0 \qquad \text{as} \quad ||\Pi_n|| \to 0 . \end{aligned}$$

The following corollary is an extension of a lemma in Berman [1].

COROLLARY 1.1. *Let $f \in L_1(R)$; then the family $f(X(t))$, $a \leqq t \leqq b$, is uniformly integrable and is continuous in $L_1(\Omega, P)$. Furthermore,*

$$\lim_{n\to\infty} E|R(f(X), \Pi_n) - \int_a^b f(X(t))\, dt| = 0$$

and

$$\lim_{n\to\infty} E|B(f(X), \Pi_n) - \int_a^b f(X(t))\, dt| = 0 .$$

If $f \in L_1^{\text{loc}}(R)$, then instead of convergence in L_1, we have

$$\lim_{n\to\infty} R(f(X), \Pi_n) = \int_a^b f(X(t))\, dt \quad \textit{in probability}$$

and

$$\lim_{n\to\infty} B(f(X), \Pi_n) = \int_a^b f(X(t))\, dt \quad \textit{in probability} .$$

PROOF. Let $f \in L_1(R)$, then

$$\int_{|f(X(t))|>N} |f(X(t))|\, dP \leqq (2\pi a)^{-\frac{1}{2}} \int_{|f(y)|>N} |f(y)|\, dy \to 0$$

independent of t as $N \to \infty$.

For $f \in C_0^\infty$, $\lim_{t\to s} E|f(X(t)) - f(X(s))| = 0$ by the bounded convergence theorem. For general $f \in L_1(R)$, there exists a sequence of functions $f_n \in C_0^\infty$ such that $f_n \to f$ in $L_1(R)$. Hence

$$\begin{aligned} E|f(X(t)) - f(X(s))| &\leqq E|f(X(t)) - f_n(X(t))| + E|f_n(X(t)) - f_n(X(s))| \\ &\quad + E|f_n(X(s)) - f(X(s))| \\ &\leqq 2(2\pi a)^{-\frac{1}{2}}||f_n - f||_1 + E|f_n(X(t)) - f_n(X(s))| . \end{aligned}$$

Given $\varepsilon > 0$, we can find N such that

$$2(2\pi a)^{-\frac{1}{2}}||f_N - f||_1 < \varepsilon/2 .$$

Then the $L_1(\Omega)$ continuity of $f_N(X(t))$ will take care of the other part. So, for $f \in L_1(R)$ the family $f(X(t))$, $a \leqq t \leqq b$, is unifomly integrable and is continuous in $L_1(\Omega)$. The rest is easy.

For $f \in L_1^{\mathrm{loc}}(R)$, knowing $P\{|X(t)| > N$ for some $t \in [a, b]\} \to 0$ as $N \to \infty$, we can disregard a set of small measure and treat f as a L_1 function on a compact set.

THEOREM 1.3. *Let f be a locally absolutely continuous function*, i.e., *there exists a point c such that*

$$f(x) = f(c) + \int_c^x f'(u)\, du$$

for all x, and let $f' \in L_2(R)$. Then

$$\lim_{n\to\infty} E|Q^2(f(X), \Pi_n) - \int_a^b (f'(X(t)))^2\, dt| = 0 .$$

PROOF.

$$\begin{aligned} (\Delta f_k^n)^2 &= \left(\int_{X(t_k^n)}^{X(t_{k+1}^n)} f'(u)\, du\right)^2 \\ &= (\eta_k^n)^2 \left(\frac{1}{|\eta_k^n|} \int_{X(t_k^n)}^{X(t_{k+1}^n)} f'(u)\, du\right)^2 \qquad (1.11) \\ &\leqq (\eta_k^n)^2 (Mf'(X(t_k^n)))^2 . \end{aligned}$$

Let $A_{N,n,k} = \{|X(t_k^n)| > N\}$. Remember that $||Mf'||_2 \leqq C||f'||_2 < \infty$ (see [5]). Then, by (1.11),

$$\begin{aligned} E|\textstyle\sum_{k=1}^{l(n)-1} (\Delta f_k^n)^2 I_{A_{N,n,k}}| &\leqq E \textstyle\sum_{k=1}^{l(n)-1} (\eta_k^n)^2 (Mf')^2 (X(t_k^n)) I_{A_{N,n,k}} \\ &= \textstyle\sum_{k=1}^{l(n)-1} E(\eta_k^n)^2 E((Mf')^2(X(t_k^n)) I_{A_{N,n,k}}) \\ &\leqq (b-a) \sup_{t_k^n} E((Mf')^2(X(t_k^n)) I_{A_{N,n,k}}) \\ &\leqq \frac{b-a}{(2\pi a)^{\frac{1}{2}}} \int_{|y|>N} (Mf')^2(y)\, dy \to 0 \end{aligned}$$

independent of n as $N \to \infty$.

We can also show $E|\sum_{k=0}^{l(n)-1} (\Delta f_k^n)^2 I_{A_{N,n,k+1}}| \to 0$ is independent of n as $N \to \infty$.

Further, let $A_N = \{|X(t)| > N$, for some t in $[a, b]\}$. We know $P(A_N) \to 0$ as $N \to \infty$. And

$$E \int_a^b (f')^2(X(t))\, dt \leqq (2\pi a)^{-\frac{1}{2}} \int_a^b ||f'||_2\, dt < \infty .$$

Hence $E(I_{A_N} \int_a^b (f')^2(X(t))\, dt) \to 0$ as $N \to \infty$. Then

$$\lim_{N\to\infty} E|Q^2(f(X), \Pi_n) - Q^2((fI_{[-N,N]})(X), \Pi_n)| = 0 ,$$

and

$$\lim_{N\to\infty} E|\int_a^b (f')^2(X(u))\,du - \int_a^b (f'I_{[-N,N]})^2(X(u))\,du| = 0\,,$$

independent of n. Thus we can assume, without loss of generality, that f is a function of compact support. By using Ergorov's theorem we have $(f(y)-f(x))/(y-x) \to f'(x)$ uniformly as $y \to x$ except on a set B of small measure. Clearly, $f' \in L_2$ implies $m\{x\,|\,|f'(x)| > N\} \to 0$ as $N \to \infty$. We let $G = B^c \cap \{x\,|\,|f'(x)| < N_0\}$, where the measure of set B and number N_0 are properly chosen, to satisfy

(i) $\int_{G^c} |f'(x)|^2\,dx < \varepsilon$,
(ii) $\int_{G^c} |(Mf')(x)|^2\,dx < \varepsilon$.

We can find δ and δ' such that (iii) and (vi) hold.

(iii) $|y-x| < \delta$ and $x \in G$ implies

$$\left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < \frac{\varepsilon}{2(N_0+1)}\,.$$

This in turn implies

$$|(f(y)-f(x))^2 - (f'(x)(y-x))^2| < \varepsilon(y-x)^2\,.$$

(iv) $0 < t < \delta'$ implies

$$E[(X(t))^2 I_{[\delta,\infty)}(|X(t)|)] < \varepsilon t\,.$$

Clearly,

$$\begin{aligned} Q^2(f,\Pi_n) &= \textstyle\sum_{k=1}^{l(n)-1} I_G(X(t_k^n)) I_{[0,\delta)}(|\eta_k^n|)(\Delta f_k^n)^2 \\ &\quad + \textstyle\sum_{k=1}^{l(n)-1} I_G(X(t_k^n)) I_{[\delta,\infty)}(|\eta_k^n|)(\Delta f_k^n)^2 + \sum_{k=1}^{l(n)-1} I_{G^c}(X(t_k^n))(\Delta f_k^n)^2 \\ &= I(n) + II(n) + III(n)\,. \end{aligned}$$

$$\begin{aligned} E|II(n)| &\leqq E \textstyle\sum_{k=1}^{l(n)-1} I_{[\delta,\infty)}(|\eta_k^n|)(\Delta f_k^n)^2 \\ &\leqq E \textstyle\sum_{k=1}^{l(n)-1} I_{[\delta,\infty)}(|\eta_k^n|)(\eta_k^n)^2 (Mf')^2(X(t_k^n)) \quad \text{(by (1.11))} \\ &= \textstyle\sum_{k=1}^{l(n)-1} E[I_{[\delta,\infty)}(|\eta_k^n|)(\eta_k^n)^2] E(Mf')^2(X(t_k^n)) \\ &\leqq \frac{\varepsilon(b-a)}{(2\pi a)^{\frac{1}{2}}} ||Mf'||_2 \qquad \text{when } ||\Pi_n|| < \delta'\,. \end{aligned}$$

Similarly,

$$\begin{aligned} E|III(n)| &\leqq E \textstyle\sum_{k=1}^{l(n)-1} I_{G^c}(X(t_k^n))(\eta_k^n)^2 (Mf')^2(X(t_k^n)) \\ &\leqq \textstyle\sum_{k=1}^{l(n)-1} E(\eta_k^n)^2 E[I_{G^c}(X(t_k^n))(Mf')^2(X(t_k^n))] \\ &\leqq (b-a)\sup_{t\in[a,b]} E[I_{G^c}(Mf')^2](X(t)) \\ &\leqq \frac{b-a}{(2\pi a)^{\frac{1}{2}}}\,\varepsilon \quad \text{by (ii)}\,. \end{aligned}$$

To deal with $I(n)$, we define

$$I'(n) = \textstyle\sum_{k=1}^{l(n)-1} I_G(X(t_k^n)) I_{[0,\delta)}(|\eta_k^n|)(f'(X(t_k^n))\eta_k^n)^2$$
$$I''(n) = \textstyle\sum_{k=1}^{l(n)-1} I_G(X(t_k^n))(f'(X(t_k^n))\eta_k^n)^2\,.$$

Then

$$(1.12)\qquad E|I(n) - I'(n)| \leqq E \sum_{k=1}^{l(n)-1} I_G(X(t_k^n)) I_{[0,\delta)}(|\eta_k^n|)\varepsilon(\eta_k^n)^2 \quad \text{by (iii)}$$
$$< \varepsilon(b-a),$$

and

$$E|I'(n) - I''(n)| \leqq E \sum_{k=1}^{l(n)-1} I_{[\delta,\infty)}(|\eta_k^n|)(f'(X(t_k^n))\eta_k^n)^2$$
$$(1.13)\qquad = \sum_{k=1}^{l(n)-1} E[I_{[\delta,\infty)}(|\eta_k^n|)(\eta_k^n)^2] E(f'(X(t_k^n)))^2$$
$$< \frac{b-a}{(2\pi a)^{\frac{1}{2}}} ||f'||_2 \varepsilon \qquad \text{when} \quad ||\Pi_n|| < \delta' .$$

Further

$$E|I''(n) - B((f')^2, \Pi_n)| \leqq \sum_{k=1}^{l(n)-1} E I_{G^c}(X(t_k^n))(f'(X(t_k^n))\eta_k^n)^2$$
$$(1.14)\qquad = \sum_{k=1}^{l(n)-1} E[(f')^2 I_{G^c}](X(t_k^n)) E(\eta_k^n)^2$$
$$\leqq \frac{b-a}{(2\pi a)^{\frac{1}{2}}} ||(f')^2 I_{G^c}||_1 < \frac{b-a}{(2\pi a)^{\frac{1}{2}}} \varepsilon \quad \text{by (i)} .$$

Combining (1.12), (1.13) and (1.14), we conclude

$$\lim_{n\to\infty} E|Q^2(f, \Pi_n) - B((f')^2, \Pi_n)| = 0 .$$

Applying Corollary 1.1, we get

$$\lim_{n\to\infty} E|Q^2(f, \Pi_n) - \textstyle\int_a^b (f')^2(X(t))\, dt| = 0 .$$

COROLLARY 1.2. *Let f be locally absolutely continuous and $f' \in L_2^{\text{loc}}(R)$; then $Q^2(f(X)) = \int_a^b (f')^2(X(t))\, dt$ in probability.*

The proof is easy, hence omitted. We have the following theorem concerning the quadratic variation of a class of additive functionals. We put $Q^2(A) = \lim_{n\to\infty} \sum_{k=1}^{l(n)-1} (A(t_{k+1}^n) - A(t_k^n))^2$, when the latter exists in some sense.

THEOREM 1.4. *Let $A(t) = f(X(t)) - f(X(0)) + \int_0^t g(X(u))\, dX(u)$ with*

(i) *f locally absolutely continuous and $f' \in L_2(R)$ (or $f' \in L_2^{\text{loc}}(R)$),*

(ii) *$g \in L_2(R)$ (or $g \in L_2^{\text{loc}}(R)$).*

Then

$$Q^2(A) = \textstyle\int_a^b (f')^2(X(u))\, du + \int_a^b g^2(X(u))\, du + 2 \int_a^b (f'g)(X(u))\, du$$
$$\text{in} \quad L_1(\Omega, P) \quad (\text{or in probability}) .$$

PROOF. Let $G(t) = \int_0^t g(X(u))\, dX(u)$ and $\Delta g_k^n = \int_{t_k^n}^{t_{k+1}^n} g(X(u))\, dX(u)$. Then $Q^2(A, \Pi_n) = Q^2(f(X), \Pi_n) + 2 \sum_{k=1}^{l(n)-1} (\Delta f_k^n)(\Delta g_k^n) + Q^2(G, \Pi_n)$. By Theorem 1.3, $Q^2(f(X)) = \int_a^b (f')^2(X(u))\, du$ in $L_1(\Omega, P)$. From [9] $Q^2(G) = \int_a^b g^2(X(u))\, du$ in $L_2(\Omega, P)$. We only need to show

$$\lim_{n\to\infty} \sum_{k=1}^{l(n)-1} (\Delta f_k^n)(\Delta g_k^n) = \textstyle\int_a^b (f'g)(X(u))\, du \qquad \text{in} \quad L_1(\Omega, P) .$$

$$(1.15)\qquad E|\sum_{k=1}^{l(n)-1} (\Delta f_k^n)((\Delta g_k^n) - g(X(t_k^n))\eta_k^n)|$$
$$\leqq \sum_{k=1}^{l(n)-1} E|(\Delta f_k^n)(\Delta g_k^n - g(X(t_k^n))\eta_k^n)|$$
$$\leqq \sum_{k=1}^{l(n)-1} [E(\Delta f_k^n)^2 E(\Delta g_k^n - g(X(t_k^n))\eta_k^n)^2]^{\frac{1}{2}}$$
$$\leqq (\sum_{k=1}^{l(n)-1} E(\Delta f_k^n)^2)^{\frac{1}{2}} (\sum_{k=1}^{l(n)-1} E(\Delta g_k^n - g(X(t_k^n))\eta_k^n)^2)^{\frac{1}{2}} .$$

By Theorem 1.3, $\lim_{n\to\infty} \sum_{k=1}^{l(n)-1} E(\Delta f_k^n)^2 = E \int_a^b (f')^2(X(u))\,du$, hence $\sum_{k=1}^{l(n)-1} E(\Delta f_k^n)^2$ is bounded independent of n. If $g \in C_0^\infty$, then

$$\begin{aligned}\sum_{k=1}^{l(n)-1} & E(\Delta g_k^n - g(X(t_k^n))\eta_k^n)^2 \\ &= \sum_{k=1}^{l(n)-1} E \int_{t_k^n}^{t_{k+1}^n} (g(X(u)) - g(X(t_k^n)))^2\,du \\ &= \sum_{k=1}^{l(n)-1} \int_{t_k^n}^{t_{k+1}^n} E(g(X(u)) - g(X(t_k^n)))^2\,du \to 0 \qquad \text{as} \quad \|\Pi_n\| \to 0\,,\end{aligned}$$

by the bounded convergence theorem and the compactness of $[a, b]$. For general $g \in L_2(R)$, we use an approximation procedure similar to that given in Corollary 1.1 to conclude $\lim_{n\to\infty} \sum_{k=1}^{l(n)-1} E(\Delta g_k^n - g(X(t_k^n))\eta_k^n)^2 = 0$. Thus

$$\lim_{n\to\infty} E\left|\sum_{k=1}^{l(n)-1} (\Delta f_k^n)(\Delta g_k^n) - \sum_{k=1}^{l(n)-1} (\Delta f_k^n) g(X(t_k^n))\eta_k^n\right| = 0\,. \tag{1.16}$$

On the other hand

$$\begin{aligned}E&\left|\sum_{k=1}^{l(n)-1} (\Delta f_k^n g(X(t_k^n))\eta_k^n - f'(X(t_k^n))\eta_k^n g(X(t_k^n))\eta_k^n)\right| \\ &\leqq \sum_{k=1}^{l(n)-1} (E(\Delta f_k^n - f'(X(t_k^n))\eta_k^n)^2 E(g(X(t_k^n))\eta_k^n)^2)^{\frac{1}{2}} \\ &\leqq \Big(\sum_{k=1}^{l(n)-1} E(\Delta f_k^n - f'(X(t_k^n))\eta_k^n)^2\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{l(n)-1} E(g(X(t_k^n))\eta_k^n)^2\Big)^{\frac{1}{2}} \to 0\,.\end{aligned} \tag{1.17}$$

Since

$$\lim_{n\to\infty} \sum_{k=1}^{l(n)-1} E(g(X(t_k^n))\eta_k^n)^2 = \lim_{n\to\infty} \sum_{k=1}^{l(n)-1} E(g^2(X(t_k^n)))\tau_k^n = E \int_a^b g^2(X(u))\,du$$

by Corollary 1.1, $\sum_{k=1}^{l(n)-1} E(g(X(t_k^n))\eta_k^n)^2$ is bounded independent of n. And we can show the other part goes to zero by the same procedure as we did in Theorem 1.3, namely, by splitting Δf_k^n into $\Delta f_k^n I_G(X(t_k^n)) I_{[0,\delta)}(|\eta_k^n|)$, $\Delta f_k^n I_G(X(t_k^n)) I_{[\delta,\infty)}(|n_k^n|)$ and $I_{G^c}(X(t_k^n))\Delta f_k^n$. (1.16) and (1.17) give us (1.15). Note that it is still enough to consider f as a function of compact support.

COROLLARY 1.3. *In Theorem* 1.4, *let* $g = -f'$, *then* $Q^2(A) = 0$ *in* $L_1(\Omega, P)$ (*or in probability*).

REMARKS. The probability measure P we used here is frequently denoted by P^0, to indicate that $X(0) = 0$ with probability one.

By the same kinds of arguments, we can show that all of our statements hold for any P^x, where P^x corresponds to the Brownian motion starting at x. In particular, the additive functionals studied in Corollary 1.3 have $Q^2(A) = 0$ in a suitable sense for all P^x. In [8] we show that any continuous additive functional $A(t)$ of one-dimensional Brownian motion $X(t)$ satisfying $P^x\{A(t)$ is of bounded variation in any finite interval$\} = 1$ for all x has a representation

$$A(t) = f(X(t)) - f(X(0)) - \int_0^t f'(X(u))\,dX(u)$$

with $f''(x) = d\mu$, where μ is a locally finite measure.

According to Corollary 1.3, if A is an additive functional of this form, and if the function f is locally absolutely continuous with $f' \in L_2^{\text{loc}}(R)$, but if f'' is not a locally-finite signed measure, then A would not be of bounded variation but would still have $Q^2(A) = 0$. For such f, $f(X_t)$ is not a semi-martingale. However its quadratic variation is still given by $\int_a^b (f')^2(X(u))\,du$.

2. Higher dimensional and space-time case. For the simplicity of notation, we shall only present our theorems in the case of R^2. The arguments, however, apply to the general case.

Let $((X(t), Y(t)), \mathscr{F}_{t,t})$ be a standard two-dimensional Brownian motion with $P((X(0), Y(0)) = (0, 0)) = 1$, where $\mathscr{F}_{s,t}$ is a two-parameter family of σ-fields such that $\mathscr{F}_{s,t} \subseteqq \mathscr{F}_{u,v}$ whenever $s \leqq u$, $t \leqq v$. Further, we have $X(t) - X(s)$ independent of $\mathscr{F}_{s,u}$ for any u and $Y(t) - Y(s)$ independent of $\mathscr{F}_{u,s}$ for any u, whenever $t > s$. Let $\{\Pi_n\}$ be a family of partitions of the interval $[a, b]$ with $a > 0$ as given in Section 1. We put

$$Z(t) = (X(t), Y(t)) , \tag{2.1}$$

$$\eta_k{}^n = X(t_{k+1}^n) - X(t_k{}^n) , \tag{2.2}$$

$$\xi_k{}^n = Y(t_{k+1}^n) - Y(t_k{}^n) . \tag{2.3}$$

Let f be a real-valued function defined on R^2, then

$$\Delta f_k{}^n = f(Z(t_{k+1}^n)) - f(Z(t_k{}^n)) , \tag{2.4}$$

$$\Delta^{(1)} f_k{}^n = f(X(t_{k+1}^n), Y(t_{k+1}^n)) - f(X(t_k{}^n), Y(t_{k+1}^n)) , \tag{2.5}$$

$$\Delta^{(2)} f_k{}^n = f(X(t_k{}^n), Y(t_{k+1}^n)) - f(X(t_k{}^n), Y(t_k{}^n)) . \tag{2.6}$$

Clearly, $\Delta f_k{}^n = \Delta^{(1)} f_k{}^n + \Delta^{(2)} f_k{}^n$.

We let $Q^2(f, \Pi_n) = \sum_{k=1}^{l(n)-1} (\Delta f_k{}^n)^2$, $f_1 = \partial f/\partial x$ and $f_2 = \partial f/\partial y$. We use $F(s, t)$ to denote a stochastic process adapted to $\mathscr{F}_{s,t}$. Just as in the one-dimensional case, we have the following propositions:

Proposition 2.1. *Let $E\{(F(s, t))^2\} \leqq M$ for all $(s, t) \in [a, b]^2$, where M is a constant, then*

$$\lim_{n\to\infty} E\{[\textstyle\sum_{k=1}^{l(n)-1} F(t_k{}^n, t_{k+1}^n)((\eta_k{}^n)^2 - \tau_k{}^n)]^2\} = 0$$

and

$$\lim_{n\to\infty} E\{[\textstyle\sum_{k=1}^{l(n)-1} F(t_k{}^n, t_k{}^n)((\xi_k{}^n)^2 - \tau_k{}^n)]^2\} = 0 .$$

Proposition 2.2. *Let $F(s, t)$, $(s, t) \in [a, b]^2$, be uniformly integrable, then*

$$\lim_{n\to\infty} E|\textstyle\sum_{k=1}^{l(n)-1} F(t_k{}^n, t_{k+1}^n)((\eta_k{}^n)^2 - \tau_k{}^n)| = 0$$

and

$$\lim_{n\to\infty} E|\textstyle\sum_{k=1}^{l(n)-1} F(t_k{}^n, t_k{}^n)((\xi_k{}^n)^2 - \tau_k{}^n)| = 0 .$$

Proposition 2.3. *Let $F(s, t)$, $(s, t) \in [a, b]^2$, be jointly continuous in $L_1(\Omega, P)$*, i.e., $\lim_{(u,v)\to(t,s)} E|F(u, v) - F(t, s)| = 0$. *Then*

$$\lim_{n\to\infty} E|\textstyle\sum_{k=1}^{l(n)-1} F(t_k{}^n, t_{k+1}^n)\tau_k{}^n - \int_a^b F(t, t)\, dt| = 0$$

and

$$\lim_{n\to\infty} E|\textstyle\sum_{k=1}^{l(n)-1} F(t_k{}^n, t_k{}^n)\tau_k{}^n - \int_a^b F(t, t)\, dt| = 0 .$$

Proposition 2.4. *Let $f \in L_1(R^2)$ and $F(s, t) = f(X(s), Y(t))$. Then the conclusions of Propositions* 2.2 *and* 2.3 *hold. Furthermore,*

$$\lim_{n\to\infty} E|\textstyle\sum_{k=1}^{l(n)-1} f(X(t_k{}^n), Y(t_{k+1}^n))(\eta_k{}^n)^2 - \int_a^b f(X(t), Y(t))\, dt| = 0$$

and

$$\lim_{n\to\infty} E|\sum_{k=1}^{l(n)-1} f(X(t_k^n), Y(t_k^n))(\xi_k^n)^2 - \int_a^b f(X(t), Y(t))\, dt| = 0 .$$

The proof is similar to Corollary 1.1 and is left for the reader. Before we go to Theorem 2.1, we prove an easy result concerning a kind of maximal function. We define

$$(M_1 f)(x, y) = \sup_r \frac{1}{|r|} \int_x^{x+r} |f(u, y)|\, du , \tag{2.7}$$

$$(M_2 f)(x, y) = \sup_r \frac{1}{|r|} \int_y^{y+r} |f(x, u)|\, du . \tag{2.8}$$

If $f \in L_2(R^2)$, then $f(\cdot, y)$ are in $L_2(R)$ for almost all y, hence

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (M_1 f)^2(x, y)\, dx\, dy = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} (M_1 f)^2(x, y)\, dx)\, dy$$
$$\leqq \int_{-\infty}^{\infty} (c \int_{-\infty}^{\infty} (f)(x, y)\, dx)\, dy \leqq c||f||^2_{L_2(R^2)} .$$

Similarly, $||M_2 f||_{L_2(R^2)} \leqq c||f||_{L_2(R^2)}$.

THEOREM 2.1. *Let f be a real-valued function on R^2 which is locally absolutely continuous along almost all lines parallel to the axes. Let* $|\text{grad } f| \in L_2(R^2)$. (*Or* $|\text{grad } f| \in L_2^{\text{loc}}(R^2)$. *Note that* grad f *exists* a.e.) *Then*

$$\lim_{n\to\infty} E|Q^2(f(Z), \Pi_n) - \int_a^b |\text{grad } f|^2(Z(u))\, du| = 0$$

(*or* $Q^2(f(Z)) = \int_a^b |\text{grad } f|^2(Z(u))\, du$ *in probability*).

PROOF. Let $H = \{(x, y) \mid f(\cdot, y)$ or $f(x, \cdot)$ is not absolutely continuous$\}$. Since $P\{X(t_k^n) \in H$ for some $t_k^n \in \Pi_n\} = 0$, we can disregard H in our computation below.

$$Q^2(f(Z), \Pi_n) = \sum_{k=1}^{l(n)-1} (\Delta f_k^n)^2 = \sum_{k=1}^{l(n)-1} (\Delta^{(1)} f_k^n + \Delta^{(2)} f_k^n)^2$$
$$= \sum_{k=1}^{l(n)-1} (\Delta^{(1)} f_k^n)^2 + 2 \sum_{k=1}^{l(n)-1} (\Delta^{(1)} f_k^n)(\Delta^{(2)} f_k^n) + \sum_{k=1}^{l(n)-1} (\Delta^{(2)} f_k^n)^2$$
$$= I(n) + 2II(n) + III(n) .$$

We can prove

$$\lim_{n\to\infty} |I(n) - \int_a^b (f_1)^2(Z(t))\, dt| = 0 ,$$

and $\lim_{n\to\infty} E|III(n) - \int_a^b (f_2)^2(Z(t))\, dt| = 0$, by an approach similar to that given in Theorem 1.3. We shall show $\lim_{n\to\infty} E|II(n)| = 0$.

To handle $II(n)$, it is enough to consider those f's which are of compact support. We define:

$$II'(n) = \sum_{k=1}^{l(n)-1} f_1(X(t_k^n), Y(t_{k+1}^n))\eta_k^n(\Delta^{(2)} f_k^n) ,$$
$$II''(n) = \sum_{k=1}^{l(n)-1} f_1(X(t_k^n), Y(t_{k+1}^n))\eta_k^n f_2(X(t_k^n), Y(t_k^n))\xi_k^n$$

and

$$II'''(n) = \sum_{k=1}^{l(n)-1} f_1(X(t_k^n), Y(t_k^n))\eta_k^n f_2(X(t_k^n), Y(t_k^n))\xi_k^n .$$

Clearly,

$$E|II(n)| \leqq E|II(n) - II'(n)| + E|II'(n) - II''(n)|$$
$$+ E|II''(n) - II'''(n)| + E|II'''(n)| .$$

By using Cauchy's inequality several times, we can easily show

$$\lim_{n\to\infty} E|II(n) - II'(n)| = \lim_{n\to\infty} E|II'(n) - II''(n)| = 0\,.$$

Further,

$$\begin{aligned} E|II''(n) - II'''(n)| &\leqq \sum_{k=1}^{l(n)-1} (E(f_1(X(t_k^n), Y(t_{k+1}^n)) - f_1(X(t_k^n), Y(t_k^n)))^2(\eta_k^n)^2)^{\frac{1}{2}} \\ &\quad \times (E(f_2(X(t_k^n), Y(t_k^n))\xi_k^n)^2)^{\frac{1}{2}} \\ &\leqq (\sum_{k=1}^{l(n)-1} E(f_1(X(t_k^n), Y(t_{k+1}^n)) - f_1(X(t_k^n), Y(t_k^n)))^2(\eta_k^n)^2)^{\frac{1}{2}} \\ &\quad \times (\sum_{k=1}^{l(n)-1} E(f_2(X(t_k^n), Y(t_k^n))\xi_k^n)^2)^{\frac{1}{2}}\,. \end{aligned}$$

If $f_1 \in C_0^\infty$, by using the dominated convergence theorem we can prove $\sum_{k=1}^{l(n)-1} E(f_1(X(t_k^n), Y(t_{k+1}^n)) - f_1(X(t_k^n), Y(t_k^n)))^2(\eta_k^n)^2 \to 0$. For general f_1, we can use C_0^∞ functions to approximate f_1 in $L_2(R^2)$ as we did in Corollary 1.1. But the other term is bounded, hence $\lim_{n\to\infty} E|II''(n) - II'''(n)| = 0$. Finally,

$$\begin{aligned} E(II'''(n))^2 &= E\sum_{k=1}^{l(n)-1} (f_1(X(t_k^n), Y(t_k^n))\eta_k^n f_2(X(t_k^n), Y(t_k^n))\xi_k^n)^2 \\ &\quad + 2\sum_{k>j} E(f_1(X(t_k^n), Y(t_k^n))\eta_k^n f_2(X(t_k^n), Y(t_k^n))\xi_k^n) \\ &\quad \times (f_1(X(t_j^n), Y(t_j^n))\eta_j^n f_2(X(t_j^n), Y(t_j^n))\xi_j^n)\,. \end{aligned}$$

By conditioning, it is easy to see that the first summation is bounded by $\sum_{k=1}^{l(n)-1} M(\tau_k^n)^2$ and the second summation vanishes. Hence $\lim_{n\to\infty} E(II(n))^2 = 0$. This completes the proof.

The following result is due to Brosamler [2]. We give another proof here as an application of Theorem 2.1.

COROLLARY 2.1 (Brosamler). *Let $p(x) = \int_D g(x, y)\mu(dy)$ be a Green potential on a Green domain $D \subseteqq R^n$, $n \geqq 2$. Let $\tau = \inf\{t \mid Z(t) \in \partial D\}$ where $Z(t)$ is an n-dimensional Brownian motion on R^n and ∂D is the Martin boundary of D. Then, in time interval $[a \wedge \tau, b \wedge \tau]$ $(a > 0)$,*

$$Q^2(p(Z)) = \int_{a\wedge\tau}^{b\wedge\tau} |\text{grad}\, p|^2(Z(u))\, du$$

in P^x probability whenever $x \in D_p = \{x \mid p(y) < \infty\}$.

PROOF. Let $p(x) = \int_D g(x, y)\mu(dy)$ with the support of μ being compact. Then by the energy identity $\int_D |\text{grad}\, p|^2(y)\, dy = c\int_D p(y)\mu(dy) < \infty$. By an easy argument one can show $\text{grad}\, p \in L_2^{\text{loc}}(D)$ for any bounded potential p. Further, since p has all first weak derivatives, it has an almost everywhere equivalent version $\bar{p}$ such that $\bar{p}$ is absolutely continuous along almost all the lines parallel to the axes. We conclude $\bar{p}$ satisfies the conditions of Theorem 2.1. Let $Z(t)$ be a standard n-dimensional Brownian motion and $\tau = \inf\{t \mid Z(t) \in D^c\}$. Then $\bar{p} = p$ a.e. implies

$$\begin{aligned} Q^2(\bar{p}(Z(t\wedge\tau))) = Q^2(p(Z(t\wedge\tau))) &= \int_{a\wedge\tau}^{b\wedge\tau} (\text{grad}\, \bar{p})^2(Z(u))\, du \\ &= \int_{a\wedge\tau}^{b\wedge\tau} (\text{grad}\, p)^2(Z(u))\, du \end{aligned}$$

in probability with respect to any P^x, $x \in D$. For general potential p, we define

$T_n = \inf\{t \mid p(Z(t)) \geqq n\}$. Then for any $x \in D_p$,

$$P^x\{|Q^2((p \wedge n)(Z), \Pi_m) - Q^2(p(Z), \Pi_m)| > 0 \text{ for some } m\}$$
$$\leqq P^x(T_n < \infty) \to 0 \quad \text{as} \quad n \to \infty\,, \quad \text{where} \quad D_p = \{x \mid p(x) < \infty\}\,.$$

On the other hand, if grad p exists at point $x \in D$ and $p(x) < n$, then p is continuous along the coordinate directions in a neighborhood of x, hence grad $(p \wedge n)(x)$ exists and equals grad $p(x)$. Then

$$P^x\{\textstyle\int_{a\wedge\tau}^{b\wedge\tau} (\operatorname{grad} p \wedge n)^2(Z(u))\, du - \int_{a\wedge\tau}^{b\wedge\tau} (\operatorname{grad} p)^2(Z(u))\, du\}$$
$$\leqq P^x(T_{n-1} < \infty) \to 0 \quad \text{as} \quad n \to \infty\,.$$

Hence, for any potential p in the interval $[a, b]$ $(a > 0)$ $Q^2(p(Z)) = \int_{a\wedge\tau}^{b\wedge\tau} (\operatorname{grad} p)^2(Z(u))\, du$ in P^x probability for all $x \in D_p$.

LEMMA 2.1. *Let $g = (g^{(1)}, g^{(2)})$ be a function from R^2 to R^2. Let $E \int_0^t (g^{(i)}(Z(u)))^2\, du < \infty$ (or $P\{\int_0^t (g^{(i)}Z(u))^2\, du < \infty\} = 1$) for $i = 1, 2$.*

Define $G(t) = \int_0^t g(Z(u)) \cdot dZ(u)$; then in the interval $[0, t]$, $Q^2(G) = \int_0^t |g(Z(u))|^2\, du$ in $L_1(\Omega, p)$ (or in probability).

The proof is routine, and will be left to the reader.

THEOREM 2.2. *Let f be a function from R^2 to R^2 which satisfies the conditions of Theorem 2.1. Let g be a function from R^2 to R^2 which satisfies the conditions of Lemma 2.1. Define $A(t) = f(Z(t)) - f(Z(0)) + \int_0^t g(Z(u)) \cdot dZ(u)$. Considering partitions in the interval $[a, b]$ with $a > 0$, we have*

$$Q^2(A) = \int_a^b |\operatorname{grad} f|^2(Z(u))\, du + \int_a^b |g(Z(u))|^2\, du + 2 \int_a^b \operatorname{grad} f(Z(u)) \cdot g(Z(u))\, du$$

in $L_1(\Omega, p)$ (or in probability).

The proof is similar to that of Theorem 1.4 and will be omitted.

THEOREM 2.3. *Let f be a real-valued function defined on $[a, b] \times R$ such that $f(\cdot, x)$ is absolutely continuous. Further, let $f_1 \in L_2^{\text{loc}}([a, b] \times R)$ and f_2 be continuous in $[a, b] \times R$. Let $X(t)$ be a standard one-dimensional Brownian motion, and let $F(t) = f(t, X(t))$. Then $Q^2(F) = \int_a^b (f_2)^2(t, X(t))\, dt$ in probability.*

PROOF.

$$\begin{aligned} Q^2(F, \Pi_n) &= \textstyle\sum_{k=1}^{l(n)-1} (f(t_{k+1}^n, X(t_{k+1}^n)) - f(t_k^n, X(t_k^n)))^2 \\ &= \textstyle\sum_{k=1}^{l(n)-1} (f(t_{k+1}^n, X(t_{k+1}^n)) - f(t_k^n, X(t_{k+1}^n)))^2 \\ &\quad + \textstyle\sum_{k=1}^{l(n)-1} (f(t_k^n, X(t_{k+1}^n)) - f(t_k^n, X(t_k^n)))^2 \\ &\quad + 2 \textstyle\sum_{k=1}^{l(n)-1} (f(t_{k+1}^n, X(t_{k+1}^n)) - f(t_k^n, X(t_{k+1}^n)))(f(t_k^n, X(t_{k+1}^n)) \\ &\quad - f(t_k^n, X(t_k^n))) \\ &= I(n) + II(n) + 2III(n)\,. \end{aligned}$$

Since $P\{|X(t)| < N$ for some $t \in [a, b]\} \to 0$ as $N \to \infty$, by neglecting a set of small measure we can restrict our attention to a compact set $[a, b] \times K$, where

K is compact in R. We still use f to denote $f|[a, b] \times K$. Then

$$\begin{aligned} E|I(n)| &\leqq \sum_{k=1}^{l(n)-1} E\Big(\int_{t_k^n}^{t_{k+1}^n} f_1(u, X(t_{k+1}^n))\, du\Big)^2 \\ &\leqq \sum_{k=1}^{l(n)-1} E(t_{k+1}^n - t_k^n) \int_{t_k^n}^{t_{k+1}^n} (f_1)^2(u, X(t_{k+1}^n))\, du \\ &\leqq (b-a) \max_{1\leqq k\leqq l(n)-1} E \int_{t_k^n}^{t_{k+1}^n} (f_1)^2(u, X(t_{k+1}^n))\, du \\ &\leqq \frac{b-a}{(2\pi a)^{\frac{1}{2}}} \max_{1\leqq k\leqq l(n)-1} \int_{t_k^n}^{t_{k+1}^n} \int_{-\infty}^{\infty} (f_1)^2(u, v)\, du\, dv \\ &\to 0 \qquad \text{as} \quad ||\Pi_n|| \to 0 \end{aligned}$$

because $f_1 \in L_2([a, b] \times K)$.

Further, on $[a, b] \times K$, f_2 is uniformly continuous and bounded. Hence given any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in [a, b]$, $|f(t, y) - f(t, x) - f_2(t, x)(y - x)| < \varepsilon/(2M + 1)$ whenever $|y - x| < \delta$, where x and y are in K and $M = \operatorname{Max}\{|f_2(t, y)|; (t, y) \in [a, b] \times K\}$. Then $|(f(t, y) - f(t, x))^2 - (f_2)^2(t, x)(y - x)^2| < \varepsilon$. Hence

$$\begin{aligned} &E|II(n) - \textstyle\sum_{k=1}^{l(n)-1} (f_2)^2(t_k^n, X(t_k^n))(\eta_k^n)^2| \\ &\quad\leqq \textstyle\sum_{k=1}^{l(n)-1} E|(f(t_k^n, X(t_{k+1}^n)) - f(t_k^n, X(t_k^n)))^2 - (f_2)^2(t_k^n, X(t_k^n))(\eta_k^n)^2| \\ &\quad\leqq \textstyle\sum_{k=1}^{l(n)-1} E|(f(t_k^n, X(t_{k+1}^n)) - f(t_k^n, X(t_k^n)))^2 I_{[0,\delta)}(|\eta_k^n|) - (f_2)^2(t_k^n, X(t_k^n))(\eta_k^n)^2| \\ &\qquad + \textstyle\sum_{k=1}^{l(n)-1} E|(f(t_k^n, X(t_{k+1}^n)) - f(t_k^n, X(t_k^n)))^2 I_{[\delta,\infty)}(|\eta_k^n|)| = A(n) + B(n)\,. \end{aligned}$$

But

$$\begin{aligned} A(n) &\leqq \textstyle\sum_{k=1}^{l(n)-1} E|\varepsilon(\eta_k^n)^2 I_{[0,\delta)}(|\eta_k^n|)| + \sum_{k=1}^{l(n)-1} E|I_{[\delta,\infty)}(|\eta_k^n|)(f_2)^2(t_k^n, X(t_k^n))(\eta_k^n)^2| \\ &\leqq \varepsilon(b-a) + M^2\varepsilon(b-a) \qquad \text{when} \quad ||\Pi_n|| < \delta'\,, \end{aligned}$$

where δ' is chosen such that $0 < t < \delta'$ implies $E|(X(t))^2 I_{[\delta,\infty)}(|X(t)|)| < \varepsilon t$; and

$$\begin{aligned} B(n) &= \textstyle\sum_{k=1}^{l(n)-1} E\Big(\int_{X(t_k^n)}^{X(t_{k+1}^n)} f_2(t_k^n, u)\, du\Big)^2 I_{[\delta,\infty)}(|\eta_k^n|) \\ &\leqq \textstyle\sum_{k=1}^{l(n)-1} EM^2(\eta_k^n)^2 I_{[\delta,\infty)}(|\eta_k^n|) \leqq (b-a)M^2\varepsilon\,. \end{aligned}$$

Hence $\lim_{n\to\infty} E|II(n) - \sum_{k=1}^{l(n)-1} (f_2)^2(t_k^n, X(t_k^n))(\eta_k^n)^2| = 0$. But f_2 is bounded and uniformly continuous on $[a, b] \times K$, so $F_2(t, X(t))$, $t \in [a, b]$, is a uniformly integrable family of functions and it is also continuous in $L_1(\Omega, P)$. By Theorems 1.1 and 1.2,

$$\lim_{n\to\infty} E\Big|\sum_{k=1}^{l(n)-1} (f_2)^2(t_k^n, X(t_k^n))(\eta_k^n)^2 - \int_a^b (f_2)^2(t, X(t))\, dt\Big| = 0\,.$$

Combining the above results, we conclude $Q^2(f) = \int_a^b (f_2)^2(t, X(t))\, dt$ in $L_1(\Omega, P)$ when f is restricted on $[a, b] \times K$. This implies $Q^2(Y) = \int_a^b (f_2)^2(t, X(t))\, dt$ in probability.

Obvious generalization of Theorem 2.3 holds for $(t, x) \in R \times R^n$.

3. Remarks. So far, we restricted our attention to interval $[a, b]$ with $a > 0$. If $a = 0$, we define $\Pi_n = \{t_1^n, t_2^n, \cdots, t_{l(n)}^n\}$ such that $0 < t_1^n < t_2^n < \cdots < t_{l(n)}^n = b$ with $\lim_{n\to\infty} t_1^n = 0$ and $\lim_{n\to\infty} ||\Pi_n|| = 0$. We have some remarks:

(i) The conclusion of Corollary 1.1 is not true for $a = 0$. For example, let $a = 0$, $b = 1$,

$$f(x) = x^{-\frac{1}{2}}e^{-x} \quad \text{when} \quad x \geqq 0$$
$$= 0 \quad \text{otherwise}$$

and

$$\Pi_n = \{1/n^4, 1/n^8 + 1/n, 1/n^8 + 2/n, \cdots\}.$$

Then

$$\lim_{n\to\infty} ER(f, \Pi_n) = \infty, \qquad \text{but} \quad E\int_0^1 f(X(t))\,dt < \infty.$$

(ii) Let $f \in L_1(R)$ and $|f(x)| \leqq C$ for $x \in (-\varepsilon, \varepsilon)$ where $\varepsilon > 0$ and C is a constant, then in $[0, 1]$

$$\lim_{n\to\infty} E|B(f(X), \Pi_n) - R(f(X), \Pi_n)| = 0$$

and

$$\lim_{n\to\infty} E|B(f(X), \Pi_n) - \int_0^1 f(X(t))\,dt| = 0.$$

(iii) For any $f \in L_1(R)$, if we have nice partitions to eliminate the singularity of $(2\pi t)^{-\frac{1}{2}}e^{-y^2/2t}$ as $t \to 0$, then we still have $\lim_{n\to\infty} E|B(f(X), \Pi_n) - R(f(X), \Pi_n)| = 0$ and $\lim_{n\to\infty} E|B(f(X), \Pi_n) - \int_0^1 f(X(t))\,dt| = 0$ for any $f \in L_1(R)$. For example, let $t_k^n = k/n$; then

$$\begin{aligned} E|B(f(X), \Pi_n) &- R(f(X), \Pi_n)| \\ &= E|\textstyle\sum_{t_k^n<\delta} f(X(t_k^n))((\eta_k^n)^2 - \tau_k^n)| + E|\sum_{t_k^n\geqq\delta} f(X(t_k^n))((\eta_k^n)^2 - \tau_k^n)| \\ &= I(n) + II(n). \end{aligned}$$

$$\begin{aligned} I(n) &\leqq \sum_{k/n<\delta} E\left|f\left(X\left(\frac{k}{n}\right)\right)\right| E((\eta_k^n)^2 + \tau_k^n) \leqq \sum_{k/n<\delta} n^{\frac{1}{2}}(2\pi k)^{-\frac{1}{2}}\|f\|_1 \frac{2}{n} \\ &\leqq \sum_{k<n\delta} 2^{\frac{1}{2}}(n\pi)^{-\frac{1}{2}}k^{-\frac{1}{2}}\|f\|_1 \leqq \left(\frac{2}{n\pi}\right)^{\frac{1}{2}}((n\delta)^{\frac{1}{2}} + 1)\|f\|_1 \sim c\delta^{\frac{1}{2}} \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Given $\varepsilon > 0$, we can choose δ small to make $I(n) < \frac{1}{2}\varepsilon$, then we apply Corollary 1.1 to have $II(n) < \frac{1}{2}\varepsilon$. Similarly,

$$\begin{aligned} E|R(f, \Pi_n) - \textstyle\int_0^1 f(X(t))\,dt| &\leqq E|\textstyle\sum_{t_k^n<\delta} f(X(t_k^n))\eta_k^n| + E|\int_0^\delta f(X(t))\,dt| \\ &\quad + E|\textstyle\sum_{t_k^n\geqq\delta} f(X(t_k^n))\eta_k^n - \int_\delta^1 f(X(t))\,dt|. \end{aligned}$$

Then we can prove each part goes to zero.

(iv) Let f be an absolutely continuous function and $f' \in L_2(R)$ (or $f' \in L_2^{\text{loc}}(R)$). Either we require $|f'| \leqq C$ in $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$ or we use nice partitions as indicated in (iii). Then, in interval $[0, b]$, we have

$$Q^2(f(X)) = \int_0^b (f')^2(X(t))\,dt \quad \text{in} \quad L_1(\Omega, P) \qquad \text{(or in probability)}.$$

Under either condition, Theorem 1.4 also holds. The proofs are easy modifications of Theorem 1.3 and Theorem 1.4. Let f be a function from R^n to R. Let $|\text{grad } f|$ be bounded in a neighborhood of (x, y) and f satisfy the conditions of Theorem 2.1, then $Q^2(f(Z)) = \int_0^b |\text{grad } f|^2(Z(u))\,du$ in proper sense with respect to measure $P^{(x,y)}$. Under this conditions, Theorem 2.2 holds too.

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