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EDGEWORTH EXPANSIONS FOR INTEGRALS OF SMOOTH FUNCTIONS

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Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables with $E(X_1) = 0$, $E(X_1^2) = 1$, and $E(X_1^4) < \infty$, and for $n = 1, 2, \cdots$ let P_n be the distribution of $n^{-\frac{1}{2}} \sum_{i=1}^n X_i$. If f is a function with bounded uniformly continuous derivative of order 4, then $\int f\, dP_n$ has an asymptotic expansion in terms of $n^{-\frac{1}{2}}$ with a remainder term of $o(n^{-1})$. This remains true even if P_1 is purely discrete and nonlattice.

1. Introduction and summary. Let $\mathscr{P}$ be the family of all probability measures P on the Borel field $\mathscr{B}$ of the real line with $\int xP(dx) = 0$, $\int x^2P(dx) = 1$, and $\int x^4P(dx) < \infty$, and for $P \in \mathscr{P}$ and $n \in \mathbb{N}$ let P_n be the distribution of $n^{-\frac{1}{2}} \sum_{i=1}^n X_i$, where $X_1, \cdots, X_n$ are independent random variables with distribution P. Let, furthermore, ψ be a measurable function satisfying

$$\sup\{(1 + x^4)^{-1}|\psi(x)| : x \in \mathbb{R}\} < \infty . \tag{1.1}$$

If $P \in \mathscr{P}$, then P_n, $n \in \mathbb{N}$, is said to have an Edgeworth expansion (of order $o(n^{-1})$) at ψ if

$$|\int \psi\, dP_n - \int \psi(t)\varphi(t)\{1 + n^{-\frac{1}{2}}p_1(t) + n^{-1}p_2(t)\}\, dt| = o(n^{-1}) , \tag{1.2}$$

where

$$\begin{aligned} \varphi(t) &= (2\pi)^{-\frac{1}{2}} \exp(-t^2/2) \\ p_1(t) &= \rho(t^3 - 3t)/6 \\ p_2(t) &= \rho^2t^6/72 + (\tau/24 - 5\rho^2/24)t^4 + (-\tau/4 + 5\rho^2/8)t^2 + \tau/8 - 5\rho^2/24 \\ \rho &= \int x^3P(dx) \\ \tau &= \int x^4P(dx) - 3 . \end{aligned} \tag{1.3}$$

For $P \in \mathscr{P}$ let $\mathscr{F}(P)$ be the family of all ψ satisfying (1.1) such that P_n, $n \in \mathbb{N}$, has an Edgeworth expansion at ψ. Note that assumption (1.1) implies that $\int \psi\, dP_n$ exists for all $n \in \mathbb{N}$.

Two measures μ, ν on $\mathscr{B}$ are called orthogonal if there exists a set $A \in \mathscr{B}$ with $\mu(A) = 0$ and $\nu(A^c) = 0$, where A^c denoted the complement of A. A probability measure P on $\mathscr{B}$ is called not purely singular (n.p.s.) if P and the Lebesgue measure are not orthogonal. A probability measure P on $\mathscr{B}$ satisfies Cramér's condition if

$$\limsup_{|t|\to\infty} |\int e^{itx}P(dx)| < 1 .$$

If $P \in \mathscr{P}$ satisfies Cramér's condition, then $\mathscr{F}(P)$ contains the class Ψ_1 of all

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bounded monotone functions. This result is due to Cramér (1928). In this case $\mathscr{F}(P)$ also contains the class Ψ_2 of all functions ψ satisfying (1.1), with derivatives ψ' for which $x \to |\psi'(x)|$ is bounded by a polynomial. This follows from Bhattacharya and Rao (1976), page 208, Theorem 20.1. If $P \in \mathscr{P}$ is n.p.s. then $\mathscr{F}(P)$ contains the class Ψ_3 of all bounded measurable functions (see Bikjalis (1964)). In fact, if $P \in \mathscr{P}$ is n.p.s., then $\mathscr{F}(P)$ contains the class Ψ_4 of all measurable functions satisfying (1.1). This is Theorem 19.5 in [1].

For lattice distributions $P \in \mathscr{P}$ there exist asymptotic formulas for $P_n\{k\}$, k a lattice point (see Esseen (1945), page 63). Using the Euler summation formula one can derive $\psi \in \mathscr{F}(P)$ for a certain class Ψ_5 of smooth functions ψ. There are, however, well-known families Ψ for which $\Psi \subset \mathscr{F}(P)$ holds for all $P \in \mathscr{P}$, i.e., even if P is a discrete nonlattice distribution.

(a) The characteristic functions of P_n, $t \to \int e^{itx} P_n(dx)$, $n \in \mathbb{N}$, have asymptotic expansions of the type (1.3) (see Hsu (1945)). Hence for $m \in \mathbb{N}$ the family Ψ_6 of functions

$$t \to \sum_{i=1}^{m} (a_i \cos \alpha_i t + b_i \sin \beta_i t) ,$$

$a_i, b_i, \alpha_i, \beta_i \in \mathbb{R}$, $i = 1, \cdots, m$, is contained in $\mathscr{F}(P)$.

(b) Von Bahr (1965) showed that absolute moments of P_n, $n \in \mathbb{N}$, have asymptotic expansions of the type discussed here: The family Ψ_7 of all functions $t \to |t|^\alpha$, $\alpha \in (1, 4]$, is contained in $\mathscr{F}(P)$. Here we show that for arbitrary $P \in \mathscr{P}$ (i.e., even for discrete nonlattice distributions) the set $\mathscr{F}(P)$ contains the family Ψ_8 of all functions which have a bounded uniformly continuous derivative of order 4. Note that for every function $\psi \in \Psi_8$ there exist $A_i \in \mathbb{R}$ with $|\psi^{(i)}(x)| \leqq A_i(1 + |x|^{4-i})$, $i = 0, 1, 2, 3$, and for $i = 0$ this yields (1.1). This means (not unexpectedly) that Ψ_8 is a smaller class than the class Ψ_2 one obtains when the distribution P satisfies Cramér's condition. On the other hand, we have $\Psi_6 \subset \Psi_8$.

If Ψ is a class of functions ψ for which (1.1) and (1.2) hold uniformly for $\psi \in \Psi$, then every pointwise limit of a sequence in Ψ belongs to $\mathscr{F}(P)$ (see Remark 2.6). Using this we can show $\psi \in \mathscr{F}(P)$ for all $P \in \mathscr{P}$ for a somewhat larger class Ψ_9 of functions ψ. But we still have $\Psi_9 \subsetneqq \mathscr{F}(P)$, since $\Psi_9 \cap \Psi_7$ contains only the functions $x \to x^2$ and $x \to x^4$ (see Remark 2.9). Note that $\mathscr{F}(P)$ is linear, and hence $\mathscr{F}(P)$ contains the linear space Ψ_{10} spanned by $\Psi_9 \cup \Psi_7$. The results are formulated for classes Ψ of smooth functions ψ where (1.2) holds uniformly for $\psi \in \Psi$.

2. Results. Here we state the main results. The proofs are contained in Section 3.

Let $\Psi_{M,M'}$ be a family of measurable functions which satisfies the following conditions:

(2.1) $$\sup\{|\psi(x)|(1 + x^4)^{-1} : x \in \mathbb{R}, \psi \in \Psi_{M,M'}\} = M < \infty ,$$

(2.2) for all $\psi \in \Psi_{M,M'}$, ψ has a derivative of order 4, $\psi^{(4)}$, such that

(i) $\sup\{|\psi^{(4)}(x)| : x \in \mathbb{R}, \psi \in \Psi_{M,M'}\} = M' < \infty$

(ii) for every $\varepsilon > 0$ there exists a positive number δ such that
$$\sup\{|\psi^{(4)}(t') - \psi^{(4)}(t)| : t', t \in \mathbb{R}, |t' - t| < \delta, \psi \in \Psi_{M,M'}\} \leqq \varepsilon .$$

(2.3) THEOREM. *Let X_i, $i \in \mathbb{N}$, be a sequence of independent identically distributed random variables with $E(X_1) = 0$, $E(X_1^2) = 1$, and $E(X_1^4) < \infty$, and for $n \in \mathbb{N}$, P_n be the distribution of $n^{-\frac{1}{2}} \sum_{i=1}^{n} X_i$. Then*

$$\text{(2.4)} \qquad \sup\{|\textstyle\int \psi \, dP_n - \int \psi(t)\varphi(t)(1 + n^{-\frac{1}{2}}p_1(t) + n^{-1}p_2(t))\, dt| : \psi \in \Psi_{M,M'}\} = o(n^{-1}) ,$$

where p_1 and p_2 are defined in (1.3).

If $E(X_1^3) = 0$ and $E(X_1^4) = 3$, a nonuniform version of Theorem 2.1 follows from Theorem 3 of Butzer, Hahn and Westphal (1975), page 335.

(2.5) REMARK. If we only assume $E(|X_1|^3) < \infty$, a corresponding result holds: in (2.4), $o(n^{-1})$ must be replaced by $o(n^{-\frac{1}{2}})$; in (1.3) $p_2(t) \equiv 0$; and in (2.2) the third derivatives $\psi^{(3)}$ are assumed to be uniformly bounded and uniformly equicontinuous. We do not know whether similar results hold for higher order Edgeworth expansions. The order $o(n^{-1})$ is, however, sufficient for computing Hodges and Lehmann deficiencies using smooth loss functions.

(2.6) REMARK. The set $\bar{\Psi}_{M,M'}$ of all pointwise limits of sequences in $\Psi_{M,M'}$ satisfies

$$\sup\{|\textstyle\int \psi \, dP_n - \int \psi(t)\varphi(t)(1 + n^{-\frac{1}{2}}p_1(t) + n^{-1}p_2(t))\, dt| : \psi \in \bar{\Psi}_{M,M'}\} = o(n^{-1}) .$$

This is an immediate consequence of Theorem 2.3 and Lebesgue's dominated convergence theorem. However, $\bar{\Psi}_{M,M'}$ will contain only functions having a bounded Schwarzian difference quotient:

(2.7) PROPOSITION. *For every $\psi \in \bar{\Psi}_{M,M'}$, $x, h \in \mathbb{R}$ we have*

$$\text{(2.8)} \qquad |-\psi(x - 3h) + 4\psi(x - 2h) - 5\psi(x - h) + 5\psi(x + h) - 4\psi(x + 2h) + \psi(x + 3h)| \leqq 50|h|^4 M'/3 .$$

PROOF. That (2.8) holds for $\psi \in \Psi_{M,M'}$ follows from a Taylor expansion and (2.2(i)). Since the set of all functions on $\mathbb{R}$ satisfying (2.8) is closed in the topology of pointwise convergence on $\mathbb{R}$, (2.8) holds for every $\psi \in \bar{\Psi}_{M,M'}$.

(2.9) REMARK. For $\alpha \in (0, 4)$, $\alpha \neq 2$, the function $t \to |t|^\alpha$ does not fulfill (2.8). Hence for arbitrary $P \in \mathscr{P}$ the union Ψ_9 of all $\bar{\Psi}_{M,M'}$, $M, M' > 0$, does not cover $\mathscr{F}(P)$, since it does not cover the class Ψ_7 of all functions considered by von Bahr.

The following example shows that $\psi \in \mathscr{F}(P)$ does not hold if ψ is only piecewise smooth, even if $P \in \mathscr{P}$ has finite moments of all orders:

(2.10) EXAMPLE. Let P be the probability measure defined by $P\{-1\} = P\{1\} = \frac{1}{2}$, and $\psi(t) = |t|$, $t \in \mathbb{R}$.

Then

$$\text{(2.11)} \qquad \limsup_{n\in\mathbb{N}} n\left|\int \psi\, dP_n - \int \psi(t)\varphi(t)(1+n^{-1}p_2(t))\, dt\right| > 0\,,$$

where $p_2(t) = -(t^4 - 6t^2 + 3)/12$.

PROOF. (i) For the sake of easy reference we state the following results: If $f = \varphi p$ with p a polynomial, then uniformly for $m \in \mathbb{N}$

$$\int_0^{\alpha(m+1)} f(x)\, dx = \alpha \sum_{k=1}^m f(\alpha k) + \alpha[f(0) + f(\alpha(m+1))]/2 - \alpha^2[f'(\alpha(m+1)) - f'(0)]/12 + O(\alpha^3) \quad \text{for } \alpha > 0,\ \alpha \to 0\,.$$

This is a special version of the Euler summation formula.

(ii) Let $n \in \mathbb{N}$ be even, $n = 2m$, say. Let, furthermore, A_n be the set of non-negative atoms of P_n. Then $\# A_n = m + 1$ and $\sup A_n = n^{\frac{1}{2}}$. We know from Esseen (1945), page 63, Theorem 5, that

$$\sup\{|P_n\{\xi\} - 2n^{-\frac{1}{2}}\varphi(\xi)(1 + \textstyle\sum_{k=2}^4 n^{-k/2}p_k(\xi))| : \xi \in A_n\} = o(n^{-\frac{3}{2}})\,,$$

where p_k are polynomials, $k = 3, 4$. Hence

$$\int \psi\, dP_n = 2\sum_{\xi\in A_n} \xi P_n\{\xi\} = 2\sum_{\xi\in A_n} \xi 2n^{-\frac{1}{2}}\varphi(\xi)(1 + \textstyle\sum_{k=2}^4 n^{-k/2}p_k(\xi)) + o(n^{-1})\,.$$

Using (i) we obtain with $a_n = (n+2)n^{-\frac{1}{2}}$

$$\begin{aligned}\int \psi\, dP_n &= 2\int_0^{a_n} x\varphi(x)\, dx + 2n^{-\frac{1}{2}}a_n\varphi(a_n) + 2n^{-1}\{(-a_n^2 + 1)\varphi(a_n) - \varphi(0)\}/3 \\ &\quad + n^{-1}\int_{-\infty}^{\infty} |x|\varphi(x)p_2(x)\, dx + o(n^{-1}) \\ &= \int_{-\infty}^{\infty} |x|\varphi(x)\, dx - n^{-1}2\varphi(0)/3 + n^{-1}\int_{-\infty}^{\infty}|x|\varphi(x)p_2(x)\, dx + o(n^{-1})\,.\end{aligned}$$

This proves (2.11).

3. Proof of the theorem. The proof is based on the following proposition.

(3.1) PROPOSITION. *Let $P, Q \in \mathscr{P}$ with $\int x^k(P-Q)(dx) = 0$, $k = 3, 4$. Then*

$$\text{(3.2)} \qquad \sup_{\psi\in\Psi_{M,M'}} \left|\int \psi\, d(P_n - Q_n)\right| = o(n^{-1})\,,$$

where for $n \in \mathbb{N}$ P_n and Q_n are the distributions of $n^{-\frac{1}{2}}\sum_{i=1}^n X_i$, and X_i, $i \in \mathbb{N}$, are independent identically distributed random variables with distribution P and Q, respectively.

PROOF. According to Feller (1971), page 258, (3.10) (see also Trotter (1959), pages 229–230) we have

$$\text{(3.3)} \qquad \left|\int \psi\, d(P_n - Q_n)\right| \leqq n \sup\{|\textstyle\int \psi(xn^{-\frac{1}{2}} - t)(P-Q)(dx)| : t \in \mathbb{R}\}\,.$$

Let $\varepsilon > 0$ and choose $d \in \mathbb{R}$ such that $\int x^4 1_{\{|x|>d\}}(x)(P+Q)(dx) < \varepsilon$. From a Taylor expansion we obtain that for $t \in \mathbb{R}$ there exists $\xi_t(\psi, x, n) \in \mathbb{R}$ with $|\xi_t(\psi, x, n) + t| \leqq |x|n^{-\frac{1}{2}}$ such that

$$\psi(xn^{-\frac{1}{2}} - t) - \sum_{i=0}^4 \psi^{(i)}(-t)(xn^{-\frac{1}{2}})^i/i! = n^{-2}x^4(\psi^{(4)}(\xi_t(\psi, x, n)) - \psi^{(4)}(-t))/24\,.$$

This together with (2.2(i)) implies that for all $x \in \mathbb{R}$

$$\sup\{|\psi(xn^{-\frac{1}{2}} - t) - \textstyle\sum_{i=0}^4 \psi^{(i)}(-t)(xn^{-\frac{1}{2}})^i/i!| : t \in \mathbb{R}, \psi \in \Psi_{M,M'}\} \leqq n^{-2}M'x^4/12\,.$$

Since $\lim_{n\in\mathrm{N}} \sup\{|\xi_t(\psi, x, n) + t| : |x| \leqq d, t \in \mathbb{R}, \psi \in \Psi_{M,M'}\} = 0$, we obtain with (2.2(ii))

$$\lim_{n\in\mathrm{N}} \sup\{|\psi^{(4)}(\xi_t(\psi, x, n)) - \psi^{(4)}(-t)| : t \in \mathbb{R}, |x| \leqq d, \psi \in \Psi_{M,M'}\} = 0 .$$

Hence

$$\begin{aligned} |\textstyle\int \psi(xn^{-\frac12} - t)(P - Q)(dx)| & \\ = |\textstyle\int (\psi(xn^{-\frac12} - t) - \sum_{i=0}^{4} \psi^{(i)}(-t)/i!\,(xn^{-\frac12})^i)(P - Q)(dx)| & \\ \leqq n^{-2}d^4 \sup\{|\psi^{(4)}(\xi_t(\psi, x, n)) - \psi^{(4)}(-t)| : t \in \mathbb{R}, |x| \leqq d, \psi \in \Psi_{M,M'}\} & \\ \times (P + Q)\{|x| \leqq d\}/24 + n^{-2}M' \textstyle\int x^4 1_{\{|x|\geqq d\}}(P + Q)(dx)/12 & \end{aligned}$$

yields

$$\limsup_{n\in\mathrm{N}} n^2 \sup\{|\textstyle\int \psi(xn^{-\frac12} - t)(P - Q)(dx)| : t \in \mathbb{R}, \psi \in \Psi_{M,M'}\} \leqq \varepsilon M'/12 .$$

Since $\varepsilon > 0$ was arbitrary, this together with (3.3) implies (3.2).

With Theorem 19.1 in [1] we obtain that for n.p.s. probability measure $P \in \mathscr{P}$ we have

$$(3.4) \qquad \textstyle\int (1 + |x|^4)|P_n - G_n|(dx) = o(n^{-1}) ,$$

where G_n has Lebesgue-density $t \to \varphi(t)(1 + \sum_{k=1}^{2} n^{-k/2}p_k(t))$, and for $B \in \mathscr{B}$

$$|P_n - G_n|(B) = \sup\{|P_n(A) - G_n(A)| : A \in \mathscr{B}, A \subset B\} .$$

Hence the statement (2.4) is true in this special case. For a given $P \in \mathscr{P}$ we shall therefore look for n.p.s. probability measures $Q \in \mathscr{P}$ with $\int x^k(P - Q)(dx) = 0$, $k = 3, 4$. In the following we shall show that such a n.p.s. probability measure can always be found if P is not concentrated on two points. To do this we need the following lemma.

(3.5) LEMMA. *Let $Q \in \mathscr{P}$ and $\mathscr{Q} = \{P \in \mathscr{P}: \int x^3(P - Q)(dx) = 0\}$. Then $\int x^4Q(dx) = \inf\{\int x^4P(dx) : P \in \mathscr{Q}\}$ iff Q is concentrated on two points.*

PROOF. For notational convenience let $a = \int x^3P(dx)$ and $b(P) = \int x^4P(dx)$ for $P \in \mathscr{P}$. Ljapounov's theorem implies that for $p \in \mathscr{P}$ $(\int (x - a/2)^2P(dx))^2 \leqq \int (x - a/2)^4P(dx)$, and equality holds iff there exists $c \in \mathbb{R}$ with $(x - a/2)^2 = c$ P-a.e.

Hence $(1 + a^2/4)^2 \leqq b(P) - 2a^2 + \frac32 a^2 + a^4/16$ or

$$b(P) \geqq 1 + a^2 ,$$

and equality holds iff $(x - a/2)^2 = c$ P-a.e. Therefore, $b(P) = 1 + a^2$ implies that P is concentrated on the two points $a/2 + c^{\frac12}$ and $a/2 - c^{\frac12}$. If, on the other hand, $P \in \mathscr{P}$ is concentrated on two points, x_1, x_2, say, then $P \in \mathscr{P}$ implies $x_1 = -(p/q)^{\frac12}$ and $x_2 = (p/q)^{\frac12}$, where $q = 1 - p$ and $p \in (0, 1)$ is the solution of

$$a = (q - p)(pq)^{-\frac12} .$$

Then $b(P) = (1 - 3pq)/(pq) = 1 + a^2$.

(3.6) PROPOSITION. *If $P \in \mathscr{P}$ is not concentrated on two points, then there exists a* n.p.s. *probability measure $Q \in \mathscr{P}$ with $\int x^k(P - Q)(dx) = 0$, $k = 3, 4$.*

PROOF. Let $p \in (0, 1)$ be the solution of $\int x^3 P(dx) = (q - p)(pq)^{-\frac{1}{2}}$, where $q = 1 - p$, and let P_0 be the probability measure defined by $P_0\{-(p/q)^{\frac{1}{2}}\} = q$, $P_0\{(q/p)^{\frac{1}{2}}\} = p$. Then $P_0 \in \mathscr{P}$, and $\int x^3(P - P_0)(dx) = 0$. From Lemma 3.5 we obtain that $\beta = \int x^4(P - P_0)(dx) > 0$. Let $\bar{p} \in (0, 1)$ be the solution of $\int x^3 P(dx) = 2^{-\frac{3}{2}}(\bar{q} - \bar{p})(\bar{p}\bar{q})^{-\frac{1}{2}}$, where $\bar{q} = 1 - \bar{p}$, and let $t > 0$ be a solution of $4 \int x^4 P(dx) = (t + 2)(t + 3)/[t(t + 1)]$. Let X and Y be independent rv's such that the distribution of X has Lebesgue-density $x \to (2\Gamma(t))^{-1}|x|^{t-1}e^{-|x|}$ and the distribution $\bar{P}$ of Y is defined by $\bar{P}\{-(\bar{p}/\bar{q})^{\frac{1}{2}}\} = \bar{q}$ and $\bar{P}\{(\bar{q}/\bar{p})^{\frac{1}{2}}\} = \bar{p}$. Then the distribution of $Z = 2^{-\frac{1}{2}}([t(t + 1)]^{-\frac{1}{2}}X + Y)$, say $\bar{Q}$, is n.p.s. and satisfies $\bar{Q} \in \mathscr{P}$, $\int x^3(P - \bar{Q})\, dx = 0$, and $r = \int x^4 \bar{Q}(dx) > \int x^4 P(dx) = s$. Let $\alpha = \beta/(r - s + \beta)$. Then $\alpha \in (0, 1)$, and with $Q = \alpha\bar{Q} + (1 - \alpha)P_0$ we found a n.p.s. probability measure in $\mathscr{P}$ with $\int x^k(P - Q)(dx) = 0$ for $k = 3, 4$. This proves the proposition.

Now we shall consider the case when $P \in \mathscr{P}$ is concentrated on two points. For notational convenience let

$$g_n(P, t) = \varphi(t)(1 + n^{-\frac{1}{2}}p_1(t) + n^{-1}p_2(t)), \qquad t \in \mathbb{R},\ n \in \mathbb{N},$$

where $\varphi(t)$, $p_1(t)$, and $p_2(t)$ are defined in (1.3). Let $\mathbb{N}_2$ be the set of even positive integers. The following result is an immediate consequence of Proposition 3.6.

(3.7) PROPOSITION. *For every* $P \in \mathscr{P}$

$$\text{(3.8)} \qquad \lim_{n \in \mathbb{N}_2} n \sup\{|\textstyle\int \psi\, dP_n - \int \psi(t) g_n(t, P)\, dt| : \psi \in \Psi_{M,M'}\} = 0\,.$$

PROOF. For $k \in \mathbb{N}$ let $Y_k = 2^{-\frac{1}{2}}(X_{2k} + X_{2k-1})$, where X_i, $i \in \mathbb{N}$, are independent random variables with distribution P. Then Y_k, $k \in \mathbb{N}$, is a sequence of independent identically distributed random variables such that the distribution $\hat{P}$ of Y_1 is not concentrated on two points. According to Proposition 3.6 there exists a n.p.s. probability measure $Q \in \mathscr{P}$ with $\int x^k(\hat{P} - Q)(dx) = 0$, $k = 3, 4$. For $n \in \mathbb{N}$ let P_n be the distribution of $n^{-\frac{1}{2}} \sum_{i=1}^n Y_i$, and Q_n be the distribution of $n^{-\frac{1}{2}} \sum_{i=1}^n Z_i$, where Z_i, $i \in \mathbb{N}$, are independent random variables with distribution Q. As mentioned above,

$$\text{(3.9)} \qquad \lim_{n \in \mathbb{N}} n \sup\{|\textstyle\int \psi\, dQ_n - \int \psi(t) g_n(t, Q)\, dt| : \psi \in \Psi_{M,M'}\} = 0\,.$$

From Proposition 3.1 we obtain with (3.9)

$$\lim_{n \in \mathbb{N}} n \sup\{|\textstyle\int \psi\, d\hat{P}_n - \int \psi(t) g_n(t, \hat{P})\, dt| : \psi \in \Psi_{M,M'}\} = 0\,.$$

This together with $\hat{P}_n = P_{2n}$ and $g_n(t, \hat{P}) = g_{2n}(t, P)$, $n \in \mathbb{N}$, implies (3.8).

The following proposition deals with the case of odd sample sizes n and probability measures $P \in \mathscr{P}$ which are concentrated on two points.

(3.10) PROPOSITION. *Let* $P \in \mathscr{P}$ *be concentrated on two points. Then*

$$\text{(3.11)} \qquad \lim_{k \in \mathbb{N}} k \sup\{|\textstyle\int \psi\, dP_{2k+1} - \int \psi(t) g_{2k+1}(t, P)\, dt| : \psi \in \Psi_{M,M'}\} = 0\,.$$

PROOF. Let $a, b \in \mathbb{R}$ be the two points with $P\{a, b\} = 1$, and let Ψ^* be the family of all functions $t \to \psi((2k/(2k + 1))^{\frac{1}{2}}t + (2k + 1)^{-\frac{1}{2}}s)$, $\psi \in \Psi_{M,M'}$, $k \in \mathbb{N}$, and $s \in \{a, b\}$.

For some $\bar{M} > M$, $\bar{M}' > M'$ we have $\Psi^* \subset \Psi_{\bar{M},\bar{M}'}$. Using Proposition 3.7 for $\Psi_{\bar{M},\bar{M}'}$ instead of $\Psi_{M,M'}$ we obtain uniformly for $\psi \in \Psi_{M,M'}$

$$\int \psi \, dP_{2k+1} = \int\int \psi((2k/(2k+1))^{\frac{1}{2}}t + (2k+1)^{-\frac{1}{2}}s)g_{2k}(t, P)\, dt\, P(ds) + o(k^{-1}) .$$

For notational convenience let $a_k = (2k/(2k+1))^{\frac{1}{2}}$, $b_k = (2k+1)^{-\frac{1}{2}}$, and let $u = a_k^{-1}t + b_k s$, whence $t = a_k(u - b_k s)$. Since P is concentrated on two points, we have uniformly for $\psi \in \Psi_{M,M'}$

$$\begin{aligned}
&\int\int \psi(u)g_{2k}(t, P)\, dt\, P(ds) \\
&\quad = a_k \int\int \psi(u)g_{2k}(a_k(u - b_k s), P)\, du\, P(ds) \\
&\quad = a_k \int\int \psi(u)[g_{2k}(a_k u, P) - a_k b_k s g_{2k}'(a_k u, P) + \tfrac{1}{2}(a_k b_k)^2 s^2 g_{2k}''(a_k u, P)]\, du\, P(ds) \\
&\qquad + o(k^{-1}) \\
&\quad = a_k \int \psi(u)[g_{2k}(a_k u, P) + g_{2k}''(a_k u, P)/(4k)]\, du + o(k^{-1}) .
\end{aligned}$$

Since $\int 1_{\{|u| \leq \log k\}} g_{2k}^{(i)}(a_k u, P)(1 + u^4)\, du = o(k^{-1})$ for $i = 0$ and 2, we obtain with (2.1) that uniformly for $\psi \in \Psi_{M,M'}$

$$\begin{aligned}
&\int\int \psi(u)g_{2k}(t, P)\, dt\, P(ds) \\
&\quad = a_k \int \psi(u)[g_{2k}(a_k u, P) + g_{2k}''(a_k u, P)/(4k)]1_{\{|u| \leq \log k\}}(u)\, du + o(k^{-1}) \\
&\quad = a_k \int \psi(u)[g_{2k}(u, P) + u g_{2k}'(u, P)/(4k) + g_{2k}''(u, P)/(4k)]1_{\{|u| \leq \log k\}}(u)\, du + o(k^{-1}) \\
&\quad = a_k \int \psi(u)[g_{2k}(u, P) + u\varphi'(u)/(4k) + \varphi''(u)/(4k)]1_{\{|u| \leq \log k\}}(u)\, du + o(k^{-1}) \\
&\quad = a_k \int \psi(u)[g_{2k}(u, P) - \varphi(u)/(4k)]1_{\{|u| \leq \log k\}}(u)\, du + o(k^{-1}) \\
&\quad = \int \psi(u)[(1 + 1/(4k))g_{2k}(u, P) - \varphi(u)/(4k)]1_{\{|u| \leq \log k\}}(u)\, du + o(k^{-1}) \\
&\quad = \int \psi(u)g_{2k}(u, P)1_{\{|u| \leq \log k\}}(u)\, du + o(k^{-1}) \\
&\quad = \int \psi(u)g_{2k}(u, P)\, du + o(k^{-1}) .
\end{aligned}$$

On the other hand,

$$\int |g_{2k+1}(u, P) - g_{2k}(u, P)|(1 + u^4)\, du = o(k^{-1}) ,$$

and therefore we have uniformly for $\psi \in \Psi_{M,M'}$

$$\int \psi \, dP_{2k+1} = \int \psi(t)g_{2k+1}(t, P)\, dt + o(n^{-1})$$

which is the assertion (3.11).

PROOF OF THEOREM 2.3. If $P \in \mathscr{P}$ is concentrated on two points, then (2.4) follows from Propositions 3.7 and 3.10. If $P \in \mathscr{P}$ is not concentrated on two points, Proposition 3.6 implies the existence of a n.p.s. probability measure $Q \in \mathscr{P}$ with $\int x^k(P - Q)(dx) = 0$, $k = 3, 4$. Using Proposition 3.1 it suffices to show that (2.4) holds for Q. This, however, follows from (3.4).

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