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AN IMPROVED UPPER BOUND FOR STANDARD p-FUNCTIONS

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For standard p-functions, an upper bound for $M = p(1)$, for a given value m of $m(p) = \min\{p(t), 0 < t \leqq 1\}$, was proved in a previous paper by the author. The bound implied that $\nu_0 \leqq .590$, ν_0 being the constant defined by

$$I_M = \inf\{m(p) \mid p(1) = M\}, \qquad \nu_0 = \inf\{M \mid I_M > 0\},$$

in which p varies over the class of standard p-functions. In the present paper both of these upper bounds are sharpened by a refinement of the argument, the limit for ν_0 being reduced to .560.

1. Introduction. An upper bound for $M = p(1)$, for a given value m of $m(p) = \min\{p(t), 0 < t \leqq 1\}$ for $p \in \mathscr{P}$, where $\mathscr{P}$ denotes the class of standard p-functions, was obtained in [2]. The new bound implied that $\nu_0 \leqq .590$, ν_0 being the constant defined by

$$I_M = \inf_{p \in \mathscr{P}}\{m(p) \mid p(1) = M\},$$
$$\nu_0 = \inf\{M \mid I_M > 0\}.$$

Thus, the range within which ν_0 was known to lie was reduced to $.368 \doteqdot e^{-1} \leqq \nu_0 \leqq .590$, where $\doteqdot$ denotes approximate equality.

In this note, by a refinement of the argument in [2], sharper upper bounds are obtained for M and ν_0, the latter being .560. In the course of the argument an upper bound for $p(t)$ for any given value of t is obtained which appears to be of intrinsic interest.

2. Main result. In [2], we consider an increasing sequence $t_1 < t_2 < t_3 < \cdots < t_n$ in which $t_n = 1$, and t_1 satisfies the restrictions $p(t_1) = \alpha$, $p(1 - t_1) \leqq \alpha$ or $p(t_1) \geqq \alpha$, $p(1 - t_1) = \alpha$ (cf. (14)*). (N. B.: Here and throughout the following, an asterisk means that the relation referred to is the relation with that number in [2].) In modification of the argument in [2], consider a sequence $t_1 < t_2 < \cdots < t_n$ in which t_n has any positive value t and there are no restrictions on t_1 apart from $t_1 < t$ and, as in (15)*,

$$p(t_n - t_{r+1}) \geqq p(t_n - t_r) > 0, \qquad r = 2, 3, \cdots, (n-1). \tag{1}$$

The whole argument in [2] continues to hold if M is taken to be the value of

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$p(t)$ instead of $p(1)$. In place of (29)* we obtain

$$(2) \qquad 1 - p(t) \geqq \alpha\beta_{n-2} - \alpha p(t - t_1) + \beta_{n-2}\left\{(1 - \beta_1) + \sum_{r=1}^{n-3}\left(1 - \frac{\beta_{r+1}}{\beta_r}\right)\right\}$$

where $\alpha = p(t_1)$ and $\beta_r = p(t_n - t_{n-r})$, $r = 1, 2, \cdots, (n-2)$.

As shown in [2], for a given value of t_1 the right-hand side of (2) is maximized when

$$(3) \qquad \beta_k = \beta_1{}^k, \qquad k = 1, 2, \cdots, (n-2).$$
$$\beta_1 = 1 - \frac{1}{n-1}(1 - \alpha).$$

Hence it follows from (2) that (cf. (32)*)

$$(4) \qquad 1 - p(t) \geqq -p(t_1)p(t - t_1) + \left\{1 - \frac{1}{n-1}[1 - p(t_1)]\right\}^{n-1}.$$

Letting $n \to \infty$ in (4), we obtain

$$(5) \qquad 1 - p(t) \geqq -p(t_1)p(t - t_1) + \exp\{-1 + p(t_1)\}.$$

Kingman's second order inequality $0 \leqq g(t_2)$ (cf. (9 ii)*) gives for the sequence $\{t_1, t\}$

$$(6) \qquad 1 - p(t) \geqq -p(t_1)p(t - t_1) + p(t_1).$$

As

$$(7) \qquad \exp[-1 + p(t)] \geqq p(t),$$

(5) provides a sharper bound than (6). The improvement over the upper bound in [2] is effected by using the lower bound in (5) in place of that in (6).

Next let $t_n = 1$ as in [2]. Then $p(t_n) = p(1) = M$. Consider the sequence $\{v_1, v_2\}$ where $v_1 = t_n - t_{n-1}$, $v_2 = t_n$. Substituting in (5) $t_n - t_{n-1}$ for t_1 and t_n for t, we obtain

$$(8) \qquad p(t_{n-1}) \geqq \frac{\exp\{-1 + p(t_n - t_{n-1})\}}{p(t_n - t_{n-1})} - \frac{1 - M}{p(t_n - t_{n-1})}.$$

Similarly, substitution in (5) of $t_n - t_{n-j-1}$ for t_1 and t_n for t yields

$$(9) \qquad p(t_{n-j-1}) \geqq \frac{\exp\{-1 + p(t_n - t_{n-j-1})\}}{p(t_n - t_{n-j-1})} - \frac{1 - M}{p(t_n - t_{n-j-1})}.$$

The whole of the inductive argument in [2] remains valid on substituting the lower bounds in (8) and (9) for those in (18)* and (23)* respectively. Thus, we obtain (cf. (26)*)

$$(10) \qquad \frac{1 - M}{p(t_n - t_2)} \geqq p(t_1)\left[1 - \frac{p(t_n - t_1)}{p(t_n - t_2)}\right]$$
$$+ \sum_{k=1}^{n-2} \frac{\exp\{-1 + p(t_n - t_{n-k})\}}{p(t_n - t_{n-k})}\left[1 - \frac{p(t_n - t_{n-k})}{p(t_n - t_{n-k+1})}\right]$$

in which $p(0) = 1$.

For any $t_1 \in (0, 1)$, either (a) $p(t_1) \geqq p(1 - t_1)$ or (b) $p(t_1) < p(1 - t_1)$. If (b) holds, we replace t_1 by $(1 - t_1)$. Thus t_1 can always be so chosen that

$$p(t_1) \geqq p(1 - t_1)\,. \tag{11}$$

Using (11) and substituting β_r for $p(t_n - t_{n-r})$, we obtain from (10),

$$1 - M \geqq \alpha\beta_{n-2} - \alpha^2 + \beta_{n-2} \sum_{k=1}^{n-2} \frac{\exp(-1 + \beta_k)}{\beta_k} \left(1 - \frac{\beta_k}{\beta_{k-1}}\right) \tag{12}$$

in which $\beta_0 = 1$.

Because of (7), (12) provides a sharper lower bound for $(1 - M)$ than (29)*. The argument is continued in [2], by maximizing the right-hand side of (29)* in $\beta_1, \beta_2, \cdots, \beta_{n-2}$ for fixed α. The maximum of the right-hand side of (12) cannot be derived explicitly. But since (12) holds for all β_i subject to (1), we can assign to β_i the same values as in [2], i.e., the values in (3), and obtain

$$1 - M \geqq -\alpha^2 + \left[1 - \frac{(1-\alpha)}{n-1}\right]^{n-1} \times \left\{\alpha + \frac{(1-\alpha)}{n-1} \sum_{k=1}^{n-2} \frac{\exp\left\{-1 + \left[1 - \frac{(1-\alpha)}{n-1}\right]^k\right\}}{\left(1 + \frac{1-\alpha}{n-1}\right)^k}\right\}. \tag{13}$$

Now take limits as $n \to \infty$. In the right-hand side of (13) put

$$k = (n-1)z\,.$$

For fixed z as $n \to \infty$,

$$\left(1 - \frac{1-\alpha}{n-1}\right)^k \to \exp[-(1-\alpha)z]\,,$$

and

$$\exp\left\{-1 + \left[1 - \frac{(1-\alpha)}{n-1}\right]^k\right\} \to \exp\{-1 + \exp[-(1-\alpha)z]\}\,.$$

As k varies over the sequence $1, 2, \cdots, (n-2)$, z assumes the values $1/(n-1)$, $2/(n-1), \cdots, (n-2)/(n-1)$ which have a common difference of $1/(n-1)$. Hence the second term within the braces in (13) becomes in the limit

$$\begin{aligned} &= (1-\alpha) \int_0^1 \frac{\exp\{-1 + \exp[-(1-\alpha)z]\}}{\exp[-(1-\alpha)z]}\, dz \\ &= (1-\alpha) \int_0^1 \exp\{-1 + (1-\alpha)z + \exp[-(1-\alpha)z]\}\, dz\,. \end{aligned}$$

Thus on taking limits, (13) yields

$$\begin{aligned} 1 - M \geqq -\alpha^2 + \exp(-1+\alpha)\{\alpha + (1-\alpha) \textstyle\int_0^1 \exp\{-1 \\ + (1-\alpha)z + \exp[(-1+\alpha)z]\}\, dz\}\,. \end{aligned} \tag{14}$$

Set $u(\alpha) =$ integral in the right-hand side of (14). Then (14) reduces to

$$1 - M \geqq h(\alpha)\,, \tag{15}$$

where

$$h(\alpha) = -\alpha^2 + \exp(-1 + \alpha)[\alpha + (1 - \alpha)u(\alpha)] .$$

It can be shown that $h(\alpha)$ is maximized for a unique value α_1 of α and $h'(\alpha) < 0$ for $\alpha > \alpha_1$. An upper bound for M can thus be obtained in terms of the function u. But the upper bound so obtained is not numerically computable.

A numerically computable bound is obtained by expanding $\exp\{-(1 - \alpha)z\}$ in powers of $(1 - \alpha)z$. We have

$$\exp\{-(1 - \alpha)z\} \geqq 1 - (1 - \alpha)z + \frac{(1 - \alpha)^2}{2!} z^2 - \frac{(1 - \alpha)^3}{3!} z^3 . \tag{16}$$

Hence in the right-hand side of (15),

$$\begin{aligned} &\exp\{-1 + (1 - \alpha)z + \exp[-(1 - \alpha)z]\} \\ &\qquad \geqq \exp\left\{\frac{(1 - \alpha)^2}{2} z^2 - \frac{(1 - \alpha)^3}{6} z^3\right\} \\ &\qquad \geqq 1 + \frac{(1 - \alpha)^2}{2} z^2 - \frac{(1 - \alpha)^3}{6} z^3 . \end{aligned} \tag{17}$$

Substituting by (17) in (14) and integrating,

$$1 - M \geqq g(\alpha) , \tag{18}$$

where

$$g(\alpha) = -\alpha^2 + \exp(-1 + \alpha)\left\{1 + \frac{(1 - \alpha)^3}{6} - \frac{(1 - \alpha)^4}{24}\right\} .$$

It is easily verified that $g''(\alpha) < 0$ for $\alpha \in (0, 1)$. Hence $g'(\alpha) = 0$ for a unique value K_1 of α and $g'(\alpha) < 0$ for $\alpha > K_1$. Hence $g(\alpha)$ is maximized for $\alpha = K_1$ if $m \leqq K_1$ and for $\alpha = m$ if $m > K_1$. This gives the upper bounds

$$\begin{aligned} &\text{if} \quad m \leqq K_1 , \\ &M \leqq 1 + K_1^2 - \exp(-1 + K_1)\left\{1 + \frac{(1 - K_1)^3}{6} - \frac{(1 - K_1)^4}{24}\right\} ; \\ &\text{if} \quad m > K_1 , \\ &M \leqq 1 + m^2 - \exp(-1 + m)\left\{1 + \frac{(1 - m)^2}{6} - \frac{(1 - m)^4}{24}\right\} . \end{aligned} \tag{19}$$

In (19), K_1 is the value of α for which

$$\begin{aligned} 0 &= g'(\alpha) \\ &= -2\alpha + \exp(-1 + \alpha)\left\{1 - \frac{(1 - \alpha)^2}{2} + \frac{(1 - \alpha)^3}{3} - \frac{(1 - \alpha)^4}{24}\right\} . \end{aligned} \tag{20}$$

It follows that

$$\nu_0 \leqq 1 + K_1^2 - \exp(-1 + K_1)\left\{1 + \frac{(1 - K_1)^3}{6} - \frac{(1 - K_1)^4}{24}\right\} . \tag{21}$$

It is found from (20) that $K_1 \doteqdot .1834$ and hence by (21)

$$\nu_0 \leqq .560 .$$

REMARK. In (16), the expansion in powers of $(1 - \alpha)z$ can be continued to higher powers. Secondly, the refinement of the computation can be carried to a second stage. In (5), an upper bound for $p(t)$ is obtained by a slight modification in the argument in [2], viz., by keeping $t_n = t$ and removing the restriction on t_1. The same modifications in the argument from (6) to (21), yield the following sharper bound for $p(t)$:

$$
\begin{aligned}
(22) \qquad 1 - p(t) \geqq & -p(t_1)p(1 - t_1) \\
& + \exp[-1 + p(t_1)]\left\{1 + \frac{[1 - p(t_1)]^3}{6} - \frac{[1 - p(t_1)]^4}{24}\right\},
\end{aligned}
$$

which is slightly lower than the upper bound in (5). Hence (22) can be used in place of (5). However, calculations (the details of which are not given here) show that together these refinements improve the upper limit for ν_0 by less than .001, so that to the 3rd decimal place the upper limit remains .560.

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