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THE ORDER OF THE NORMAL APPROXIMATION FOR LINEAR COMBINATIONS OF ORDER STATISTICS WITH SMOOTH WEIGHT FUNCTIONS

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A Berry-Esseen bound of order $n^{-\frac{1}{2}}$ is established for linear combinations of order statistics. The theorem requires a "smooth" weight function, and the underlying distribution function must not have "too much weight in the tails." The distribution function need not be continuous.

1. Introduction and main result. Linear combinations of order statistics play an important role in the theory of estimation. Many authors (see, e.g., Stigler (1974) and Shorack (1972)) have established their asymptotic normality under different sets of conditions. However, though usual statistical practice ignores this fact, such limit theorems are useless for applications, unless one is willing to believe that these asymptotic assertions provide a good approximation for finite sample sizes. Recently attention has been paid to this problem of the accuracy of the normal approximation. Bjerve (1977) has obtained a Berry–Esseen type bound of order $n^{-\frac{1}{2}}$ (n being the sample size) for trimmed linear combinations of order statistics. His result admits quite general weights on observations between the αth and βth sample percentile $(0 < \alpha < \beta < 1)$, but he does not allow weights to be put on the remaining observations. In addition the distribution function (df) must satisfy a quite severe smoothness condition. The purpose of this paper is to establish a Berry–Esseen type bound of order $n^{-\frac{1}{2}}$ for linear combinations of order statistics, which allows weights to be put on all the observations. From the standpoint of probability theory our result can be viewed as a contribution to the problem of extending the Berry–Esseen theorem to certain sums of dependent random variables.

Before we state our main result let us first introduce some notation. Let for each $n \geqq 1$, $T_n = n^{-1} \sum_{i=1}^{n} J(i/(n+1))X_{in}$, where X_{in}, $i = 1, \cdots, n$ denotes the ith order statistic of a random sample $X_1, \cdots, X_n$ of size n from a distribution with df F and J is a bounded measurable weight function on (0, 1). The inverse of a df will always be the left-continuous one. Let $F_n^*(x) = P(T_n^* \leqq x)$ for $-\infty < x < \infty$, where

$$T_n^* = (T_n - \mathscr{E}(T_n))/\sigma(T_n) . \tag{1.1}$$

In Theorem 2 of Stigler (1974), it is shown that T_n^* is asymptotically $N(0, 1)$-distributed as $n \to \infty$, if J is bounded and continuous a.e. F^{-1}, $\mathscr{E}X_1^2 < \infty$ and

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$\sigma^2(J, F) > 0$, where

(1.2) $$\sigma^2(J, F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(F(x))J(F(y))(F(\min(x, y)) - F(x)F(y))\, dx\, dy\,.$$

In addition these assumptions imply that $\lim_{n\to\infty} n\sigma^2(T_n) = \sigma^2(J, F)$ (see Theorem 1 of Stigler (1974)).

In the following theorem we establish a Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for the normal approximation of F_n^*. Let Φ denote the df of the standard normal distribution.

THEOREM 1. *Suppose*

(1) *J is bounded and continuous on* (0, 1). *The derivative J' exists, except possibly at a finite number of points; J' satisfies a Lipschitz condition of order $> \frac{1}{2}$ on the open intervals where it exists. The inverse F^{-1} satisfies a Lipschitz condition of order $> \frac{1}{2}$ on neighbourhoods of the points where J' does not exist.*

(2) $\mathscr{E}|X_1|^3 < \infty$ *and* $\int_0^1 |J'(s)|\, dF^{-1}(s) < \infty$.

Then $\sigma^2(J, F) > 0$ implies that there exists a constant C, depending on J and F but not on n, such that for all $n \geqq 1$

$$\sup_x |F_n^*(x) - \Phi(x)| \leqq Cn^{-\frac{1}{2}}\,.$$

Theorem 1 is the first general theorem establishing a Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for linear combinations of order statistics, which allows weights to be put on all the observations. The theorem requires a "smooth" weight function, and the underlying df must not have "too much weight in the tails." The df need not be continuous.

In Section 2 we shall approximate T_n^* by a random variable (rv) S_n^* such that $T_n^* - S_n^*$ is of negligible order for our purposes. A Berry-Esseen bound of order $n^{-\frac{1}{2}}$ for S_n^* is established in Section 3 using a technique based on characteristic functions due to Bickel (1974) (see also Bjerve (1977)).

2. Approximation by S_n^*. Let, for each $n \geqq 1$, $U_1, \cdots, U_n$ be independent uniform (0, 1) rv's and let $U_{in} (1 \leqq i \leqq n)$ denote the ith order statistic of $U_1, \cdots, U_n$. It is well known that the joint distribution of $X_1, \cdots, X_n$ is the same as that of $(F^{-1}(U_1), \cdots, F^{-1}(U_n))$ for any df F. Therefore we shall identify X_i with $F^{-1}(U_i)$ and also X_{in} with $F^{-1}(U_{in})$. Throughout we shall assume that all rv's are defined on the same probability space $(\Omega, \mathscr{A}, \mathscr{P})$. For any rv X with $0 < \sigma(X) < \infty$ we denote by X^* the rv $(X - \mathscr{E}(X))/\sigma(X)$. $\chi_E(\cdot)$ denotes the indicator of a set E.

Define, for each $n \geqq 1$, the rv S_n by

(2.1) $$S_n = I_{1n} + I_{2n}$$

where

(2.2) $$I_{1n} = -n^{-1} \sum_{i=1}^n \int_0^1 J(s)(\chi_{(0,s]}(U_i) - s)\, dF^{-1}(s)$$

and

(2.3) $$I_{2n} = -n^{-2} \sum_{i=1}^n \sum_{j=1}^{i-1} \int_0^1 J'(s)(\chi_{(0,s]}(U_i) - s)(\chi_{(0,s]}(U_j) - s)\, dF^{-1}(s)\,.$$

In this section we shall prove that under appropriate conditions $T_n^* - S_n^*$ is of negligible order for our purposes:

$$P(|T_n^* - S_n^*| \geqq n^{-\frac{1}{2}}) = \mathcal{O}(n^{-\frac{1}{2}}) \qquad \text{as} \quad n \to \infty . \tag{2.4}$$

For the purpose of our proofs we start by stating a very simple but useful lemma.

LEMMA 2.1. *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be two sequences of* rv's *(defined on the same probability space* $(\Omega, \mathscr{A}, P)$*), such that*

(1) $\sigma^2(X_n - Y_n) = \mathcal{O}(n^{-\frac{5}{2}})$ *as* $n \to \infty$, *and*

(2) *either* $0 < \lim_{n\to\infty} n\sigma^2(X_n) < \infty$ *or* $0 < \lim_{n\to\infty} n\sigma^2(Y_n) < \infty$ *holds.*

Then for any $a > 0$ $P(|X_n^* - Y_n^*| \geqq an^{-\frac{1}{2}}) = \mathcal{O}(n^{-\frac{1}{2}})$ *as* $n \to \infty$.

PROOF. To start with the proof we note that

$$X_n^* - Y_n^* = \frac{X_n - Y_n - \mathscr{E}(X_n - Y_n)}{\sigma(X_n)} + (Y_n - \mathscr{E}(Y_n))\frac{(\sigma(Y_n) - \sigma(X_n))}{\sigma(X_n)\sigma(Y_n)} \tag{2.5}$$

and hence that

$$\sigma^2(X_n^* - Y_n^*) \leqq 2\sigma^{-2}(X_n)\sigma^2(X_n - Y_n) + 2\sigma^2(Y_n)\left(\frac{\sigma(Y_n) - \sigma(X_n)}{\sigma(X_n)\sigma(Y_n)}\right)^2 . \tag{2.6}$$

Obviously we may assume that $0 < \lim_{n\to\infty} n\sigma^2(X_n) < \infty$. Hence we know that $\sigma^{-2}(X_n) = \mathcal{O}(n)$ as $n \to \infty$. Because also $\sigma^2(X_n - Y_n) = \mathcal{O}(n^{-\frac{5}{2}})$ as $n \to \infty$, we have shown that the first term on the right-hand side of (2.6) is $\mathcal{O}(n^{-\frac{3}{2}})$ as $n \to \infty$. To proceed with the second term on the right-hand side of (2.6) we note that it follows from our assumptions that $0 < \lim_{n\to\infty} n\sigma^2(Y_n) < \infty$. Now

$$\begin{aligned} 2\sigma^2(Y_n)\left(\frac{\sigma(Y_n) - \sigma(X_n)}{\sigma(X_n)\sigma(Y_n)}\right)^2 &= 2\,\frac{(\sigma^2(Y_n) - \sigma^2(X_n))^2}{\sigma^2(X_n)(\sigma(X_n) + \sigma(Y_n))^2} \\ &\leqq 2\,\frac{(2\sigma(X_n)\sigma(Y_n - X_n) + \sigma^2(Y_n - X_n))^2}{\sigma^2(X_n)(\sigma(X_n) + \sigma(Y_n))^2} \end{aligned}$$

and we can use the preceding results to find that

$$\begin{aligned} &2\,\frac{(2\sigma(X_n)\sigma(Y_n - X_n) + \sigma^2(Y_n - X_n))^2}{\sigma^2(X_n)(\sigma(X_n) + \sigma(Y_n))^2} \\ &\qquad = \mathcal{O}(n^2)(\mathcal{O}(n^{-\frac{1}{2}})\mathcal{O}(n^{-\frac{5}{4}}) + \mathcal{O}(n^{-\frac{5}{2}}))^2 = \mathcal{O}(n^{-\frac{3}{2}}) \qquad \text{as} \quad n \to \infty . \end{aligned}$$

Hence we have shown that $\sigma^2(X_n^* - Y_n^*) = \mathcal{O}(n^{-\frac{3}{2}})$ as $n \to \infty$. An application of Chebyshev's inequality completes the proof. □

In order to prove that (2.4) holds under appropriate conditions we need two more lemmas. In our second lemma we approximate T_n by a rv V_n given by

$$V_n = \textstyle\int_0^1 J(s)F_n^{-1}(s)\, ds = \sum_{i=1}^n \int_{(i-1)/n}^{i/n} J(s)\, ds \cdot X_{in} , \tag{2.7}$$

where F_n denotes the empirical df based on $X_1, \cdots, X_n$. We shall show that $T_n^* - V_n^*$ is of negligible order for our purposes. Let $||f|| = \sup_{0<t<1} |f(t)|$ for any function f on (0, 1). In certain cases the function f is defined on (0, 1) except at a finite number of points. Then $||f||$ will denote the supremum of $|f|$ on the domain of f.

LEMMA 2.2. *Let $\mathscr{E}X_1^2 < \infty$ and suppose that condition* (1) *of Theorem* 1 *is satisfied. Then $\sigma^2(J, F) > 0$ implies that for any $a > 0$ $P(|T_n^* - V_n^*| \geqq an^{-\frac{1}{2}}) = \mathscr{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. The assumption that J' satisfies a Lipschitz condition of order $> \frac{1}{2}$ can be replaced by the boundedness of J' on the open intervals where it exists. The Lipschitz condition for F^{-1} may be of order $\geqq \frac{1}{2}$.*

PROOF. It follows from $\mathscr{E}X_1^2 < \infty$ that $\mathscr{E}X_{in}^2 < \infty$ for any $1 \leqq i \leqq n$. Furthermore it is well known (see Esary, Proschan and Walkup (1967)) that for any x, y, i, j, n and F we have $P(X_{in} \leqq x, X_{jn} \leqq y) \geqq P(X_{in} \leqq x)P(X_{jn} \leqq y)$. Using a representation of the covariance of two random variables given in Lehmann (1966), page 1139, this result implies directly that the covariance between X_{in} and X_{jn} is finite and nonnegative for all $1 \leqq i \neq j \leqq n$. Obviously this implies that

$$\sigma^2(\textstyle\sum_{i=1}^n a_i X_{in}) \leqq \sigma^2(\sum_{i=1}^n b_i X_{in}) \tag{2.8}$$

holds, provided $a_i a_j \leqq b_i b_j$ for all $1 \leqq i, j \leqq n$. This inequality is due to W. R. van Zwet and will be very useful in what follows.

Since the assumptions of this lemma imply those of Theorem 1 of Stigler (1974) (see our introduction) we know that $\lim_{n\to\infty} n\sigma^2(T_n) = \sigma^2(J, F)$. By assumption we have also that $\sigma^2(J, F) > 0$, whereas a simple application of (2.8) yields $\sigma^2(T_n) \leqq n^{-1}||J||^2\sigma^2(X_1)$. Because $||J|| < \infty$ and $\sigma^2(X_1) < \infty$ by the assumptions of the lemma these results imply that $0 < \lim_{n\to\infty} n\sigma^2(T_n) = \sigma^2(J, F) < \infty$. Application of Lemma 2.1 shows that it suffices now to prove that

$$\sigma^2(T_n - V_n) = \mathscr{O}(n^{-\frac{3}{2}}) \qquad \text{as} \quad n \to \infty . \tag{2.9}$$

To prove (2.9) we distinguish two cases: (i) J is everywhere differentiable on (0, 1), and (ii) J' fails to exist at a finite number of points.

We first prove (2.9) in case (i). Using (2.7) and (2.8) we see that

$$\sigma^2(T_n - V_n) \leqq \sigma^2\left(\textstyle\sum_{i=1}^n X_{in}\left|\frac{J(i/(n+1))}{n} - \int_{(i-1)/n}^{i/n} J(s)\, ds\right|\right) . \tag{2.10}$$

Applying (2.8) again and using the condition for J we find that

$$\sigma^2(T_n - V_n) \leqq n^{-3}||J'||^2\sigma^2(X_1) . \tag{2.11}$$

Because $||J'|| < \infty$ and $\sigma^2(X_1) < \infty$ by the assumptions of the lemma the proof of case (i) of the lemma is now complete.

Suppose now that we are in case (ii). Without any loss of generality we assume that J' does not exist at only one point, say $s = s_1$. Let $j = [ns_1] + 1$. Using inequality (2.8) twice we see that

$$\begin{aligned}\sigma^2(T_n - V_n) \leqq{}& 2\sigma^2\left(\textstyle\sum_{i=1;i\neq j}^n X_{in}\left|\frac{J(i/(n+1))}{n} - \int_{(i-1)/n}^{i/n} J(s)\, ds\right|\right) \\ &+ 2\sigma^2\left(X_{jn}\left|\frac{J(j/(n+1))}{n} - \int_{(j-1)/n}^{j/n} J(s)\, ds\right|\right) .\end{aligned} \tag{2.12}$$

Using condition (1) of Theorem 1 and applying (2.8) once more we obtain that

$$\sigma^2(T_n - V_n) \leqq 2n^{-3}\|J'\|^2\sigma^2(X_1) + 8n^{-2}\|J\|^2\sigma^2(X_{jn}) \,. \tag{2.13}$$

Hence it remains to prove that $\sigma^2(X_{jn}) = \mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. Let g_n denote the beta-density of the uniform order statistic U_{jn} (with $j = [ns_1] + 1$) and let E_n be the set

$$E_n = \left\{u : \left|u - \frac{[ns_1 + 1]}{n+1}\right| \leqq (mn^{-1}\log n)^{\frac{1}{2}}, 0 < u < 1\right\} \tag{2.14}$$

for some fixed $m > 0$. The complement of E_n in (0, 1) will be denoted by E_n^c. Then we have that

$$\begin{aligned}\sigma^2(X_{jn}) &\leqq \mathscr{E}\left(X_{jn} - F^{-1}\left(\frac{j}{n+1}\right)\right)^2 \\ &= \int_{E_n}\left(F^{-1}(u) - F^{-1}\left(\frac{j}{n+1}\right)\right)^2 g_n(u)\,du \\ &\quad + \int_{E_n^c}\left(F^{-1}(u) - F^{-1}\left(\frac{j}{n+1}\right)\right)^2 g_n(u)\,du \,.\end{aligned} \tag{2.15}$$

Because $\mathscr{E}X_1^2 < \infty$ we can use Lemma 4 of Stigler (1969) to see that the second integral on the right-hand side of (2.15) is $\mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$, provided we choose m sufficiently large. The Lipschitz condition of F^{-1} in a neighbourhood of s_1 can be used to treat the first integral on the right-hand side of (2.15). Since $(j-1)/n \leqq s_1 < j/n$ we have for sufficiently large n and some constant $B > 0$ that

$$\int_{E_n}\left(F^{-1}(u) - F^{-1}\left(\frac{j}{n+1}\right)\right)^2 g_n(u)\,du \leqq B \cdot \mathscr{E}\left|U_{jn} - \frac{j}{n+1}\right| . \tag{2.16}$$

It follows directly from this and the well-known fact that, as $\lim_{n\to\infty} j/n = s_1$, for $0 < s_1 < 1$, $\mathscr{E}|U_{jn} - j/(n+1)| = \mathcal{O}(n^{-\frac{1}{2}})$, that also the first integral on the right-hand side of (2.15) is $\mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. Hence we can conclude that $\sigma^2(X_{jn}) = \mathcal{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. This and (2.13) implies that $\sigma^2(T_n - V_n) = \mathcal{O}(n^{-\frac{5}{2}})$ as $n \to \infty$, in case (ii). This completes the proof of the lemma. □

Define for $0 < u < 1$ the function

$$\psi(u) = \int_u^1 J(s)\,ds - (1-u)\int_0^1 J(s)\,ds \tag{2.17}$$

and let $c = \int_0^1 J(s)\,ds$. Then it is easy to check (see Shorack (1972) for a similar approach) that

$$V_n = \int_0^1 \psi(\Gamma_n(s))\,dF^{-1}(s) + cn^{-1}\sum_{i=1}^n F^{-1}(U_i) \,, \tag{2.18}$$

holds with probability 1. We use the fact that, almost surely, none of the rv's $U_1, \cdots, U_n$ take values corresponding to the discontinuities of F^{-1}. Here Γ_n denotes the empirical df based on $U_1, \cdots, U_n$. This representation of V_n will be very useful.

In our third lemma we use representation (2.18) to show that $V_n^* - S_n^*$ is of negligible order for our purposes. This will be achieved by Taylor expanding ψ so that the remainder term is negligible order for our purposes and by noting that certain quadratic terms in our Taylor expansion are also of sufficiently small order of magnitude.

LEMMA 2.3. *Let $\mathscr{E}|X_1|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$ and suppose that condition* (1) *of Theorem* 1 *is satisfied. Then $\sigma^2(J, F) > 0$ implies that for any $a > 0$ $P(|V_n^* - S_n^*| \geqq an^{-\frac{1}{2}}) = \mathscr{O}(n^{-\frac{1}{2}})$ as $n \to \infty$.*

PROOF. It follows directly from the proof of Lemma 2.2 that $0 < \lim_{n\to\infty} n\sigma^2(V_n) = \sigma^2(J, F) < \infty$. Application of Lemma 2.1 shows that it suffices now to prove that

$$\sigma^2(V_n - S_n) = \mathscr{O}(n^{-\frac{3}{2}}) \qquad \text{as} \quad n \to \infty\,. \tag{2.19}$$

For the purpose of this proof we define, for each $n \geqq 1$, the rv W_n given by

$$W_n = \int_0^1 \left(\psi(s) + (\Gamma_n(s) - s)\psi'(s) + \frac{(\Gamma_n(s) - s)^2}{2}\psi''(s)\right) dF^{-1}(s) + cn^{-1} \textstyle\sum_{i=1}^n F^{-1}(U_i)\,. \tag{2.20}$$

Note that the assumptions of the lemma guarantee that W_n is well defined. It will be convenient to prove

$$\sigma^2(V_n - W_n) = \mathscr{O}(n^{-\frac{3}{2}}) \qquad \text{as} \quad n \to \infty \tag{2.21}$$

and

$$\sigma^2(W_n - S_n) = \mathscr{O}(n^{-\frac{3}{2}}) \qquad \text{as} \quad n \to \infty \tag{2.22}$$

rather than (2.19). We first prove (2.22). Using (2.17) we find that

$$\begin{aligned} W_n = {} & \int_0^1 \psi(s)\, dF^{-1}(s) - \int_0^1 J(s)(\Gamma_n(s) - s)\, dF^{-1}(s) \\ & - \int_0^1 J'(s) \frac{(\Gamma_n(s) - s)^2}{2}\, dF^{-1}(s) \\ & + c \int_0^1 (\Gamma_n(s) - s)\, dF^{-1}(s) + cn^{-1} \textstyle\sum_{i=1}^n F^{-1}(U_i)\,. \end{aligned} \tag{2.23}$$

Because $\Gamma_n(s) = n^{-1} \sum_{i=1}^n \chi_{(0,s]}(U_i)$ for all $0 < s < 1$ and $n \geqq 1$ we have

$$\begin{aligned} & \int_0^1 (\Gamma_n(s) - s)\, dF^{-1}(s) \\ & \quad = n^{-1} \textstyle\sum_{i=1}^n \left(\int_{(0,U_i)} (-s)\, dF^{-1}(s) + \int_{[U_i,1)} (1 - s)\, dF^{-1}(s)\right)\,. \end{aligned} \tag{2.24}$$

Now integration by parts, the finiteness of $\mathscr{E}|X_1|$ and the fact that, almost surely, none of the rv's $U_1, \cdots, U_n$ take values corresponding to the discontinuities of F^{-1} shows that

$$\int_0^1 (\Gamma_n(s) - s)\, dF^{-1}(s) = -n^{-1} \textstyle\sum_{i=1}^n F^{-1}(U_i) + \int_0^1 F^{-1}(s)\, ds \tag{2.25}$$

holds with probability 1.

Thus

$$\begin{aligned} &W_n - \mathscr{E}(W_n) \\ (2.26)\qquad &=_{\text{a.s.}} -n^{-1} \textstyle\sum_{i=1}^n \int_0^1 J(s)(\chi_{(0,s]}(U_i) - s)\, dF^{-1}(s) \\ &\quad - 2^{-1}n^{-2} \textstyle\sum_{i=1}^n \sum_{j=1}^n \int_0^1 J'(s)(\chi_{(0,s]}(U_i) - s)(\chi_{(0,s]}(U_j) - s)\, dF^{-1}(s) \\ &\quad + 2^{-1}n^{-1} \textstyle\int_0^1 J'(s)s(1-s)\, dF^{-1}(s)\,. \end{aligned}$$

Combining (2.26) with (2.1), (2.2) and (2.3) and using the assumptions of the lemma together with Fubini's theorem to verify that $\mathscr{E}S_n = 0$ we find that

$$\begin{aligned} (2.27)\qquad &W_n - S_n - \mathscr{E}(W_n - S_n) \\ &\quad =_{\text{a.s.}} -2^{-1}n^{-2} \textstyle\sum_{i=1}^n \int_0^1 J'(s)((\chi_{(0,s]}(U_i) - s)^2 - s(1-s))\, dF^{-1}(s)\,, \end{aligned}$$

and hence that

$$(2.28)\qquad \sigma^2(W_n - S_n) = 2^{-2}n^{-3}\sigma^2\left(\textstyle\int_0^1 J'(s)(\chi_{(0,s]}(U_1) - s)^2\, dF^{-1}(s)\right)\,.$$

To see that the variance on the right-hand side of (2.28) is finite note that

$$\begin{aligned} &\sigma^2\left(\textstyle\int_0^1 J'(s)(\chi_{(0,s]}(U_1) - s)^2\, dF^{-1}(s)\right) \\ &\quad \leqq \mathscr{E}\left(\textstyle\int_0^1 J'(s)(\chi_{(0,s]}(U_1) - s)^2\, dF^{-1}(s)\right)^2 \\ (2.29)\qquad &\quad = \mathscr{E} \textstyle\int_0^1\int_0^1 J'(s)J'(v)(\chi_{(0,s]}(U_1) - s)^2(\chi_{(0,v]}(U_1) - v)^2\, dF^{-1}(s)\, dF^{-1}(v) \\ &\quad = \textstyle\int_0^1\int_0^1 J'(s)J'(v)\mathscr{E}(\chi_{(0,s]}(U_1) - s)^2(\chi_{(0,v]}(U_1) - v)^2\, dF^{-1}(s)\, dF^{-1}(v) \\ &\quad \leqq \textstyle\int_0^1\int_0^1 |J'(s)J'(v)|(\mathscr{E}(\chi_{(0,s]}(U_1) - s)^4\mathscr{E}(\chi_{(0,v]}(U_1) - v)^4)^{\frac12}\, dF^{-1}(s)\, dF^{-1}(v) \\ &\quad \leqq 2\left(\textstyle\int_0^1 |J'(s)|(s(1-s))^{\frac12}\, dF^{-1}(s)\right)^2\,, \end{aligned}$$

where the interchange of the expectation and the integrals is a consequence of Fubini's theorem. The validity of this application of Fubini's theorem can be inferred from the moment condition of the lemma, the boundedness of J' on its domain and the continuity of F^{-1} at the points where J' is undefined. These conditions also imply that $\int_0^1 |J'(s)|(s(1-s))^{\frac12}\, dF^{-1}(s)$ is finite.

Thus we have shown that $\sigma^2(W_n - S_n) = \mathscr{O}(n^{-3})$ as $n \to \infty$. This completes the proof of (2.22).

Next we prove (2.21). As in the second part of the proof of Lemma 2.2 we distinguish two cases. First we assume (case (i)) that J is everywhere differentiable on (0, 1). Using (2.18), (2.20) and Taylor's theorem, together with the Lipschitz condition for J' on (0, 1), we see that for all $n \geqq 1$ and some constant $A > 0 |V_n - W_n| \leqq A \int_0^1 |\Gamma_n(s) - s|^{\frac52}\, dF^{-1}(s)$ and hence that

$$(2.30)\qquad \sigma^2(V_n - W_n) \leqq \mathscr{E}(V_n - W_n)^2 \leqq A^2\mathscr{E}\left(\textstyle\int_0^1 |\Gamma_n(s) - s|^{\frac52}\, dF^{-1}(s)\right)^2\,.$$

Applying Fubini's theorem, the Cauchy–Schwarz inequality, and making some simple moment calculations it follows that for some constant $B > 0$

$$(2.31)\qquad \sigma^2(V_n - W_n) \leqq Bn^{-\frac52}\left(\textstyle\int_0^1 (s(1-s))^{\frac12}\, dF^{-1}(s)\right)^2\,.$$

The moment assumption of the lemma ensures that the integral on the right-hand side of (2.31) is finite. This completes the proof of (2.21) for case (i).

Suppose now that J' fails to exist at a finite number of points (case (ii)). To prove (2.21) in this case is somewhat more delicate. It seems convenient to introduce at this point the well-known Kolmogorov–Smirnov statistic $D_n = n^{\frac{1}{2}} \sup_{0<s<1} |\Gamma_n(s) - s|$. It was shown by Dvoretzky, Kiefer and Wolfowitz (1956) that $P(D_n \geqq \lambda_n) \leqq c \exp(-2\lambda_n^2)$, for all $n \geqq 1$, $\lambda_n \geqq 0$ and a positive constant c independent of n and λ_n. Obviously this implies that $P(D_n \geqq (2^{-1}m \log n)^{\frac{1}{2}}) = \mathscr{O}(n^{-m})$ as $n \to \infty$, for any fixed $m > 0$. Let us denote by χ_n the indicator of the set $\{D_n \geqq (2^{-1}m \log n)^{\frac{1}{2}}\}$. Define $\chi_n^c = 1 - \chi_n$. Without loss of generality we assume that J' does not exist at only one point $s_1 \in (0, 1)$.

We first show that $\mathscr{E}(V_n - W_n - \mathscr{E}(V_n - W_n))^2\chi_n = \mathscr{O}(n^{-\frac{5}{2}})$ as $n \to \infty$ holds for an appropriate value of m. Since $\sigma^2(W_n - S_n) = \mathscr{O}(n^{-3})$ as $n \to \infty$, and hence that $\mathscr{E}(W_n - S_n - \mathscr{E}(W_n - S_n))^2\chi_n = \mathscr{O}(n^{-3})$ as $n \to \infty$ for any $m > 0$, was obtained earlier in this proof, it suffices to show that

$$\mathscr{E}(V_n - S_n - \mathscr{E}(V_n - S_n))^2\chi_n = \mathscr{O}(n^{-\frac{5}{2}}) \quad \text{as} \quad n \to \infty \,, \tag{2.32}$$

To prove (2.32) we apply Hölder's inequality to obtain for any $0 < \eta < 1$

$$\begin{aligned}&\mathscr{E}(V_n - S_n - \mathscr{E}(V_n - S_n))^2\chi_n \\ &\quad \leqq (\mathscr{E}(V_n - S_n - E(V_n - S_n))^{2+2\eta})^{1/(1+\eta)}(\mathscr{E}\chi_n^{1+1/\eta})^{\eta/(1+\eta)} \,,\end{aligned}$$

and hence, using the c_r-inequality (see, e.g., Loève (1955), page 155), that

$$\begin{aligned}&\mathscr{E}(V_n - S_n - \mathscr{E}(V_n - S_n))^2\chi_n \\ &\quad \leqq 16(\mathscr{E}|V_n|^{2+2\eta} + \mathscr{E}|S_n|^{2+2\eta})^{1/(1+\eta)}(P(\chi_n = 1))^{\eta/(1+\eta)} \,.\end{aligned} \tag{2.33}$$

Since $P(\chi_n = 1) = \mathscr{O}(n^{-m})$ as $n \to \infty$, it follows that $(P(\chi_n = 1))^{\eta/1+\eta} = \mathscr{O}(n^{-\frac{5}{2}})$ as $n \to \infty$, provided we choose $m > 5/\eta$. Now using (2.1), (2.2), (2.3) and (2.7) and applying integration by parts we see that

$$|V_n| \leqq n^{-1}||J|| \textstyle\sum_{i=1}^n |F^{-1}(U_i)|$$

and

$$|S_n| \leqq n^{-1}(||J|| + ||J'||) \textstyle\sum_{i=1}^n (|F^{-1}(U_i)| + \int_0^1 |F^{-1}(s)|\, ds)$$

holds for all $n \geqq 1$ with probability one. Combining this result with the finiteness of $\mathscr{E}|X_1|^{2+2\eta}$ for any $0 < \eta < \varepsilon/2$ and some $\varepsilon > 0$ satisfying the moment condition of the lemma and applying the c_r-inequality we find that the expectations on the right of (2.33) are uniformly bounded in n for any $\eta \in (0, \varepsilon/2)$. Hence we have shown that (2.32) holds for any fixed $m > (10/\varepsilon)$.

To complete the proof of (2.21) in case (ii) it remains to show that $\mathscr{E}(V_n - W_n - \mathscr{E}(V_n - W_n))^2\chi_n^c = \mathscr{O}(n^{-\frac{5}{2}})$ as $n \to \infty$ for some fixed $m > (10/\varepsilon)$. It follows from (2.18) and (2.20) that $V_n - W_n = \int_0^1 g_n(s)\, dF^{-1}(s)$ where

$$g_n(s) = \psi(\Gamma_n(s)) - \psi(s) - (\Gamma_n(s) - s)\psi'(s) - \frac{(\Gamma_n(s) - s)^2}{2}\psi''(s) \,, \tag{2.34}$$

for all $0 < s < 1$, except $s = s_1$, and any $n \geqq 1$. Note that the fact that g_n remains undefined in $s = s_1$ causes no problem because F^{-1} puts no mass at s_1.

Taking the set E_n as in (2.14) we write

$$V_n - W_n = \int_{E_n} g_n(s)\, dF^{-1}(s) + \int_{E_n{}^c} g_n(s)\, dF^{-1}(s)\,, \tag{2.35}$$

and hence that

$$\mathscr{E}(V_n - W_n)^2 \chi_n{}^c \leqq 2\mathscr{E}\left(\int_{E_n} g_n(s)\, dF^{-1}(s) \cdot \chi_n{}^c\right)^2 + 2\mathscr{E}\left(\int_{E_n{}^c} g_n(s)\, dF^{-1}(s) \cdot \chi_n{}^c\right)^2 . \tag{2.36}$$

On the set where $\chi_n{}^c = 1$, we have that $|g_n(s)| = \mathscr{O}((\Gamma_n(s) - s)^2) = \mathscr{O}(n^{-1}\log n)$ as $n \to \infty$, uniformly for all $0 < s < 1$ except $s = s_1$. Using the Lipschitz condition for F^{-1} we find that the first right-hand term of (2.36) is of order $\mathscr{O}(n^{-\frac{3}{2}})$ as $n \to \infty$. On the set $E_n{}^c$ we have that, because $\chi_n{}^c = 1$, s_1 is not contained in the closed interval between s and $\Gamma_n(s)$ for all sufficiently large n. Together with (2.34) and the Lipschitz condition for J' this implies for $s \in E_n{}^c$ and some constant $A > 0$ that $|g_n(s)| \leqq A|\Gamma_n(s) - s|^{\frac{3}{2}}$ for all sufficiently large n. Now we can simply repeat the argument given in (2.30), (2.31) and the remark following it. Hence we can conclude that $\mathscr{E}(V_n - W_n)^2\chi_n{}^c = \mathscr{O}(n^{-\frac{3}{2}})$ as $n \to \infty$. From this we find easily that also $\mathscr{E}(V_n - W_n - \mathscr{E}(V_n - W_n))^2\chi_n{}^c = \mathscr{O}(n^{-\frac{3}{2}})$ as $n \to \infty$. This completes the proof of the lemma. □

To conclude this section we remark that to show that $P(|T_n{}^* - S_n{}^*| \geqq n^{-\frac{1}{2}}) = \mathscr{O}(n^{-\frac{1}{2}})$ as $n \to \infty$, first use Lemma 2.2 to see that $P(|T_n{}^* - V_n{}^*| \geqq 2^{-1}n^{-\frac{1}{2}}) = \mathscr{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. Next apply Lemma 2.3 to find that $P(|V_n{}^* - S_n{}^*| \geqq 2^{-1}n^{-\frac{1}{2}}) = \mathscr{O}(n^{-\frac{1}{2}})$ as $n \to \infty$. Hence, since the conditions of Lemma 2.3 imply those of Lemma 2.2, $P(|T_n{}^* - S_n{}^*| \geqq n^{-\frac{1}{2}}) = \mathscr{O}(n^{-\frac{1}{2}})$ as $n \to \infty$, is shown to hold under the conditions of Lemma 2.3.

3. The order of normal approximation for $S_n{}^*$. In this section we shall show that the conditions of Theorem 1 ensure that the normal approximation for $S_n{}^*$ is of order $n^{-\frac{1}{2}}$. As we have already shown in Section 2 that, under the conditions of Lemma 2.3, we may approximate $T_n{}^*$ by $S_n{}^*$, the proof of Theorem 1 will then be completed.

The rv $S_n{}^*$ is given by $S_n{}^* = \mathscr{I}_{1n} + \mathscr{I}_{2n}$, where $\mathscr{I}_{mn} = I_{mn}/\sigma(S_n)$ for $m = 1, 2$ and all $n \geqq 1$. For convenience we shall write $\sigma_n = \sigma(S_n)$. Since our proof will depend on characteristic functions (ch.f.) let us denote by $\rho_n{}^*$ and ρ_{1n} the ch.f. of $S_n{}^*$ and $\mathscr{I}_{1n}$. The ch.f. of a summand of $n\sigma_n\mathscr{I}_{1n}$, that is of

$$-\int_0^1 J(s)(\chi_{(0,s]}(U_1) - s)\, dF^{-1}(s) \tag{3.1}$$

will be denoted by ρ. Clearly we have $\rho_{1n}(t) = \rho^n(t/n\sigma_n)$ for all t and $n \geqq 1$.

Following Bickel (1974) we shall first show that there exist $\varepsilon_1 > 0$, D_1 and a natural number n_1, depending on J and F but not on n, such that for all $n \geqq n_1$

$$\int_{|t| < \varepsilon_1 n^{\frac{1}{2}}} |\rho_{1n}(t) - e^{-t^2/2}| \cdot |t|^{-1}\, dt \leqq D_1 n^{-\frac{1}{2}} . \tag{3.2}$$

Secondly we show that there exist $\varepsilon_2 > 0$, D_2 and a natural number n_2, depending on J and F but not on n, such that for all $n \geqq n_2$

$$\int_{|t| < \varepsilon_2 n^{\frac{1}{2}}} |\rho_n{}^*(t) - \rho_{1n}(t)| \cdot |t|^{-1}\, dt \leqq D_2 n^{-\frac{1}{2}} . \tag{3.3}$$

The Berry–Esseen bound of order $n^{-\frac{1}{2}}$ for S_n^* then follows directly from (3.2), (3.3) and the usual argument based on Esseen's smoothing lemma (see, e.g., Feller (1966)).

We first prove (3.2):

LEMMA 3.1. *Let* $\mathscr{E}|X_1|^3 < \infty$ *and suppose that condition* (1) *of Theorem* 1 *is satisfied. Then* $\sigma^2(J, F) > 0$ *implies* (3.2).

PROOF. To start with the proof we note that the conditions of Lemma 2.3 are satisfied. Since it was already shown in the proof of Lemma 2.3 that $0 < \lim_{n\to\infty} n\sigma^2(V_n) = \sigma^2(J, F) < \infty$ and that $\sigma^2(V_n - S_n) = \mathscr{O}(n^{-\frac{3}{2}})$ as $n \to \infty$ it follows that $0 < \lim_{n\to\infty} n\sigma_n^{\,2} = \sigma^2(J, F) < \infty$. However, to prove the lemma we shall need the more precise result that

$$\frac{\sigma^2(J, F)}{n\sigma_n^{\,2}} = 1 + \mathscr{O}(n^{-1}) \qquad \text{as} \quad n \to \infty . \tag{3.4}$$

To see that (3.4) holds, note first that, using the boundedness of J and J' (on its domain), the continuity of F^{-1} at the points where J' does not exist, the finiteness of $\mathscr{E}|X_1|^{2+\varepsilon}$ for some $\varepsilon > 0$, and applying Fubini's theorem, we find that $\mathscr{E}I_{1n} = \mathscr{E}I_{2n} = 0$ and $\mathscr{E}I_{1n}I_{2n} = 0$. Hence the covariance between I_{1n} and I_{2n} is zero. This implies that

$$\sigma_n^{\,2} = \sigma^2(I_{1n}) + \sigma^2(I_{2n}) . \tag{3.5}$$

Note also that $\sigma^2(I_{1n}) = n^{-1}\sigma^2(J, F)$ and $\sigma^2(I_{2n}) = \mathscr{O}(n^{-2})$ as $n \to \infty$. Combining this with (3.5) we have proved (3.4). Hence

$$\mathscr{I}_{1n} = \tau_n I_{1n}^* \tag{3.6}$$

where $I_{1n}^* = I_{1n}/\sigma(I_{1n})$ and $\tau_n = 1 + \mathscr{O}(n^{-1})$ as $n \to \infty$. Remark that I_{1n}^* is a properly standardized sum of independent, identically distributed, random variables.

Secondly we will show that the summands of $n\sigma_n\tau_n I_{1n}^*$ (that is of (3.1)) have finite absolute third moment. Note that

$$\begin{aligned} &|\textstyle\int_0^1 J(s)(\chi_{(0,s]}(U_1) - s)\, dF^{-1}(s)| \\ &\qquad \leqq \|J\|(\textstyle\int_{(0,U_1)} s\, dF^{-1}(s) + \int_{[U_1,1)} (1 - s)\, dF^{-1}(s)) . \end{aligned} \tag{3.7}$$

Using integration by parts, the finiteness of $\mathscr{E}|X_1|^3 < \infty$, and applying the c_r-inequality (see Loève (1955)), we find that

$$\begin{aligned} \mathscr{E}(\textstyle\int_{(0,U_1)} s\, dF^{-1}(s))^3 &= \mathscr{E}|U_1F^{-1}(U_1) - \textstyle\int_0^{U_1} F^{-1}(s)\, ds|^3 \\ &\leqq 4(\mathscr{E}|U_1F^{-1}(U_1)|^3 + \mathscr{E}(\textstyle\int_0^{U_1} |F^{-1}(s)|\, ds)^3) \\ &\leqq 4(\mathscr{E}|X_1|^3 + (\mathscr{E}|X_1|)^3) < \infty . \end{aligned} \tag{3.8}$$

Now (3.7), (3.8) and a symmetry argument ensure that the summands of $n\sigma_n\tau_n I_{1n}^*$ have finite absolute third moment.

We are now in a position to prove (3.2). Remark first that using (3.6) and applying a change of variables we get

$$(3.9)\qquad \int_{|t|<\varepsilon_1 n^{\frac{1}{2}}} |\rho_{1n}(t) - e^{-t^2/2}|\,|t|^{-1}\,dt \leqq \int_{|t|<\varepsilon_1 n^{\frac{1}{2}}\tau_n} |\mathscr{E} e^{itI_{1n}^*} - e^{-t^2/2}|\,|t|^{-1}\,dt + \int_{|t|<\varepsilon_1 n^{\frac{1}{2}}\tau_n} |e^{-t^2/2} - e^{-t^2/2\tau_n^2}|\,|t|^{-1}\,dt\,.$$

Since I_{1n}^* is a properly standardized sum of independent, identically distributed, random variables with finite absolute third moment and $\tau_n = \mathscr{O}(1)$ as $n \to \infty$, we can simply follow the argument leading to the Berry–Esseen theorem (Feller (1966)) to see that the first integral on the right-hand side of (3.9) is $\mathscr{O}(n^{-\frac{1}{2}})$, as $n \to \infty$.

To treat the second integral on the right-hand side of (3.9) we note that because $\tau_n = 1 + \mathscr{O}(n^{-1})$ as $n \to \infty$ we have from an application of the mean value theorem that for all sufficiently large n

$$\int_{|t|<\varepsilon_1 n^{\frac{1}{2}}\tau_n} |e^{-t^2/2} - e^{-t^2/2\tau_n^2}|\,|t|^{-1}\,dt \leqq An^{-1} \int_{-\infty}^{\infty} |t| e^{-t^2/4}\,dt$$

holds for some constant $A > 0$. This completes the proof of the lemma. □

Next we shall be concerned with the problem of showing that (3.3) holds under appropriate conditions. To estimate $|\rho_n^*(t) - \rho_{1n}(t)|$ is a rather delicate matter. We start with the very simple remark that since $|\rho_n^*(t) - \rho_{1n}(t)| = |\mathscr{E} e^{it\mathscr{I}_{1n}}(e^{it\mathscr{I}_{2n}} - 1)|$ we have (see Bickel (1974)) for all t and any m and $n \geqq 1$

$$(3.10)\qquad |\rho_n^*(t) - \rho_{1n}(t)| \leqq \left|\sum_{l=1}^{2m-1} \frac{(it)^l}{l!} \mathscr{E} e^{it\mathscr{I}_{1n}}(\mathscr{I}_{2n})^l\right| + \frac{t^{2m}}{(2m)!}\mathscr{E}(\mathscr{I}_{2n})^{2m}\,.$$

Estimates for $|\mathscr{E} e^{it\mathscr{I}_{1n}}(\mathscr{I}_{2n})^l|$ and $|\mathscr{E}(\mathscr{I}_{2n})^{2m}|$ which are adequate for our purposes will be given in the following lemma. The basic idea of this lemma is similar to that of Lemmas 6.2 and 6.3 of Bickel (1974) (see also Bjerve (1977) where the same idea is exploited).

LEMMA 3.2. *Suppose the conditions* (1) *and* (2) *of Theorem* 1 *are satisfied. Then* $\sigma^2(J, F) > 0$ *implies that there exists a constant* $A > 0$, *depending on J and F but not on l, m and n, such that for all t and any* $n \geqq 1$

(i) $|\mathscr{E} e^{it\mathscr{I}_{1n}}\mathscr{I}_{2n}| \leqq At^2 n^{-\frac{1}{2}} |\rho(t/n\sigma_n)|^{n-2}$

(ii) $|\mathscr{E} e^{it\mathscr{I}_{1n}}(\mathscr{I}_{2n})^l| \leqq A^l n^{\bar{l}/2} |\rho(t/n\sigma_n)|^{n-2l}$ *for* $1 \leqq 2l \leqq n$,

(iii) $\mathscr{E}(\mathscr{I}_{2n})^{2m} \leqq A^{2m} n^{-m} m^{2m}$ *for* $1 \leqq m \leqq n$.

PROOF. For convenience we shall write

$$(3.11)\qquad g(U_i) = -\int_0^1 J(s)(\chi_{(0,s]}(U_i) - s)\,dF^{-1}(s) \qquad \text{for} \quad 1 \leqq i \leqq n$$

and

$$(3.12)\qquad h(U_i, U_j) = -\int_0^1 J'(s)(\chi_{(0,s]}(U_i) - s)(\chi_{(0,s]}(U_j) - s)\,dF^{-1}(s) \qquad \text{for} \quad 1 \leqq j < i \leqq n\,.$$

It follows from this, (2.1)—(2.3) and the definitions of $\mathscr{I}_{1n}$ and $\mathscr{I}_{2n}$ given earlier in this section that

$$(3.13)\qquad \mathscr{I}_{1n} = (n\sigma_n)^{-1} \sum_{i=1}^n g(U_i)\,, \qquad \mathscr{I}_{2n} = (n^2\sigma_n)^{-1} \sum_{i=1}^n \sum_{j=1}^{i-1} h(U_i, U_j)\,.$$

To prove statement (i) we follow Bickel's idea (see Bickel (1974)) and remark that

$$\begin{aligned}|\mathscr{E}e^{it\mathscr{I}_{1n}}\mathscr{I}_{2n}| &= |(n^2\sigma_n)^{-1}\textstyle\sum_{i=1}^n\sum_{j=1}^{i-1}\mathscr{E}e^{it\mathscr{I}_{1n}}h(U_i, U_j)| \\ &\leqq \sigma_n^{-1}\cdot|\rho(t/n\sigma_n)|^{n-2} \\ &\quad\times \textstyle\int_0^1 |J'(s)|\cdot|\mathscr{E}e^{(it/n\sigma_n)g(U_1)}(\chi_{(0,s]}(U_1)-s)|^2\,dF^{-1}(s)\,,\end{aligned} \tag{3.14}$$

where the interchange of expectation and integral follows from an application of Fubini's theorem. The validity of this application follows from the finiteness of $\mathscr{E}|X_1|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$ (as implied by condition (2)), the boundedness of J' on its domain and the continuity of F^{-1} at the points where J' is undefined. Thus we have for $0 < s < 1$ and $n \geqq 1$

$$\begin{aligned}|\mathscr{E}e^{(it/n\sigma_n)g(U_1)}(\chi_{(0,s]}(U_1)-s)|^2 &\leqq t^2(n\sigma_n)^{-2}(\mathscr{E}|g(U_1)|\cdot|\chi_{(0,s]}(U_1)-s|)^2 \\ &\leqq t^2(n\sigma_n)^{-2}\mathscr{E}g^2(U_1)\cdot s(1-s)\,.\end{aligned} \tag{3.15}$$

Because as in the proof of Lemma 3.1 the conditions of Lemma 2.3 are satisfied we can repeat the argument given in the first part of that proof to find that $0 < \lim_{n\to\infty} n\sigma_n^2 = \sigma^2(J, F) < \infty$, $\sigma^2(I_{1n}) = n^{-1}\sigma^2(J, F)$ and hence that $\sigma^2(g(U_1)) = \sigma^2(J, F)$. We can conclude that for some constant $A > 0$ the left-hand side of (3.15) is bounded by $At^2n^{-1}s(1-s)$ for $0 < s < 1$, all t and $n \geqq 1$. In view of (3.14) we have obtained statement (i).

To prove statement (ii) we note that for $l \geqq 1$

$$(\mathscr{I}_{2n})^l = (n^2\sigma_n)^{-l}\textstyle\sum_{(i_\nu, j_\nu);\nu=1,\cdots,l}\prod_{\nu=1}^l h(U_{i_\nu}, U_{j_\nu})\,,$$

where the summation is over all pairs (i_ν, j_ν), $1 \leqq j_\nu < i_\nu \leqq n$, $\nu = 1, \cdots, l$. Following again Bickel's idea (see Bickel (1974)) we note that this implies

$$|\mathscr{E}e^{it\mathscr{I}_{1n}}(\mathscr{I}_{2n})^l| \leqq (n^2\sigma_n)^{-l}|\rho(t/n\sigma_n)|^{n-2l}\cdot\mathscr{E}(\textstyle\sum_{i=1}^n\sum_{j=1}^{i-1}|h(U_i, U_j)|)^l\,. \tag{3.16}$$

Applying the c_r-inequality (see Loève (1955)) and using (3.12) we find

$$\mathscr{E}(\textstyle\sum_{i=1}^n\sum_{j=1}^{i-1}|h(U_i, U_j)|)^l \leqq n^{2l}\mathscr{E}|h(U_1, U_2)|^l\,. \tag{3.17}$$

Finally note that it follows from (3.12) that

$$\mathscr{E}|h(U_1, U_2)|^l \leqq (\textstyle\int_0^1 |J'(s)|\,dF^{-1}(s))^l\,. \tag{3.18}$$

Combining this with condition (2) of Theorem 1 and using (3.16) and (3.17) we have proved statement (ii).

The proof of statement (iii) is essentially that of Lemma 6.2 of Bickel (1974). We use (3.18) and condition (2) of Theorem 1 to guarantee the existence of some constant $B > 0$ such that $\mathscr{E}|h(U_1, U_2)|^{2m} \leqq B^{2m}$ (because in Bickel (1974) h is bounded at the outset, Bickel does not encounter this problem). This completes the proof of lemma. □

We are now in a position to prove (3.3).

LEMMA 3.3. *Suppose the conditions* (1) *and* (2) *of Theorem* 1 *are satisfied. Then* $\sigma^2(J, F) > 0$ *implies* (3.3).

PROOF. The proof is essentially Bickel's proof. See Bickel (1974), pages 17 and 18. Remark first that it follows directly from Lemma 3.2 and the conditions of this lemma that the statements (i), (ii) and (iii) of Lemma 3.2 hold.

It follows from statement (iii) of Lemma 3.2 that for $|t| < \varepsilon_2 n^{\frac{1}{2}}$

$$\frac{t^{2m}}{(2m)!}\mathscr{E}(\mathscr{J}_{2n})^{2m} \leqq \varepsilon_2^{2m} n^m (2m)^{-2m} e^{2m} A^{2m} n^{-m} m^{2m} \leqq (2\varepsilon_2 A)^{2m} .$$

Following Bickel (1974) we take $\varepsilon_2 = p/(2A)$ for some $0 < p < 1$ and $m = ([\log n] + 1)/2|\log p| \wedge n$ to obtain that

$$\frac{t^{2m}}{(2m)!}\mathscr{E}(\mathscr{J}_{2n})^{2m} \leqq p^{2m} < n^{-1} \qquad \text{for} \quad |t| < \varepsilon_2 n^{\frac{1}{2}} . \tag{3.19}$$

Because ρ is the ch.f. of a rv with expectation zero and variance $0 < \sigma^2(J, F) < \infty$ and $\lim_{n\to\infty} n\sigma_n^2 = \sigma^2(J, F)$ (see the proof of Lemma 3.1), there exists, for p sufficiently small, a $\tau > 0$ such that for $|t| < \varepsilon_2 n^{\frac{1}{2}}$

$$\log|\rho(t/n\sigma_n)| \leqq -\frac{\tau t^2}{n} . \tag{3.20}$$

From (3.10) with $m = 1$, and Lemma 3.2 (i) and (iii) we have for all t and $n \geqq 1$

$$\begin{aligned} |\rho_n^*(t) - \rho_{1n}(t)| &\leqq |t| \cdot |\mathscr{E} e^{it\mathscr{J}_{1n}} \mathscr{J}_{2n}| + \frac{t^2}{2}\mathscr{E}(\mathscr{J}_{2n})^2 \\ &\leqq A|t|^3 n^{-\frac{1}{2}} |\rho(t/n\sigma_n)|^{n-2} + A^2 t^2 n^{-1} . \end{aligned}$$

Combining this with (3.20) we find that

$$\int_{|t|<n^{\frac{1}{2}}} |\rho_n^*(t) - \rho_{1n}(t)| \cdot |t|^{-1}\, dt = \mathscr{O}(n^{-\frac{1}{2}}) \qquad \text{as} \quad n \to \infty . \tag{3.21}$$

We also have, using (3.20) and statement (ii) of Lemma 3.2, that for $n^{\frac{1}{2}} \leqq |t| < \varepsilon_2 n^{\frac{1}{2}}$ and $l < 2m$

$$|\mathscr{E} e^{it\mathscr{J}_{1n}} (\mathscr{J}_{2n})^l| \leqq A^l n^{l/2} \exp(-\tau n^{\frac{1}{2}}(1 - 4m/n)) .$$

But then we obtain for $n^{\frac{1}{2}} \leqq |t| < \varepsilon_2 n^{\frac{1}{2}}$

$$\left| \sum_{l=1}^{2m-1} \frac{(it)^l}{l!} \mathscr{E} e^{it\mathscr{J}_{1n}} (\mathscr{J}_{2n})^l \right| = \mathscr{O}(n^{-1}) \qquad \text{as} \quad n \to \infty . \tag{3.22}$$

Now combine (3.19), (3.21) and (3.22) with (3.10). This completes the proof of the lemma. □

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REFERENCES

BICKEL, P. J. (1974). Edgeworth expansions in nonparametic statistics. *Ann. Statist.* **2** 1–20.

BJERVE, S. (1977). Error bounds for linear combinations of order statistics. *Ann. Statist.* **5** 237–255.

DVORETZKY, A., KIEFER, J. and WOLFOWITZ, J. (1956). Asymptotic minimax character of sample distribution function and of the classical multinomial estimator. *Ann. Math. Statist.* **27** 642-669.

ESARY, J. D., PROSCHAN, F. and WALKUP, D. W. (1967). Association of random variables, with applications. *Ann. Math. Statist.* **38** 1466-1474.

FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications*, **2**. Wiley, New York.

LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137-1153.

LOÈVE, M. (1955). *Probability Theory*. Van Nostrand, Princeton.

SHORACK, G. R. (1972). Functions of order statistics. *Ann. Math. Statist.* **43** 412-427.

STIGLER, S. M. (1969). Linear functions of order statistics. *Ann. Math. Statist.* **40** 770-788.

STIGLER, S. M. (1974). Linear functions of order statistics with smooth weight functions. *Ann. Statist.* **2** 676-693.

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