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DISCRETE-TIME STABLE PROCESSES AND THEIR CERTAIN PROPERTIES

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In this paper we derive the characteristic functions of multivariate stable distributions; specifically the canonical representation of symmetric stable laws is given. Based on that representation, we construct linear stable processes (which include autoregressive stable processes) and stable processes with spectral representation. A sufficient condition for linear stable processes to be regular is given; the complete regularity of autoregressive stable processes is proved. Furthermore, we derive the asymptotic distribution of the Fourier transform of a sample from stable processes with spectral representation.

0. Introduction. The present paper attempts to construct various (discrete-time) stable processes through the canonical representation of multivariate stable laws: throughout this construction, concrete models which seem to be practically useful are suggested and their probabilistic properties are examined. A construction of stable processes by means of characteristic functionals can be seen in Hida (1970). His study is, however, confined to stable processes with independent increments; in this paper we try to construct stable processes with "correlation" and to investigate the conditions under which statistical inference can be made for those processes.

To be specific, in Section 1 we establish the canonical representations of multivariate stable distributions (Theorem 1.2 to 1.4). That section introduces, in particular, the concept of symmetric multivariate stable distributions and shows that their characteristic functions have a simple representation (Theorems 1.3 and 1.4). In Section 2 we construct discrete-time stable processes and examine their stationarity. In particular, we give two typical examples: linear stable processes (which include autoregressive stable processes) and stable processes with spectral representation. Namely, let $\{\varepsilon_t, t \in I\}$ be a family of i.i.d. random variables with a stable distribution of exponent α: that section proves that the equations $X_t = \sum_{i=0}^{\infty} \gamma_i \varepsilon_{t-i}$ $(t \in I)$ generate a stationary stable process $\{X_t : t \in I\}$ if $\sum_{i=0}^{\infty} |\gamma_i|^{\alpha-\delta} < \infty$ for some $\delta > 0$ $(\delta < \alpha)$ when $0 < \alpha < 1$, or if $\sum_{i=0}^{\infty} |\gamma_i| < \infty$ when $\alpha > 1$. Those processes are called linear stable processes and their regularities are examined in Section 3. On the other hand, in the same section we will show that the spectral representation known in the theory of second-order stationary processes has an analogue for a certain class of stable

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processes. These processes will be termed stable processes with spectral representation; the characteristic function $C(u_1, \cdots, u_p)$ of a finite sample from those processes will be shown to have a representation of the form $C(u_1, \cdots, u_p) = \exp[-\int_0^\pi \{|\sum_{j=1}^p \cos(\lambda_j \omega) u_j|^\alpha + |\sum_{j=1}^p \sin(\lambda_j \omega) u_j|^\alpha\} dF(\omega)]$, where F is a finite measure with a bounded, Riemann-integrable density.

The concepts of regularity and complete regularity of stationary processes were introduced by Rozanov (1967). In Section 3 we examine the regularities of linear stable processes. Theorem 3.1 provides a sufficient condition for linear stable processes to be regular; Theorem 3.2 states that autoregressive stable processes ($\alpha > 1$) are completely regular if the usual condition for eigenvalues of the difference equations is satisfied.

In Section 4, we introduce the kernel $K_n(\omega) = (2\pi)^{-1} C_n (2n+1)^{1-\alpha} |\sin\{(n+\frac{1}{2})\omega\}/\sin(\omega/2)|^\alpha$ $(1 < \alpha \leqq 2)$ and prove that it has a similar property to the Fejér kernel as far as pointwise convergence is concerned (Lemma 4.2). Furthermore, defining $D_n(\omega_j) = (2\pi)^{-1/\alpha} C_n^{1/\alpha} (2n+1)^{(1-\alpha)/\alpha} \sin\{(n+\frac{1}{2})(\omega_j - \omega)\}/\sin((\omega_j - \omega)/2))$, we show that the kernel $|\sum_{j=1}^p u_j D_n(\omega_j)|^\alpha$ reals the values of the $f(\omega_j)$ in the sense that $\int_{-\pi}^{\pi} |\sum_{j=1}^p u_j D_n(\omega_j)|^\alpha f(\omega)\, d\omega$ converges pointwise to $\sum_{j=1}^p |u_j|^\alpha f(\omega_j)$ under fairly general conditions (Theorem 4.1). That result will be employed to show that the Fourier transformation of the sample generated by stable processes with spectral representation reveals the structure of the measure F in the canonical representation to a certain extent (Theorem 4.3).

1. Multivariate stable distributions. The following theorem concerning the multivariate infinitely divisible distributions is well known (cf., for example, Parthasarathy (1967)).

THEOREM 1.1. *A characteristic function $f(\mathbf{u})$ is that of a (finite-demensional) infinitely divisible distribution if and only if it has a representation of the form*

$$\log f(\mathbf{u}) = i \sum_{j=1}^p \beta_j u_j - \tfrac{1}{2} \sum_{i=1}^p \sum_{j=1}^p \sigma_{ij} u_i u_j + \int \left(e^{i \sum u_j x_j} - 1 - \frac{i \sum_j u_j x_j}{1 + \sum_j x_j^2}\right) \frac{1 + \sum x_j^2}{\sum x_j^2} \nu(d\mathbf{x})$$

where $\mathbf{u}$ and $\mathbf{x}$ are real p-dimensional vectors the jth *element of which are denoted by u_j and x_j, $\{\sigma_{ij}; i, j = 1, 2, \cdots, p\}$ is a symmetric, nonnegative-definite matrix, and ν is a finite measure on R^p assigning zero measure to the origin ($p \geqq 1$). The β_i and the measure ν are uniquely determined in the representation above. Furthermore, if the matrix $\{\sigma_{ij}\}$ is positive definite, $\{\sigma_{ij}\}$ is also unique.*

DEFINITION 1.1. A p-dimensional random vector $\mathbf{X}$ is said to have a stable distribution if for every positive integer k, and $\mathbf{X}_1, \cdots, \mathbf{X}_k$ independent with the same distributions as $\mathbf{X}$, there are a constant $a_k > 0$ and a vector $\mathbf{b}_k$ such that

(1.1) $$\mathscr{L}(\mathbf{X}_1 + \cdots + \mathbf{X}_k) = \mathscr{L}(a_k \mathbf{X} + \mathbf{b}_k)$$

($\mathscr{L}$ denotes the distribution).

REMARK. The same line of argument used in the univariate case applies here and a_k can be expressed as k^λ for some $\lambda > 0$. This fact is given without proof.

Let $f(\mathbf{u})$ be the characteristic function of a stable distribution. Since a stable distribution is also infinitely divisible,

$$\begin{aligned}\psi(\mathbf{u}) &= \log f(\mathbf{u}) \\ &= i \sum \beta_i u_i - \tfrac{1}{2} \textstyle\sum_i \sum_j \sigma_{ij} u_i u_j \\ &\quad + \int \left(e^{i \sum u_i x_i} - 1 - \frac{i \sum u_i x_i}{1 + \sum x_i^2}\right) \frac{1 + \sum x_i^2}{\sum x_i^2} \nu(d\mathbf{x}) .\end{aligned}$$

Suppose that $\sigma_{ij} = 0$, for $i, j = 1, 2, \cdots, p$. From Definition 1.1, $[\exp\{\psi(\mathbf{u})\}]^k = \exp(i \sum_j b_{k,j} u_j) \exp[\psi(a_k \mathbf{u})]$, where $b_{k,j}$ denotes the jth element of the vector $\mathbf{b}_k$. Thus

$$k\psi(\mathbf{u}) = i \textstyle\sum_j b_{k,j} u_j + \psi(a_k \mathbf{u}) . \tag{1.2}$$

Define μ and μ_k to be measures on $\mathscr{B}(R^p)$ such that, for $B \in \mathscr{B}(R^p)$, $\mu(B) = \int_B (1 + \sum x_j^2)/\sum x_i^2 \nu(d\mathbf{x})$ and $\mu_k(B) = \mu(\mathbf{z};\, a_k \mathbf{z} \in B)$, where $\mathscr{B}$ denotes the Borel field. Between these measures there exists the following relation.

LEMMA 1.1. $k\mu(d\mathbf{x}) = \mu_k(d\mathbf{x})$.

PROOF.

$$\psi(a_k \mathbf{u}) = i a_k \sum \lambda_j u_j + \int \left(e^{i \sum u_j x_j} - 1 - \frac{i \sum u_j x_j}{1 + \sum x_j^2}\right) \mu_k(d\mathbf{x}) ,$$

for suitable constants λ_j. On the other hand, from (1.2),

$$k\psi(\mathbf{u}) = i \sum k\beta_i u_i + \int \left(e^{i \sum u_j x_j} - 1 - \frac{i \sum u_j x_j}{1 + \sum x_j^2}\right) k\mu(d\mathbf{x}) .$$

Then by the uniqueness of representation (see Theorem 1.1), the lemma follows from the relation (1.2). □

Let T_p be the unit sphere $\{\mathbf{x} : \sum_{i=1}^p x_i^2 = 1, \mathbf{x} \in R^p\}$ in R^p and let $\mathbf{y} = (y_1, \cdots, y_p) \in T_p$ where $y_j = x_j/(\sum x_j^2)^{1/2}$ and let $\mathbf{z} = (r, \mathbf{y})$ with $r = (\sum x_i^2)^{1/2}$. Consider the map $\rho : R^p \cap \{0\}' \to (0, \infty) \times T_p$ such that $\rho(\mathbf{x}) = \mathbf{z}$. The map ρ is one-to-one onto $(0, \infty) \times T_p$. Now let ξ and ξ_k be measures defined on $\mathscr{B}[(0, \infty) \times T_p]$ such that, for every $B \in \mathscr{B}[(0, \infty) \times T_p]$, $\xi(B) = \mu(\rho^{-1}(B))$, and $\xi_k(B) = \mu_k(\rho^{-1}(B))$. Then:

LEMMA 1.2. *For any $x \in R^+$, and for any D_p, Borel subset of T_p,*

$$\xi([x, \infty) \times D_p) = (1/x^{1/\lambda})\xi([1, \infty) \times D_p) .$$

PROOF. Let $\alpha > 0$. Then in view of the definition of ξ_k,

$$\xi_k([\alpha, \infty) \times D_p) = \xi\left(\left[\frac{\alpha}{a_k}, \infty\right) \times D_p\right) . \tag{1.3}$$

On the other hand, from Lemma 1.1,

$$\mu_k(\rho^{-1}([\alpha, \infty) \times D_p)) = k\xi([\alpha, \infty) \times D_p) . \tag{1.4}$$

Thus using the fact that $a_k = k^\lambda$, $\lambda > 0$, it follows from (1.3) and (1.4) that $\xi([\alpha/k^\lambda, \infty) \times D_p) = k\xi([\alpha, \infty) \times D_p)$. Putting $\alpha = (k/n)^\lambda$ with some positive integer n,

$$\xi(([k/n)^\lambda, \infty) \times D_p) = (k/n)^{-1}\xi([1, \infty) \times D_p)\,. \tag{1.5}$$

The equation (1.5) implies that for a dense subset of R^+, it is true that

$$\xi([x, \infty) \times D_p) = 1/(x^{1/\lambda})\xi([1, \infty) \times D_p)\,. \tag{1.6}$$

However, since ξ is monotone in x, (1.6) holds for all $x \in R^+$. □

LEMMA 1.3. $\lambda > \frac{1}{2}$.

Define the measure η on $\mathscr{B}(T_p)$ by $\eta(D_p) = \xi([1, \infty) \times D_p)$ for $D_p \in \mathscr{B}(T_p)$. As in the above, denote by $\mathbf{y}$ a point belonging to T_p; $r > 0$, and let $\mathbf{z}$ be the pair $(r, \mathbf{y})$. By virtue of Lemma 1.2, the next theorem is derived as a straightforward generalization of the univariate case of stable distributions. This result is also to be found in Rvaceva (1962).

THEOREM 1.2. *A characteristic function $f(\mathbf{u})$ is stable if and only if it is given by either*

$$\log f(\mathbf{u}) = i \sum \beta_j u_j + \int_{(0,\infty)\times T_p}\left(e^{ir\sum u_i y_i} - 1 - \frac{ir\sum u_i y_i}{1 + r^2}\right)\frac{dr}{r^{1+\alpha}}\,\eta(d\mathbf{y}) \tag{1.7}$$

or

$$\log f(\mathbf{u}) = i \sum \beta_j u_j - \tfrac{1}{2}\sum\sum \sigma_{ij} u_i u_j \tag{1.8}$$

where η is a finite measure on T_p and $0 < \alpha < 2$.

For various applications, Theorem 1.2 seems to be too general; in the rest of this section we explore the canonical representations of symmetric stable laws.

DEFINITION 1.2. The distribution of a random vector $\mathbf{x}$ is called symmetric with respect to $\boldsymbol{\theta}$ if there exists a vector $\boldsymbol{\theta}$ of the same dimension as $\mathbf{x}$ such that $P_r(\mathbf{x} - \boldsymbol{\theta} \in A) = P_r(-\mathbf{x} + \boldsymbol{\theta} \in A)$ for any Borel set A.

LEMMA 1.4. *A stable distribution is symmetric if and only if the measure η in (1.7) satisfies $\eta(B) = \eta(B^+)$ for all $B \in \mathscr{B}(T_p)$, where B^+ is the subset $\{\mathbf{y} \in T_p:\ -\mathbf{y} \in B\}$.*

PROOF. Let $\mathbf{x}$ be a random vector. Then the distribution of $\mathbf{x}$ is symmetric with respect to $\boldsymbol{\theta}$ if and only if for the characteristic function $f(\mathbf{u})$ of $\mathbf{x} - \boldsymbol{\theta}$ it is true that $f(\mathbf{u}) = f(-\mathbf{u})$. On the other hand, since ρ is one-to-one and onto, the measure η is uniquely determined. □

REMARK. If $p = 1$, (1.7) becomes

$$\log f(u_1) = i\beta_1 u_1 + m_1 \int_0^\infty \left(e^{iu_1x_1} - 1 - \frac{iu_1x_1}{1 + x_1^2}\right)\frac{dx_1}{x_1^{1+\alpha}} + m_2 \int_{-\infty}^0 \left(e^{iu_1x_2} - 1 - \frac{iu_1x_2}{1 + x_2^2}\right)\frac{dx_2}{|x_2|^{1+\alpha}}\,. \tag{1.9}$$

Stable distributions for which $m_1 = m_2$ are usually called symmetric; the expression (1.9) is equivalent, in this case, to

$$\log f(u_1) = i\beta u_1 - d|u_1|^\alpha , \tag{1.10}$$

for some positive constant d and a real number β (cf. Breiman (1968), pages 204–207).

THEOREM 1.3. *A stable distribution is symmetric if and only if its* $\log f(\mathbf{u})$ *is given by*

$$\log f(\mathbf{u}) = i \sum \beta_i u_i - \int_{T_p} |\sum u_i y_i|^\alpha \xi(d\mathbf{y}) , \tag{1.11}$$

where $0 < \alpha < 2$, *and* ξ *is a finite measure.*

PROOF. The symmetricity of η implies that

$$\log f(\mathbf{u}) = i \sum \gamma_i u_i + \tfrac{1}{2} \int_{T_p}\Big[\int_{-\infty}^0 \Big(e^{ir(\sum u_i y_i)} - 1 - \frac{ir(\sum u_i y_i)}{1 + r^2}\Big) \frac{dr}{|r|^{1+\alpha}}$$
$$+ \int_0^\infty \Big(e^{ir(\sum u_i y_i)} - 1 - \frac{ir(\sum u_i y_i)}{1 + r^2}\Big) \frac{dr}{r^{1+\alpha}}\Big] \eta(d\mathbf{y}) .$$

However, the term in the brackets corresponds to the last two terms in (1.9) with $m_1 = m_2 = 1$ and $u_1 = \sum u_i y_i$; thus it is equal to $\delta \sum u_i y_i - d|\sum u_i y_i|^\alpha$ for a certain constant δ in view of (1.10). Set $\xi(dy) = \frac{1}{2}\eta(d\mathbf{y})$ and $\beta_i = \gamma_i + \delta \int_{T_p} y_i \eta(d\mathbf{y})$, to obtain (1.11). The sufficiency is obvious. □

There exists another form of representing symmetric stable distributions. Namely,

THEOREM 1.4. *A multivariate distribution is symmetrically stable if and only if its characteristic function* $f(\mathbf{u})$ *is given by*

$$\log f(\mathbf{u}) = i \sum \beta_i u_i - \int_{R^p} |\sum u_i x_i|^\alpha \psi(d\mathbf{x}) \tag{1.12}$$

where ψ *is a (not necessarily finite) measure on* $\mathscr{B}(R^p)$ *such that*

$$\int |\sum x_i^2|^{\alpha/2} \psi(d\mathbf{x}) < \infty . \tag{1.13}$$

PROOF. The necessity of the condition is obvious since in Theorem 1.3, the measure ξ can be regarded as the measure over R^p giving zero measure to $\{T_p\}' \cap R^p$ (the prime denotes the complement). The sufficiency is proved as follows. First symmetrize ψ by defining ψ^* as $\psi(B) + \psi(-B)$ for $B \in \mathscr{B}(R^p)$. Then (1.12) and (1.13) still hold for ψ^*. Let ξ be the measure over $(0, \infty) \times T_p$ induced from ψ^* by the map ρ, and let

$$\xi^*(d\mathbf{y}) = \int_0^\infty r^\alpha \xi(dr, d\mathbf{y}) ; \tag{1.14}$$

then, in view of (1.13), ξ^* is a symmetric, finite measure on $\mathscr{B}(T_p)$. Using (1.14), (1.12) is rewritten as

$$\log f(\mathbf{u}) = \sum \beta_i u_i - \int_{T_p} |\sum u_i y_i|^\alpha \xi^*(d\mathbf{y}) . \tag{1.15}$$ □

EXAMPLE. Suppose $\{X_t : t = 0, \pm 1, \pm 2, \cdots\}$ is a set of independent random variables with the same log-characteristic function given by $i\beta u - c|u|^\alpha$ ($c > 0$ and $0 < \alpha < 2$). Let $\{\alpha_i : i = 0, 1, 2, \cdots\}$ be a sequence of constants such that $\sum_{i=0}^\infty |\alpha_i|^{\alpha-\varepsilon} < \infty$ for some ε ($0 < \varepsilon < \alpha$) if $0 < \alpha \leqq 1$ or $\sum_{i=0}^\infty |\alpha_i| < \infty$ if $1 < \alpha < 2$. Then Y_{t_i} $(= \sum_{j=0}^\infty \alpha_j X_{t_i-j})$, $i = 1, 2, \cdots, p$, have a symmetric multivariate distribution: let $Y_{t_i}^n = \sum_{j=0}^n \alpha_j X_{t_i-j}$; then each of the $Y_{t_i}^n$ converges a.e. in view of Kolmogorov's three series theorem. Set $Y_i = \sum_{j=0}^\infty \alpha_j X_{i-j}$, $i = 1, 2, \cdots, p$ and consider the characteristic function $g(\mathbf{u})$ of $Y_1, \cdots, Y_p$; then in view of the Lebesgue bounded convergence theorem, it holds that

$$g(\mathbf{u}) = \exp\{i\beta(\textstyle\sum_{i=0}^\infty \alpha_i)(\sum_{j=1}^p u_i) - c \sum_{k=-\infty}^p |\sum_{i=k}^p \alpha_{i-k} u_i|^\alpha\} . \tag{1.16}$$

Now let $\alpha_i = 0$ for $i < 0$. Then

$$\textstyle\sum_{k=-\infty}^p |\sum_{i=k}^p \alpha_{i-k} u_i|^\alpha = \sum_{k=-\infty}^p |\sum_{i=1}^p \alpha_{i-k} u_i|^\alpha .$$

Therefore (1.16) can be identified with (1.12) if the measure ψ in (1.12) is interpreted as giving the mass c to each point $(\alpha_{1-k}, \alpha_{2-k}, \cdots, \alpha_{p-k})$, $k = \cdots, p-1$. Though the number of the points having positive masses is not finite, it is true that, since $\alpha/2 < 1$,

$$\textstyle\sum_{k=-\infty}^p |\sum_{i=1}^p |\alpha_{i-k}|^2|^{\alpha/2} \leqq \sum_{k=-\infty}^p \sum_{i=1}^p |\alpha_{i-k}|^\alpha < \infty ,$$

which proves that (1.13) is satisfied.

2. Stable processes and stationarity. From now on, we exclusively consider symmetric stable distributions defined in the previous section. Then in view of Theorem 1.4 it may be natural to define stable processes as this:

DEFINITION 2.1. A triple $(R^\infty, \mathscr{B}(R^\infty), \mu)$ is called a (discrete-time) stable process if for any J, finite subset of I (the set of all integers), the characteristic function $f_J(\mathbf{u})$ of μ_J is represented by

$$\log f_J(\mathbf{u}) = i \textstyle\sum_{j\in J} \beta_j u_j - \int \cdots \int_{\prod_{j\in J} R_j} |\sum_{i\in J} u_j x_j|^\alpha \psi_J(d\mathbf{x}) \tag{2.1}$$

with a measure ψ_J defined on $\mathscr{B}(\prod_{i\in J} R_i)$ such that

$$\int \cdots \int |\textstyle\sum_{j\in J} x_j^2|^{\alpha/2} \psi_J(d\mathbf{x}) < \infty , \tag{2.2}$$

where $\mathscr{B}(R^\infty)$ denotes the σ-field generated from the cylinder subsets of R^∞, and $\mu_J(B)$ is defined as $\mu_J(B) = \mu(B \times \prod_{i\in I-J} R_i)$ for $B \in \mathscr{B}(\prod_{j\in J} R_j)$ $(R_i = R)$.

The Kolmogorov conditions of consistency are characterized in terms of characteristic functions as follows.

DEFINITION 2.2. Call a class of characteristic functions $\{f_J(\mathbf{u})\}$ consistent, if it satisfies the following conditions (2.3) and (2.4): if $\alpha_1, \cdots, \alpha_n$ is a permutation of $1, 2, \cdots, n$,

$$f_{t_1,\cdots,t_n}(u_1, \cdots, u_n) = f_{t_{\alpha_1},\cdots,t_{\alpha_n}}(u_{\alpha_1}, \cdots, u_{\alpha_n}) \tag{2.3}$$

and if $m < n$,

$$f_{t_1,\cdots,t_m}(u_1, \cdots, u_m) = f_{t_1,\cdots,t_n}(u_1, \cdots, u_m, 0, \cdots, 0) . \tag{2.4}$$

REMARK. Based on a consistent class of characteristic functions, the stationarity is characterized: given a consistent class of characteristic functions $\{f_J(\mathbf{u})\}$ the stochastic process consistent with that class is stationary if and only if for every $\mathbf{u} \in \prod_{j \in J} R_j$ and for any integer k, $f_J(\mathbf{u}) = f_{J+k}(\mathbf{u})$.

To give concrete examples of stable stochastic processes, first of all, linear stable processes are constructed as follows: let $\{X_i, i \in I\}$ be a sequence of independent random variables with the common log-characteristic function given by $i\beta u - c|u|^\alpha$, $c > 0$ and $0 < \alpha < 2$. The example in the previous section showed that the Y_t generated by $Y_t = \sum_{i=0}^{\infty} \gamma_i X_{t-i}$, $t = t_1, t_2, \cdots, t_k$, have a multivariate stable distribution if $\sum_{i=0}^{\infty} |\gamma_i|^{\alpha-\varepsilon} < \infty$ for some ε $(0 < \varepsilon < \alpha)$ when $0 < \alpha < 1$ or if $\sum_{i=0}^{\infty} |\gamma_i| < \infty$ when $1 < \alpha < 2$. Then it is easy to see that the class of characteristic functions given by (1.16) satisfies (2.3) and (2.4) above; moreover, the process $\{Y_t\}$ is stationary. Call the process $\{Y_t\}$ generated by the scheme $Y_t = \sum \gamma_i X_{t-i}$ *a linear stable process*. Section 3 explores conditions for the ergodicity of linear processes.

Next, consider a special but practically important case of the above example. Namely, that is a finite-order autoregressive stable process. Given real number $\xi_0, \xi_1, \cdots, \xi_p$, assume the function $\sum_{i=0}^{p} \xi_i z^i$ $(\xi_0 = 1)$ has all zeros outside the unit circle in the complex plane. Then the following expansion converges uniformly on the unit circle $\{|z| = 1\}$; that is,

$$1/(\sum \xi_i z^i) = \sum_{i=0}^{\infty} \gamma_i z^i \,. \tag{2.5}$$

Now, suppose that a linear stable process $\{Y_t\}$ is generated by $Y_t = \sum_{i=0}^{\infty} \gamma_i X_{t-i}$, where the γ_i are given by (2.5) and the characteristic function of X_t is given by $\exp\{i\beta u - c|u|^\alpha\}$ $(\alpha > 1)$. Let $Z_t = \sum_{i=0}^{p} \xi_i Y_{t-i}$; then the Z_t are independently identically stably distributed. As a result, $\{Y_t\}$ may be regarded as generated by: $\sum_{i=0}^{p} \xi_i Y_{t-i} = X_t$, where the X_t are independent stable random variables with a common symmetric stable distribution. Call the process $\{Y_t\}$ thus defined a (finite-order) *autoregressive stable process*. The complete regularity of such processes will be examined in the next section.

Denote by $\mathscr{D}$ the space of infinitely differentiable real-valued functions on $[0, \pi]$. Then a generalized stochastic process is defined to be the triple $(\mathscr{D}', \mathscr{B}(\mathscr{D}'), \mu)$ where $\mathscr{D}'$ is the dual of $\mathscr{D}$, and $\mathscr{B}(\mathscr{D}')$ is the σ-field generated by the weak topology on $\mathscr{D}'$. According to Minlos (1959), to each characteristic functional on $\mathscr{D}$, there corresponds a unique generalized stochastic process. For generalized stochastic processes, Gelfand and Vilenkin (1968) introduced the concept of "processes of independent values at every point." Namely, a generalized process $\mathbf{X}$ is a process of independent values at every point if $\langle X, \xi_1 \rangle$ and $\langle X, \xi_2 \rangle$ are independent whenever $\xi_1, \xi_2 \in \mathscr{D}$ and the supports of ξ_1 and ξ_2 are disjoint, where $\langle \ , \ \rangle$ denotes the canonical bilinear form on $\mathscr{D} \times \mathscr{D}'$. Consider the functional on $\mathscr{D}$:

$$C^*(u) = \exp\{-\textstyle\int_0^\pi |u(\omega)|^\alpha \, dF(\omega)\} \,; \qquad u \in \mathscr{D} \,, \tag{2.6}$$

where F is a finite measure on $\mathscr{B}([0, \pi])$ which has a bounded, Riemann-integrable density. Evidently $C^*(u)$ is a characteristic functional which defines a process of independent value at every point $\mathbf{Z}$. The process $\mathbf{Z}$ takes a point in $\mathscr{D}'$ as its sample path. Now let $\mathbf{Z}_1$ and $\mathbf{Z}_2$ be independent processes with the same characteristic functional given by $C^*(u)$. Let g_t and h_t be the maps from $[0, \pi]$ to R such that $g_t(\omega) = \cos \omega t$ and $h_t(\omega) = \sin \omega t$. Consider the random variables

$$X_t = \langle \mathbf{Z}_1, g_t \rangle + \langle \mathbf{Z}_2, h_t \rangle \,, \qquad t \in I \,. \tag{2.7}$$

The representation (2.7) may be regarded as the generalization of the spectral representation of second-order stationary processes to stable processes. For any $t_1, \cdots, t_k \,(\in I)$, $X_{t_1}, \cdots, X_{t_k}$ have a multivariate stable distribution whose characteristic function is given by

$$\begin{aligned} (2.8) \qquad & f(u_{t_1}, \cdots, u_{t_k}) \\ & = \exp[-\textstyle\int_0^\pi \{|\sum u_{t_j} \cos(t_j \omega)|^\alpha + |\sum u_{t_j} \sin(t_j \omega)|^\alpha\} \, dF(\omega)] \,. \end{aligned}$$

Furthermore the class of characteristic functions of the form (2.8) is shown to be consistent, though not stationary. The process $\{X_t\}$ thus obtained will be termed a stable process with spectral representation. In Section 4, we investigate that process.

3. The regularities of linear stable processes. Suppose now that $\{Y_t(\omega) : t \in I\}$ is a stationary process defined on a probability space $(\Omega, \mathscr{B}(\Omega), \mu)$; denote by $\mathscr{F}_{-\infty}^t$ the σ-field generated by subset A of the form: $A = \{\omega \in \Omega : Y_{t_1}(\omega) \in B_{t_1}, \cdots, Y_{t_k}(\omega) \in B_{t_k}\}$ where $t_1, \cdots, t_k \leqq t$, and $B_{t_1}, \cdots, B_{t_k}$ are arbitrary Borel subsets of R; $\mathscr{F}_t^\infty$ and $\mathscr{F}_{-\infty}^\infty$ are defined in a similar way. Now the regularity and the complete regularity are defined respectively as follows.

DEFINITION 3.1. A stationary process $\{Y_t(\omega): t \in I\}$ is called regular if for any $A \in \mathscr{F}_{-\infty}^\infty$,

$$\lim_{t\to\infty} \sup_{B \in \mathscr{F}_{-\infty}^t} |\mu(A \cap B) - \mu(A)\mu(B)| = 0 \,. \tag{3.1}$$

DEFINITION 3.2. A stationary process $\{Y_t(\omega): t \in I\}$ is completely regular if for any t,

$$\lim_{\tau\to\infty} \sup_{A \in \mathscr{F}_{-\infty}^t ; B \in \mathscr{F}_{t+\tau}^\infty} |\mu(A \cap B) - \mu(A)\mu(B)| = 0 \,. \tag{3.2}$$

THEOREM 3.1 *Let $\{Y_t\}$ be a linear stable process generated by $Y_t = \sum_{i=0}^\infty \gamma_i X_{t-i}$ where the X_t are independent stable random variables whose* log-*characteristic function is given by $i\beta u - c|u|^\alpha$. Then $\{Y_t\}$ is regular if $\sum_{i=0}^\infty i|\gamma_i| < \infty$ and* $1 < \alpha < 2$.

The proof of the theorem is broken into a sequence of five steps, each of which is stated as a lemma below. In those lemmas, the x_t are assumed i.i.d. random variables with finite expectation ($E|X_t| = a$) and they need not be stable.

As the notations to be used in the discussions below, let $A_{s_1,\cdots,s_l} = \bigcap_i \{\omega: Y_{s_i}(\omega) \in B_{s_i}\}$ and $A_{t_1,\cdots,t_k} = \bigcap_i \{\omega: Y_{t_i}(\omega) \in B_{t_i}\}$, where $s_1 \leqq s_2 \leqq \cdots \leqq s_l \leqq t_1 \leqq \cdots \leqq t_k$, $t_1 - s_l = m > 0$; the B_{s_i} and B_{t_i} are arbitrary Borel subsets of R. Define the set $A^\varepsilon \,(\in R)$ by $A^\varepsilon = \{x \in R : |x - y| < \varepsilon \text{ and } y \in A\}$. Dividing the sum $\sum \gamma_j X_{t_i-j}$ into parts, let $Y_{t_i,m} = \sum_{j=m+(t_i-t_1)+1}^{\infty} \gamma_j X_{t_i-j}$ and $Y^*_{t_i,m} = Y_{t_i} - Y_{t_i,m}$. Moreover, let $C_m = \{\omega : \max_{i=1,2,\cdots,k} |Y_{t_i,m}| < \varepsilon\}$ and C_m' be the complement of C_m.

LEMMA 3.1.

$$\mu(|Y_{t_i,m}| > \varepsilon) \leqq a \sum_{j=m+(t_i-t_1)+1}^{\infty} |\gamma_j|/\varepsilon .$$

LEMMA 3.2.

$$\mu\{A_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l} \cap C_m'\} \leqq a \sum_{j=m+1}^{\infty} |\gamma_i|/\varepsilon .$$

Let $A^*_{t_1,\cdots,t_k} = \bigcap_i \{Y^*_{t_i}(\omega) \in B^{\varepsilon}_{t_i}\}$ *and* $A^{2\varepsilon}_{t_1,\cdots,t_k} = \bigcap_i \{Y_{t_i}(\omega) \in B^{2\omega}_{t_i}\}$, *then*

LEMMA 3.3.

$$|\mu\{A_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l} \cap C_m\} - \mu\{A^*_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l} \cap C_m\}| \leqq \mu\{A^{2\varepsilon}_{t_1,\cdots,t_k} - A_{t_1,\cdots,t_k}\} .$$

PROOF. Denoting by $A'_{t_1,\cdots,t_k}$ the complement of $A_{t_1,\cdots,t_k}$,

$$\mu\{A^*_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l} \cap C_m\} = \mu\{A_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l} \cap C_m\} + \mu\{A'_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l} \cap C_m \cap A^*_{t_1,\cdots,t_k}\} .$$

On the other hand, it holds that $\mu\{A'_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l} \cap C_m \cap A^*_{t_1,\cdots,t_k}\} \leqq \mu\{A^{2\varepsilon}_{t_1,\cdots,t_k} - A_{t_1,\cdots,t_k}\}$. □

LEMMA 3.4.

$$|\mu\{A_{t_1,\cdots,t_k}\} - \mu\{A^*_{t_1,\cdots,t_k}\}| \leqq \mu\{A^{2\varepsilon}_{t_1,\cdots,t_k} - A_{t_1,\cdots,t_k}\} + 2a \sum_{j=m+1}^{\infty} j|\alpha_j|/\varepsilon .$$

LEMMA 3.5. *For any* $A_{t_1,\cdots,t_k}$,

$$\lim_{m\to\infty} \sup_{A_{s_1,\cdots,s_l}} |\mu(A_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l}) - \mu(A_{t_1,\cdots,t_k})\mu(A_{s_1,\cdots,s_l})| = 0 ,$$

where the supremum is taken over all Borel sets $A_{s_1,\cdots,s_l}$ *and all subsets* $\{s_1, \cdots, s_l\}$ *of* I *such that* $s_1, \cdots, s_l \leqq t_1 - m$.

PROOF. By virtue of Lemmas 3 and 4, it holds that

$$(3.3) \qquad |\mu(A_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l}) - \mu(A_{t_1,\cdots,t_k})\mu(A_{s_1,\cdots,s_l})| \leqq 2\mu(A^{2\varepsilon}_{t_1,\cdots,t_k} - A_{t_1,\cdots,t_k}) + 4a \sum_{j=m+1}^{\infty} j|\gamma_j|/\varepsilon .$$

Now given $\eta (> 0)$ arbitrarily small, ε can be chosen so small that $2\mu(A^{2\varepsilon}_{t_1,\cdots,t_k} - A_{t_1,\cdots,t_k}) < \eta/2$. Besides, since $\sum j|\gamma_j|$ converges by the assumption, $4a \sum_{j=m+1}^{\infty} j|\gamma_j|/\varepsilon$ is made less than $\eta/2$ by taking m large enough. □

In the rest of this section we demonstrate that the autoregressive stable process $\{Y_t\}$ generated by $\sum \xi_i Y_{t-i} = X_t$ is completely regular if all zeros of the function $\sum \xi_i z^i$ are assumed to be outside of the unit circle, and the X_t are i.i.d. stable random variables.

First of all, suppose $\{Y_t : t \in I\}$ is generated by a first-order autoregressive process: $Y_t - \xi Y_{t-1} = X_t$, where $|\xi| < 1$; then,

LEMMA 3.6. *The first-order autoregressive process* $\{Y_t\}$ *is completely regular.*

PROOF. Given t and τ, let $s_1 \leqq \cdots \leqq s_l \leqq t \leqq t+\tau \leqq t_1 \leqq \cdots \leqq t_k$; then, in view of the fact that $\{Y_t\}$ is a Markov process,

$$\mu(A_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l}) = \mu(Y_{t_k} \in B_{t_k} \mid Y_{t_{k-1}} \in B_{t_{k-1}})\mu(Y_{t_{k-1}} \in B_{t_{k-1}} \mid Y_{t_{k-2}} \in B_{t_{k-2}}) \cdots \mu(Y_{t_1} \in B_{t_1} \mid A_{s_1,\cdots,s_l})\mu(Y_{s_l} \in B_{s_l}, \cdots, Y_{s_1} \in B_{s_1}) .$$

Then,

$$|\mu(A_{t_1,\cdots,t_k} \cap A_{s_1,\cdots,s_l}) - \mu(A_{t_1,\cdots,t_k})\mu(A_{s_1,\cdots,s_l})| \leqq |\mu((Y_{t_1} \in B_{t_1}) \cap (Y_{s_l} \in B_{s_l})) - \mu(Y_{t_1} \in B_{t_1})\mu(Y_{s_l} \in B_{s_l})| .$$

The term in the right-hand side, however, tends to 0 as $\tau \to \infty$. □

THEOREM 3.2. *A (finite-order) autoregressive stable process is completely regular.*

PROOF. A finite-order autoregressive process is reduced to a vector-valued first-order autoregressive process to which an argument similar to that of Lemma 6 applies. □

4. Stable processes with spectral representation. Define a function $K_n(\omega)$ on $[-\pi, \pi]$ by

$$(4.1)\qquad K_n(\omega) = 2n+1 \qquad \text{for} \quad \omega = 0 ,$$
$$= (2n+1)^{1-\alpha}\left|\sin\{(n+\tfrac{1}{2})\omega\}\Big/\sin\left(\frac{\omega}{2}\right)\right|^{\alpha} \qquad \text{for} \quad \omega \neq 0 ,$$

where $1 < \alpha \leqq 2$. Obviously when $\alpha = 2$, $K_n(\omega)$ is nothing but the Fejér kernel. For this $K_n(\omega)$,

LEMMA 4.1. (i) $K_n(\omega) = K_n(-\omega)$;
(ii) $0 \leqq K_n(\omega) \leqq 2n+1$;
(iii) $K_n(\omega) \leqq (2n+1)^{1-\alpha}\pi^{\alpha}\omega^{-\alpha}$, for $0 < \omega < \pi$;
(iv) $2\pi \leqq \int_{-\pi}^{\pi} K_n(\omega)\, d\omega \leqq 2\alpha\pi$ for all n.

PROOF. Since (i), (ii), (iii) are self-evident, only the proof of (iv) is given. Let $F_n(\omega)$ be the Fejér kernel; namely $F_n(\omega)$ is defined to be $K_n(\omega)$ with α replaced by 2 in (4.1). Since $\alpha \leqq 2$, $F_n(\omega)/K_n(\omega) = (2n+1)^{\alpha-2}|\sin\{(n+\frac{1}{2})\omega\}/\sin(\omega/2)|^{2-\alpha} \leqq 1$. Thus, $\int_{-\pi}^{\pi} K_n(\omega)\, d\omega \geqq \int_{-\pi}^{\pi} F_n(\omega)\, d\omega \geqq 2\pi$. Define $h_n(\omega)$ by:

$$h_n(\omega) = 2n+1 \qquad \text{for} \quad 0 \leqq |\omega| \leqq \pi/(2n+1)$$
$$= (2n+1)^{1-\alpha}\pi^{\alpha}\omega^{-\alpha} \qquad \text{for} \quad \pi/(2n+1) < |\omega| \leqq \pi .$$

Then in view of (ii) and (iii), $K_n(\omega) \leqq h_n(\omega)$ for $0 \leqq \omega \leqq \pi$. Moreover,

$$\int_{-\pi}^{\pi} h_n(\omega)\, d\omega \leqq \alpha\pi .$$

□

Let $1/C_n = (1/2\pi)\int_{-\pi}^{\pi} K_n(\omega)\, d\omega$; then, by the lemma above, $1 \leqq C_n \leqq \alpha$. Let $f \in \mathscr{L}^1[-\pi, \pi]$; write $\phi(\lambda, \omega) = f(\lambda+\omega) + f(\lambda-\omega) - 2f(\lambda)$ and define $\Phi(\lambda, \omega) = \int_0^{\omega} |\Phi(\lambda, \nu)|\, d\nu$. Each point $\lambda \in (-\pi, \pi)$ for which $\lim_{\omega \to 0+} (1/\omega)\Phi(\lambda, \omega) = 0$ is called a Lebesgue point of f. To use Lemma 4.1, the next lemma can be demonstrated by following the usual steps of the proof for the case of the Fejér kernel.

LEMMA 4.2. *Let $f \in \mathscr{L}^1[-\pi, \pi]$, and let λ be a Lebesgue point. Then*

$$\lim_{n\to\infty} |(2\pi)^{-1}C_n \int_{-\pi}^{\pi} f(\lambda-\omega)K_n(\omega)\,d\omega - f(\lambda)| = 0 .$$

The result of Lemma 4.2 can be generalized in the following way.

THEOREM 4.1. *Given an integrable function $f(\lambda)$, $-\pi \leqq \lambda \leqq \pi$, let $\lambda_1, \lambda_2, \cdots, \lambda_p$ be distinct Lebesgue points of $f(\lambda)$, then for $1 < \alpha \leqq 2$, $\int_{-\pi}^{\pi} |\sum_{k=1}^{p} u_k D_n(\lambda_k)|^\alpha f(\lambda)\,d\lambda$ converges to $\sum_{k=1}^{p} |u_k|^\alpha f(\lambda_k)$ as n tends to infinity, where the u_k are any complex numbers and*

$$D_n(\lambda_k) = (2\pi)^{-1/\alpha} C_n^{1/\alpha}(2n+1)^{(1-\alpha)/\alpha} \sum_{j=-n}^{n} e^{i(\lambda_k-\lambda)j} .$$

PROOF. Without loss of generality, all u_k may be assumed to be nonzero. Set $\gamma_n = S(2n+1)^{\delta-1}$ for a positive number S and δ such that $1/\alpha < \delta < 1$. Let $E_n(\lambda_k)$ be the sets $[\lambda_k - \gamma_n, \lambda_k + \gamma_n]$, $k = 1, 2, \cdots, p$, and for each k, denote by $G_n(\lambda_k)$ the subset of $E(\lambda_k)$ such that for $\lambda \in G_n(\lambda_k)$, $|u_k D_n(\lambda_k)| \geqq |\sum_{l\neq k} u_l D_n(\lambda_l)|$; then, to neglect the terms converging to zero as $n \to \infty$,

$$\text{(4.2)} \qquad \begin{aligned} &\int_{-\pi}^{\pi} |\sum_{k=1}^{p} u_k D_n(\lambda_k)|^\alpha f(\lambda)\,d\lambda - \sum_{k=1}^{p} |u_k|^\alpha \int_{-\pi}^{\pi} |D_n(\lambda_k)|^\alpha f(\lambda)\,d\lambda \\ &\quad\approx \sum_{k=1}^{p} \int_{G_n(\lambda_k)} \{|\sum_{l=1}^{p} u_l D_n(\lambda_l)|^\alpha - |u_k D_n(\lambda_k)|^\alpha\} f(\lambda)\,d\lambda . \end{aligned}$$

Now in view of the relations that for any x and y $(x > y > 0)$, $|x+y|^\alpha \leqq x\alpha + 2\alpha x^{\alpha-1}y$ and $x^\alpha - \alpha x^{\alpha-1}y \leqq |x-y|^\alpha$, the sum in the right-hand side of (4.2) is seen to be absolutely bounded by

$$\sum_{k=1}^{p} M(\sum_{l=1}^{p} |u_l|)(2n+1)^{-\delta+(1/\alpha)} \int_{G_n} |u_k D_n(\lambda_k)|^{\alpha-1} f(\lambda)\,d\lambda , \qquad \text{for some } M < \infty .$$

However, since $\int_{G_n} |u_k D_n(\lambda_k)|^{\alpha-1} f(\lambda)\,d\lambda$ is bounded for each k, as n tends to infinity,

$$\int_{G_n(\lambda_k)} \{|\sum_{l=1}^{p} u_l D_n(\lambda_l)|^\alpha - |u_k D_n(\lambda_k)|^\alpha\} f(\lambda)\,d\lambda \to 0 . \qquad \square$$

Let $\{X_t\}$ be a stable process with spectral representation; suppose that the characteristic function of the sample $X_{-n}, X_{-n+1}, \cdots, X_{n-1}, X_n$ is given by

$$\text{(4.3)} \qquad \begin{aligned} &C(u_{-n}, \cdots, u_n) \\ &\quad= \exp[-\int_0^\pi \{|\sum_{j=-n}^{n} u_j \cos(\omega j)|^\alpha + |\sum_{j=-n}^{n} u_j \sin(\omega j)|^\alpha\}\,dF(\omega)] . \end{aligned}$$

Then construct statistics $I_n(\lambda_k)$ and $J_n(\lambda_l)$ respectively according to the formulas:

$$I_n(\lambda_k) = (2\pi)^{-1/\alpha} C_n^{1/\alpha}(2n+1)^{(1-\alpha)/\alpha} \sum_{j=-n}^{n} X_j \cos(\lambda_k j) ,$$
$$J_n(\lambda_l) = (2\pi)^{-1/\alpha} C_n^{1/\alpha}(2n+1)^{(1-\alpha)/\alpha} \sum_{j=-n}^{n} X_j \sin(\lambda_l j) ,$$

$k = 1, 2, \cdots, p$ and $l = p+1, \cdots, p+q$, where the λ_k and the λ_l are distinct Lebesgue points in $[0, \pi]$, but the λ_k may be equal to some λ_l.

THEOREM 4.2. *The $I_n(\lambda_k)$, $k = 1, 2, \cdots, p$, are independent with the $J_n(\lambda_l)$, $l = p+1, \cdots, p+q$.*

PROOF. The assertion of the theorem is an obvious consequence of the decomposability of the joint characteristic function of the $I_n(\lambda_k)$ and the $J_n(\lambda_l)$. □

Suppose that the measure $F(\omega)$ in (4.3) has a density $f(\omega)$. Defining $f(\omega) = f(-\omega)$ for $\omega < 0$, the density $f(\omega)$ is extended to $[-\pi, \pi]$; through this extension, the same notation $f(\omega)$ is preserved without confusion. Let the λ_k and λ_l be Lebesgue points in $(0, \pi)$.

THEOREM 4.3. *The $I_n(\lambda_k)$ and the $J_n(\lambda_l)$ are asymptotically independently distributed in stable distributions whose characteristic functions $C_I(u_k)$ and $C_J(u_l)$ $(k = 1, \cdots, p; l = p + 1, \cdots, p + q)$ are given by*

$$C_I(u_k) = \exp\left\{-\frac{f(\lambda_k)}{2^\alpha}|u_k|^\alpha\right\} \quad \textit{and}$$

$$C_J(u_l) = \exp\left\{-\frac{f(\lambda_l)}{2^\alpha}|u_l|^\alpha\right\} \quad \textit{respectively.}$$

PROOF. Write the joint characteristic function of the $I_n(\lambda_k)$ and the $J_n(\lambda_l)$ as

$$\exp\left[-\tfrac{1}{2}\int_{-\pi}^{\pi}\left\{\left|\sum_{k=1}^{p} u_k \frac{D_n(\lambda_k) + D_n(-\lambda_k)}{2}\right|^\alpha + \left|\sum_{l=p+1}^{p+q} u_l \frac{D_n(\lambda_l) - D_n(-\lambda_l)}{2i}\right|^\alpha\right\} f(\omega)\, d\omega\right],$$

and apply Theorem 4.1 to it. □

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