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A PATH DECOMPOSITION FOR MARKOV PROCESSES[1]

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Let $X = \{X_t, t > 0\}$ be a right continuous strong Markov process with state space E; let f be a continuous real valued function on $E \times E$; and let M be the time at which the process $\{f(X_{t-}, X_t)\}$ achieves its (last) ultimate minimum. Then conditional on X_M and the value of this minimum, the process $\{X_{M+t}\}$ is Markov and (conditionally) independent of events before M.

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a right continuous, strong Markov process with left limits, and with compact metric state space $(E, \mathscr{E})$. A point $\Delta \in E$ has been set aside as the terminal absorbing state; and $\zeta = \inf\{t: X_t = \Delta\}$. The transitions $P_t(x, f) = E^x f(X_t)$ are assumed Borel: if f is bounded, continuous then $x \to E^x f(X_t)$ is $\mathscr{E}$-measurable. Assume also that $P_0(x, \{x\}) = 1$ for all x. The sigma fields $\mathscr{F}_t$ are the usual right continuous completions of general Markov theory ([1]). Let f be a real, jointly continuous function on $E \times E$ and define

$$(1) \qquad I_t^0 = \inf_{s \leq t} f(X_{s-}, X_s)$$
$$I = \lim_{t\to\infty} I_t^0 = \inf_s f(X_{s-}, X_s)\,.$$

Assume

$$(2) \qquad f(x, \Delta) = \infty$$

$$(3) \qquad I > -\infty \quad \text{a.s.}$$

Define $I_t = I_{t+}^0$. Evidently $t \to I_t$ is right continuous and decreasing. Define

$$M_+ = \sup\{s: f(X_s, X_s) = I_s\}$$
$$M_0 = \sup\{s: f(X_{s-}, X_s) = I_s\}$$
$$M_- = \sup\{s: f(X_{s-}, X_{s-}) = I_s\}$$

and set

$$(4) \qquad M = \max\{M_-, M_0, M_+\}\,.$$

Because of the continuity of f, M is the last instant t at which the process X was at the ultimate minimum of $\{f(X_{s-}, X_s)\}$.

A random time R is a nonnegative $\mathscr{F}$-measurable random variable. The sigma fields $\mathscr{F}(R)$, $\mathscr{F}(R+)$ are defined by

$$(5) \qquad \mathscr{F}(R) = \sigma\{Z_R: \{Z_t\} \text{ is a well-measurable process}\}$$
$$\mathscr{F}(R+) = \sigma\{Z_R: \{Z_t\} \text{ is a progressive process}\}\,.$$

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Here $\sigma\{\cdots\}$ denotes the sigma field generated by whatever appears between the braces.

(6) THEOREM. *The process $\{X(M+t), t>0\}$ is conditionally independent (P^x) of $\mathscr{F}(M)$, given (I, X_M) and, given these variables, this post-M process is Markov relative to the right continuous sigma fields $\mathscr{F}(M+t+)$: for bounded measurable g*

$$E\{g(X(M+t))\,|\,\mathscr{F}(M+s+)\} = \int g(y)H_{t-s}(X(M+s), I, dy)\,, \qquad s>0$$
$$E\{g(X(M+t))\,|\,\mathscr{F}(M)\} = \int g(y)Q_t(X_M, I; dy)$$

where $H_t(x, a; dy)$ is for each a a family of transitions, and $Q_t(x, a; dy)$ is for each x, a an entrance law for $\{H_t\}$.

More information on the form of H_t, Q_t is given in (8), (9) below. Special cases of this decomposition have been given before. In [3], [8], a decomposition for real diffusions was given with $f(u, v) = \min\{u, v\}$; in [5] such a decomposition was given for general real Markov processes and the same f (but here the conditional independence was proved only for a broad class of processes, not all); in [6], decompositions were given for X an h-transform of k-dimensional Brownian motion, with $f=h$, but here also conditional independence was proved only for a few processes in the class considered.

The decompositions in the references just cited were proved more or less from first principles, and because of this the proofs, while long and involved, contain further valuable structural information in the special cases treated. Here the proof is quite short, and the end result is much more general; but it rests on a substantial mountain of general theory.

Decompositions of real diffusions at their ultimate minimum (i.e., $f(u, v) = v$) were used as a key but difficult tool by Williams ([8]) in his ingenious study of Brownian local time. Decompositions at other "minimal" times promise to be equally useful; in particular the decomposition of a general process at the time it is closest to a set A for the last time ($f(u, v) = \min\{g(u), g(v)\}$ with $g(u) = d(u, A)$ and d a metric on E) is of particular interest and will be studied elsewhere.

PROOF. Let $(R, \mathscr{B})$ denote the real line with the sigma field of Borel sets. For $x \in E$, $a \in R$ and bounded real g on $(E, \mathscr{E}) \times (R, \mathscr{B})$ define

$$K_t((x, a), g) = E^x g(X_t, I_t \wedge a)\,. \tag{7}$$

Then $(x, a) \rightarrow K_t((x, a), g)$ is measurable and $\{K_t\}$ is a semigroup of transition functions. Indeed, since $I_{t+s}(\omega) = I_t(\omega) \wedge I_s(\theta_t\omega)$,

$$\begin{aligned} K_{t+s}((x, a), g) &= E^x g(X_{t+s}, I_{t+s} \wedge a) \\ &= E^x\{E^x\{g(X_s(\theta_t), I_t \wedge I_s(\theta_t) \wedge a)\,|\,\mathscr{F}_t\}\} \\ &= E^x K_s((X_t, I_t \wedge a), g) \\ &= K_t K_s((x, a), g)\,. \end{aligned}$$

For each real number b the right continuous process $\{(X_t, I_t \wedge b)\}$ with state space $E \times R$ is strong Markov relative to the sigma fields $\mathscr{F}_t$, and has transitions $\{K_t\}$:

$$E^x\{g(X_{T+t}, I_{T+t} \wedge b) \mid \mathscr{F}_T\} = E^x\{g(X_t(\theta_T), I_T \wedge I_t(\theta_T) \wedge b) \mid \mathscr{F}_T\}$$
$$= K_t(X_T, I_T \wedge b, g) .$$

Moreover the random time M is a "coterminal time" for the (X_t, I_t) process (but not for the $\{X_t\}$ process; the vector process (X_t, I_t) here is not given in canonical form—in particular the relevant shift operator is not the same as that of $\{X_t\}$; we do not dwell on this familiar problem). Of course, $I_t = I$ for $t > M$. By Pittenger–Shih, Getoor–Sharpe ([7], [2]), $\{(X_{M+t}, I), t \geqq 0\}$ is conditionally independent of $\mathscr{F}_M$, given (X_M, I), and $\{(X_{M+t}, I), t > 0\}$ is strong Markov relative to the right continuous sigma fields $\mathscr{F}(M + t+)$, with transitions

(8) $\qquad H_t((x, a); dy\, dz) = P^{x,a}\{(X_t, I_t) \in dy\, dz, t < T\}P^{y,z}\{T = \infty\}/P^{x,a}(T = \infty)$

where $T = \inf\{s > 0 : f(X_s, X_s) = I_s \text{ or } f(X_{s-}, X_s) = I_s \text{ or } f(X_{s-}, X_{s-}) = I_s\}$ and where $P^{x,a}$ is the P^x distribution of the strong Markov process $\{(X_t, I_t \wedge a), t \geqq 0\}$ on its canonical function space. One can express (8) in terms of the original P^x measures. Indeed when $x = X_M$, $a = I$, (8) may be deciphered as

(9) $\qquad P^x\{X_t \in dy, t < T_a\}P^y\{T_a = \infty\}/P^x\{T_a = \infty\}$

where $T_a = \inf\{t > 0 : f(X_{t-}, X_t) \leqq a, \text{ or } f(X_t, X_t) \leqq a \text{ or } f(X_{t-}, X_{t-}) \leqq a\}$. The entrance law $Q_t((x, a), dy)$ is a regular conditional distribution of X_{M+t}, given (X_M, I).

Various extensions of the argument of Theorem (6) are possible. Some of these were pointed out to me by R. K. Getoor and J. Azéma (the latter via the Associate Editor, J. Walsh). Here is one extension. If R^k is k-dimensional Euclidean space, let h be a Borel function from $R^k \times R^k$ to R^k satisfying

$$h(x, h(y, z)) = h(h(x, y), z) .$$

Let I_t be an adapted, right continuous functional of X with values in R^k. Assume for all optional T and all real $s > 0$:

$$I_{T+s} = h(I_T, I_s \circ \theta_T) .$$

Then the calculations of Theorem (6) show, without change, that (X_t, I_t) is strong Markov. Interesting choices of I_t include additive functionals, multiplicative functionals, and, if X is real, the (vector) extremal functional

(10) $\qquad I_t = (\inf_{s \leqq t} X_s, \sup_{s \leqq t} X_s) .$

The results of [2], [7] may now be applied to the process (X_t, I_t) as in the proof of Theorem (6) to obtain interesting decompositions. For example, application to the functional (10) in a natural way yields a decomposition of X at the time at which it attains its (last) extreme value.

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