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ON CONJECTURES IN FIRST PASSAGE PERCOLATION THEORY

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We consider several conjectures of Hammersley and Welsh in the theory of first passage percolation on the two-dimensional rectangular lattice. Our results include: (i) a proof that the time constant is zero when the atom at zero of the underlying distribution is one-half or larger; (ii) almost sure existence of routes for the unrestricted first passage times; (iii) almost sure limit theorems for the first passages s_{0n} and b_{0n}, the reach processes y_t and y_t^u, and the route length processes N_n^s and N_n^b; (iv) bounds on the expected maximum height of routes for s_{0n} and t_{0n} when the atom at zero of the underlying distribution is one-half or larger.

0. Introduction. The results of this paper verify two conjectures posed by Hammersley and Welsh [4] in the theory of first passage percolation, and provide partial solutions to other conjectures. For a summary of the notation used the reader is referred to [4] and [8]. Recent progress toward the solution of the Hammersley and Welsh conjectures is contained in [8], [9], [10], and [12].

Hammersley and Welsh [4] proved that routes exist for the first passage times t_{0n} and s_{0n} for all n with probability one for every time coordinate distribution. Almost sure existence of routes was shown for a_{0n} and b_{0n} in the case where time coordinates were bounded above and bounded below away from zero. Hammersley and Welsh [4] conjectured that the almost sure existence of routes for a_{0n} and b_{0n} also held for all distributions. Smythe and Wierman [9] verified the conjecture except for distributions with an atom at zero equal to the critical percolation probability. The authors have independently proved the conjecture in full generality. Reh's proof is presented. The reader should note that in Section 1 one can do without assuming the existence of a finite first moment for U, an assumption which is tacitly made throughout the paper.

At the time of the original paper by Hammersley and Welsh, it was claimed (Sykes and Essam [11]) that the critical percolation probability C for the square lattice is $\frac{1}{2}$. This led to the conjecture [4] that the time constant is zero for Beronulli time coordinate distributions, with atom at zero of size one-half or larger. The value of the critical probability has in fact not yet been determined. Harris [5] proved that $C \geqq \frac{1}{2}$, and Hammersley [1] has given the upper bound $C \leqq 1 - 1/\lambda$, where λ is the connectivity constant of the square lattice. Smythe and Wierman [9] verified the spirit of the conjecture by showing that $\mu(U) = 0$ if $U(0) > C$ for any time coordinate distribution U. The conjecture is verified

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here by a symmetry argument, with no reference to the concept of critical probability. The dual graph technique employed also yields a bound on the expected maximum height of routes for s_{0n} and t_{0n} when $U(0) \geqq \frac{1}{2}$.

In Section 3, results of [3] on superconvolutive distributions are used to show almost sure convergence of s_{0n}/n and b_{0n}/n for any time coordinate distribution. These results are used to improve theorems on convergence of the reach processes y_t and y_t^u, and the route length processes N_n^s and N_n^b.

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1. Existence of routes. We state a theorem which solves Conjecture 8.1.5 in [4]. The terminology of Harris [5] will be used.

Let B_n denote the square $\{(x, y) : |x| \leqq n, |y| \leqq n\}$, and let $bd(A)$ denote the boundary of a subset A of the Euclidean plane.

Let the links of the square lattice L be partitioned into active links and passive links. We can make the following observation:

REMARK 1.1. If there does not exist an infinite active chain beginning at the origin, then there exists a positive integer n such that $(0, 0)$ cannot be connected with $bd(B_n)$ by an active chain.

Let P_i denote the point $(i, 0)$, and let u_i be the time coordinate corresponding to the link between P_{i-1} and P_i. For any box $B \equiv B_n$, let $t(B)$ denote the first passage time from $(0, 0)$ to $bd(B)$.

THEOREM 1.2. *Fix a positive integer n.*

(a) *Suppose there exists $x > 0$ such that $U(x) \leqq \frac{1}{2}$. Then for any positive integer k, with probability one there exists a box $B^k \equiv B_i$ for some i, such that $t(B^k) > kx$.*

(b) *Suppose $U(0) \geqq \frac{1}{2}$. Then with probability one there exists a circuit having total travel time equal to zero which contains P_i, $i = 0, 1, \cdots, n$ strictly in its interior.*

PROOF. *Case* (a): Define a link in the square lattice L to be active if the corresponding time coordinate u satisfies $u \leqq x$. This allows reference to the results of Harris [5].

Theorem 1 of [5] and Remark 1.1 yield the almost sure existence of a minimal positive integer $n_1 = n_1(\omega)$ such that $bd(B_{n_1})$ cannot be reached from the origin by an active chain. Define $B^1 \equiv B_{n_1}$.

Proceed by induction. Assume B^{k-1} to be defined with probability one as well as the random variable n_{k-1}, which is finite almost surely. Let R_j^i, $j = 1, 2, \cdots, 8i$, be the vertices of L which lie on $bd(B_i)$. Let A_i be the event that none of the vertices R_j^i is the endpoint of a connected infinite set of active links. We know from [5] that $P(A_i) = 1$. Denoting the set $\{\omega : n_{k-1} = i\}$ by $A_i(k)$ we find that for any $\omega \in \bigcup_{i=1}^{\infty} (A_i \cap A_i(k))$ there exists a minimal box B^k ($\equiv B_j$ for some j) so that $t(B^k(\omega)) > kx$. To complete the proof of (a), notice that

$$\begin{aligned} P(\textstyle\bigcup_{i=1}^{\infty} (A_i \cap A_i(k))) &= \textstyle\sum_{i=1}^{\infty} P(A_i \cap A_i(k)) \\ &= \textstyle\sum_{i=1}^{\infty} P(A_i(k)) \\ &= P(n_{k-1} \text{ finite}) \\ &= 1 . \end{aligned}$$

Case (b): Define a link to be active if the corresponding time coordinate u is positive, and passive if $u = 0$. The probability of a given link being active is then no greater than $\frac{1}{2}$. Considering the lattice dual L^* of L, from Theorem 1 of [5] we conclude that the probability of belonging to a CISAL in L^* is equal to zero for any given point $P_i^* = (i - \frac{1}{2}, \frac{1}{2})$. Thus there exists a box B^* in L^* such that $bd(B^*)$ cannot be reached from P_i^* by an active chain in L^*. Now, following the reasoning in Appendix 2 of [5], we consider the finite dual graph B of B^* which consists of (i) links of L lying strictly inside $bd(B^*)$ and (ii) links crossing $bd(B^*)$ extended to meet at a common vertex. B^* contains a special cut set S^* of passive links. Applying Whitney's theorem to B^* and B, the links of L crossing the links of S^* form a passive circuit in B lying strictly inside $bd(B^*)$. Thus a passive circuit in L exists with probability one.

The same procedure applies conditionally, given that any specified set of links near P_i^* are active. Hence, with probability one there exists a passive circuit containing P_i, $i = 0, 1, \cdots, n$ strictly in its interior simultaneously.

COROLLARY 1.3. *Routes exist for a_{0n} and b_{0n} for all positive integers n with probability one for every time coordinate distribution.*

PROOF. For n fixed define the event

$$R = \{\omega : \text{routes of } a_{0n}(\omega) \text{ and } b_{0n}(\omega) \text{ exist}\} .$$

In Case (a) for any natural number k choose B^k according to Theorem 1.2 for $\omega \in \Omega(k)$, where $P(\Omega(k)) = 1$. Any route from (0, 0) to $bd(B^k)$ has a first passage time exceeding kx for any $\omega \in \Omega(k)$ and only finitely many routes do not cross $bd(B^k)$. Hence

$$\{u_1 + \cdots + u_n \leqq kx\} \cap \Omega(k) \subseteqq R ,$$

and thus

$$P(R) \geqq \lim_{k\to\infty} U^{n*}(kx) = 1 .$$

In Case (b) obviously it is sufficient to restrict ourselves to the finitely many routes in the interior of the circuit the existence of which is provided by Theorem 1.2 with probability one. Again $P(R) = 1$.

2. Evaluation of the time constant. For a fixed positive integer n, construct a finite graph G as the portion of the two-dimensional rectangular lattice contained within $0 < x < n$ and $0 \leqq y \leqq n - 1$, including the vertices on the lines $x = 0$ and $x = n$. A dual graph G^* can be constructed by connecting the vertices $(i + \frac{1}{2}, j + \frac{1}{2})$; $i = 0, \cdots, n - 1$; $j = -1, 0, \cdots, n - 1$, by horizontal and vertical unit line segments, except those links on the lines $y = -\frac{1}{2}$ and $y = n - \frac{1}{2}$. The dual graph is isomorphic to the original graph.

Let each link in G be passive with probability p and active with probability $1 - p$, independent of all other links in G. Partition the links in G^* by the following: A link in G^* is passive if it crosses an active link in G. A link in G^* is active if it crosses a passive link in G.

REMARK 2.1. For any partition of G into active and passive links, there is either a passive chain in G connecting $x = 0$ and $x = n$ or a passive chain in G^* connecting $y = -\frac{1}{2}$ with $y = n - \frac{1}{2}$.

PROOF. The observation may be proved for any rectangle R with dual R^* by induction on the sum of the dimensions of the rectangle. Recall the definition of the reach process $y_t = \sup \{n : s_{0n} \leqq t\}$.

LEMMA 2.2. *For a time coordinate distribution with* $U(0) \geqq \frac{1}{2}$,

$$E(y_0) = \infty .$$

PROOF. Fix a positive integer n, and consider the graphs G and G^* when $p = \frac{1}{2}$. By symmetry when $p = \frac{1}{2}$, the probability of a passive chain crossing G is equal to the probability of a passive chain crossing G^*. Hence, the probability of such a chain is $\frac{1}{2}$.

To use this result in the first passage percolation theory, consider the lattice L with time coordinates from U associated with the links. Links with positive time coordinates are defined to be active. Links with zero time coordinates are passive. Thus there is probability $\frac{1}{2}$ that G is crossed from $x = 0$ to $x = n$ by a chain of links which has travel time zero.

Let $s_{0n}(i, G)$ denote the first passage time from $(0, i)$ to $x = n$ on paths which are contained entirely within G and do not intersect $x = 0$ except at the initial vertex. Then for some $i \in \{0, 1, \cdots, n-1\}$,

$$P(s_{0n}(i, G) = 0) \geqq \frac{1}{2n} .$$

Letting $s_{0n}(i)$ denote the cylinder point-to-line first passage time from $(0, i)$ to $x = n$, it is clear that

$$P(s_{0n}(i) = 0) \geqq \frac{1}{2n} .$$

By stationarity,

$$P(s_{0n} = 0) \geqq \frac{1}{2n} .$$

For all n, $P(s_{0n} = 0) = P(y_0 \geqq n)$. Therefore $\sum_{n=1}^{\infty} P(y_0 \geqq n) \geqq \sum_{n=1}^{\infty} 1/2n = \infty$.

THEOREM 2.3. *Let* U *be a time coordinate distribution with* $U(0) \geqq \frac{1}{2}$. *Then* $\mu(U) = 0$.

PROOF. The proof requires the following results.

LEMMA 2.4. *Let U be a time coordinate distribution which is bounded above. Then if $\mu(U) > 0$,*

$$\lim_{t\to\infty} E\left(\frac{y_t}{t}\right) = \lim_{t\to\infty} E\left(\frac{x_t}{t}\right) = \frac{1}{\mu(U)}.$$

NOTE. Smythe and Wierman [9] prove this result for y_t in Theorem 5.4. Fatou's lemma, together with Theorems 4.1 and 5.4 of [9], yields the conclusion for x_t.

Let $\mu(p)$ denote the time constant of the Bernoulli distribution with parameter $p = P(B = 0) = 1 - P(B = 1)$.

LEMMA 2.5. *For any time coordinate distribution U,*

$$\mu(U) \leqq \mu(U(0)) \frac{E(u_1)}{1 - U(0)}.$$

NOTE. This is Theorem 2.1 of Wierman [12].

To prove Theorem 2.3, first consider a Bernoulli distribution with $p \geqq \frac{1}{2}$. By Lemma 2.2, $E(y_0) = \infty$. Since the Bernoulli distribution is bounded, $E(y_0) = \infty$ implies $\mu(p) = 0$ by Lemma 2.4. Apply Lemma 2.5 to obtain $\mu(U) = 0$ for a non-Bernoulli distribution with $U(0) \geqq \frac{1}{2}$.

Let $h_n(r)$ denote the maximum distance of the route r for s_{0n} from the x-axis, and let R_n denote the set of routes for s_{0n}. Define the height process $\{h_n\}$ by $h_n = \min\{h_n(r) : r \in R_n\}$. Thus h_n is the maximum height of routes which remain closest to the x-axis.

THEOREM 2.6. *Let U be a time coordinate distribution with $U(0) \geqq \frac{1}{2}$. Then $\{(h_n/n)^\alpha\}$ is uniformly integrable for all $\alpha > 0$. In particular, $E(h_n/n) \leqq \frac{8}{3}$ $\forall\, n = 1, 2, 3, \cdots$.*

PROOF. For each n, consider subgraphs G_k ($k = 1, 2, \cdots$) of the square lattice, each isomorphic to G in the discussion beginning this section, where G_k is bounded by the lines $x = 0$, $x = n$, $y = kn$, and $y = (k - 1)n + 1$.

By the reasoning in the proof of Lemma 2.2, with probability at least $\frac{1}{2}$, there is a path from $x = 0$ to $x = n$ in G_k with zero travel time. Such a path is a barrier which the routes of interest cannot cross. By independence, the first index k for which G_k contains such a barrier is a random variable with a geometric distribution with parameter at least one-half. The same reasoning applies to barriers in subgraphs H_k below the x-axis. Define X as the first index k for which G_k contains a barrier, and Y as the first k for which H_k contains a barrier. Then

$$\frac{h_n}{n} \leqq \max\{X, Y\}$$

and X and Y are independent geometric random variables with parameter $\geqq \frac{1}{2}$. Thus all moments of h_n/n exist and are uniformly bounded in n, so $\{(h_n/n)^\alpha\}$ is uniformly integrable for all $\alpha > 0$. A simple calculation shows $E(\max\{X, Y\}) = \frac{8}{3}$.

3. Almost sure convergence of s_{0n}/n and b_{0n}/n. Hammersley and Welsh [4] proved that the first passage time process s_{0n}/n converges in probability to the time constant μ, and

$$\limsup_{n\to\infty} \frac{s_{0n}}{n} = \mu \quad \text{a.s.}, \tag{3.1}$$

for any time coordinate distribution. Smythe and Wierman [9] proved the corresponding results for b_{0n}/n. The main results of this section prove that for both processes the convergence is almost sure. As a consequence, almost sure convergence of the corresponding reach processes and route length processes is obtained.

For each positive integer n, define a random variable z_n, a modification of s_{0n}, as the first passage time between the origin and the line $x = n$ through paths which are contained entirely in the cylinder $0 \leqq x < n$ except for the terminal endpoint. This allows the paths for z_n to leave the origin on any of three links rather than only one as is required for s_{0n}. By the method of Section 1 in [12], the process $\{z_n\}$ is seen to have finite third moments for all n, for any time coordinate distribution U.

THEOREM 3.2. *Let U be any time coordinate distribution. Then*

$$\lim_{n\to\infty} \frac{z_n}{n} = \mu(U) \quad \text{a.s.}$$

PROOF. The proof requires the following result which is a special case of the lemma and Remark 2 in Note 7 of [3].

LEMMA 3.3. *Let X_s $(s = 1, 2, \cdots)$ be a monotone sequence of real nonnegative random variables with distribution functions F_s and finite second moments. Suppose that (for each pair of positive integers s and t) there exists a random variable $X'_{s,t}$ satisfying:*

(i) *$X'_{s,t}$ has distribution function F_t,*
(ii) *X_s and $X'_{s,t}$ are independent,*
and
(iii) *$F_{s+t} \geqq F_s * F_t$ for all s and t.*

Then there exists a constant γ such that $\lim_{s\to\infty} X_s/s = \gamma$ a.s.

In the first passage percolation model, define $z'_{n,m}$ as the first passage time from the endpoint of the route of z_n to the line $x = n + m$ through paths contained entirely in the cylinder $n \leqq x < n + m$ except for the terminal point. Then $z'_{n,m}$ has the same distribution as z_m. The observation

$$z_{m+n} \leqq z_n + z'_{n,m}$$

implies the superconvolutive property (iii) for the distributions of $\{z_n\}$. Independence of $z'_{n,m}$ from z_n is shown by the following:

Let $P(\omega) = i$ if (n, i) is the endpoint of the route of $z_n(\omega)$. For any fixed integer i, let $z'_{n,m}$ be the first passage time from (n, i) to the line $x = n + m$ through paths in the cylinder $n \leqq x < n + m$. Then for any two real numbers a and b,

$$P(z_n \leqq a, z'_{n,m} \leqq b) = \sum_{i=-\infty}^{\infty} P(z_n \leqq a, P = i, z'_{n,m} \leqq b) .$$

By independence, then stationarity,

$$\begin{aligned} P(z_n \leqq a, P = i, z'_{nm} \leqq b) &= P(z_n \leqq a, P = i)P(z'_{n,m} \leqq b) \\ &= P(z_n \leqq a, P = i)P(z_m \leqq b) \end{aligned}$$

for each i.

Therefore

$$\begin{aligned} P(z_n \leqq a, z'_{n,m} \leqq b) &= P(z_m \leqq b) \sum_{i=-\infty}^{\infty} P(z_n \leqq a, P = i) \\ &= P(z'_{n,m} \leqq b)P(z_n \leqq a) . \end{aligned}$$

Applying Lemma 3.3 with $X_n = z_n$ and $X'_{n,m} = z'_{n,m}$ we find there exists a constant γ such that

$$\lim_{n\to\infty} \frac{z_n}{n} = \gamma \quad \text{a.s.}$$

Further observe that

$$b_{0n} \leqq z_n \leqq s_{0n} ,$$

and by convergence in probability of s_{0n}/n and b_{0n}/n to $\mu(U)$, we obtain

$$\lim_{n\to\infty} \frac{z_n}{n} = \mu(U) \quad \text{in probability,}$$

and hence there exists a subsequence along which the convergence is almost sure. Therefore $\gamma = \mu(U)$.

Corollary 3.4. *For any time coordinate distribution* U,

$$\lim_{n\to\infty} \frac{s_{0n}}{n} = \mu(U) \quad \text{a.s.}$$

Proof. For each n, $z_n \leqq s_{0n} \leqq t_{0n}$, and both z_n/n and t_{0n}/n converge almost surely to $\mu(U)$.

Remark 3.5. Smythe and Wierman [10] consider time coordinate distributions which allow negative travel times. For a nonnegative time coordinate distribution function U, and a real number r, the distribution function $U \oplus r$ is defined by

$$U \oplus r(x) = U(x - r) \quad \forall\, x .$$

If $r < 0$, the time constant $\mu(U \oplus r)$ exists when $\rho(-r) < 1/\lambda$, where

$$\rho(y) = \inf_{u\geqq 0} \int_0^\infty e^{-u(x-y)}\, dU(x) ,$$

and λ is the connectivity constant of the square lattice. Where the sample point $\omega = (\omega_1, \omega_2, \cdots)$ denotes a sequence of time coordinates, $\omega \oplus r$ is the sequence $(\omega_1 + r, \omega_2 + r, \cdots)$. Using the general form of Kingmann's ergodic theorem for subadditive processes [6], it is shown that $t_{0n}(\omega \oplus r)/n$ converges almost surely to $\mu(U \oplus r)$. With this result, the methods of Theorem 3.2 and Corollary 3.4 apply to prove that $\lim_{n\to\infty} s_{0n}(\omega \oplus r)/n = \mu(U \oplus r)$ a.s.

COROLLARY 3.6. *For any time coordinate distribution U,*

$$\lim_{n\to\infty} \frac{b_{0n}}{n} = \mu(U) \quad \text{a.s.}$$

PROOF. When $\mu(U) = 0$, the conclusion holds by domination by the process $\{t_{0n}\}$, so we assume that $\mu(U) > 0$.

It was noted in [9] that the route of b_{0n} divides into two distinct parts. Let $P_n(\omega)$ denote the point on the y-axis where the route of $b_{0n}(\omega)$ last intersects the y-axis. The absolute value of the y-coordinate of P_n will be denoted by $|P_n|$. From [9] we know that when $\mu(U) > 0$

$$\lim_{n\to\infty} \frac{|P_n(\omega)|}{n} \leqq 1 \quad \text{a.s.}$$

Use $a_n(\uparrow)$ and $a_n(\downarrow)$ to denote the first passage times from (0, 0) to (0, n) and (0, $-n$) respectively. Clearly these random variables have the same distribution as a_{0n}. Thus

$$\lim_{n\to\infty} \frac{a_n(\uparrow\downarrow)}{n} = \mu \quad \text{a.s.}$$

Let $d_n(\uparrow)$ and $d_n(\downarrow)$ denote the first passage time from (1, 0) to (1, n) and (1, $-n$) respectively through paths lying entirely in the half-plane $x \geqq 1$. It is shown in [9] that

$$\lim_{n\to\infty} \frac{d_n(\uparrow\downarrow)}{n} = \mu \quad \text{a.s.}$$

The basic inequality needed for the proof appears in [9]:

$$s_{0n} \geqq b_{0n} \geqq s_{0n} - u_1 + a_{|P_n|}(\uparrow\downarrow) - d_{|P_n|}(\uparrow\downarrow)\,.$$

Let Ω' be the event where all of the following occur:

(i) Routes exist for b_{0n}, $a_n(\uparrow)$, $a_n(\downarrow)$, $d_n(\uparrow)$, and $d_n(\downarrow)$ for all n.
(ii) $\lim_{n\to\infty} a_n(\uparrow)/n = \mu(U)$ and $\lim_{n\to\infty} a_n(\downarrow)/n = \mu(U)$.
(iii) $\lim_{n\to\infty} d_n(\uparrow)/n = \mu(U)$ and $\lim_{n\to\infty} d_n(\downarrow)/n = \mu(U)$.
(iv) $\limsup_{n\to\infty} |P_n|/n \leqq 1$.
(v) The time coordinates of all links are finite.

The event Ω' has probability one.

Fix an $\varepsilon > 0$ and $\omega \in \Omega'$. Then there exists an $n(\omega)$ such that for all $n \geqq n(\omega)$:

(i) $u_1(\omega)/n < \varepsilon$,
(ii) $|a_n(\uparrow\downarrow)/n - \mu(U)| < \varepsilon$,
(iii) $|d_n(\uparrow\downarrow)/n - \mu(U) < \varepsilon$,
(iv) $|P_n|/n < 2$.

There exists $M(\omega)$ such that $\max_{k \leqq n(\omega)} d_k(\uparrow\downarrow)(\omega)/n < \varepsilon$ for $n \geqq M(\omega)$. Define $N(\omega) = \max\{n(\omega), M(\omega)\}$.

Suppose $n \geqq N(\omega)$. If $|P_n| \geqq n(\omega)$, then

$$\frac{s_{0n}}{n} \geqq \frac{b_{0n}}{n} \geqq \frac{s_{0n}}{n} - \frac{u_1}{n} + \left(\frac{a_{P_n}(\uparrow\downarrow)}{|P_n|} - \frac{d_{P_n}(\uparrow\downarrow)}{|P_n|}\right)\frac{|P_n|}{n} \geqq \frac{s_{0n}}{n} - 5\varepsilon .$$

If $|P_n| < n(\omega)$, then

$$\frac{s_{0n}}{n} \geqq \frac{b_{0n}}{n} \geqq \frac{s_{0n}}{n} - \frac{u_1}{n} - \frac{d_{P_n}(\uparrow\downarrow)}{n} \geqq \frac{s_{0n}}{n} - 2\varepsilon .$$

Therefore, on Ω', for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{s_{0n}}{n} \geqq \limsup_{n\to\infty} \frac{b_{0n}}{n} \geqq \liminf_{n\to\infty} \frac{b_{0n}}{n} \geqq \lim_{n\to\infty} \frac{s_{0n}}{n} - 5\varepsilon .$$

By almost sure convergence of s_{0n}/n to $\mu(U)$, letting ε tend to zero yields the conclusion.

REMARK 3.7. Almost sure convergence of b_{0n}/n holds for distributions $U \oplus r$ with negative time coordinates under the conditions specified in Remark 3.5.

We now strengthen the renewal theorems for the reach processes y_t and y_t^u. Here, if $\mu(U) = 0$ and $f(t) \geqq 0$, the statement "$\lim_{t\to\infty} f(t) = 1/\mu(U)$" will mean that $f(t)$ grows without bound as $t \to \infty$.

COROLLARY 3.8. *For any time coordinate distribution U,*

$$\lim_{t\to\infty} \frac{y_t}{t} = \frac{1}{\mu(U)} \quad \text{a.s.}$$

and

$$\lim_{t\to\infty} \frac{y_t^u}{t} = \frac{1}{\mu(U)} \quad \text{a.s.}$$

PROOF. Since s_{0n}/n and b_{0n}/n converge almost surely to μ, the conclusion follows from the general observation: If $\{v_n\}_{n\in N}$ is a sequence of random variables with v_n/n converging to a nonnegative constant μ almost surely, then r_t/t converges almost surely, where $r_t = \sup\{n: v_n \leqq t\}$ is the associated reach process.

The route lengths N_n^s and N_n^b are defined as the number of arcs in the shortest path which is a route for s_{0n} and b_{0n} respectively. For a time coordinate distribution U, the function $\mu(U \oplus r)$ is concave [10], and thus has left and right derivatives at each point, which we denote by $\mu^-(r)$ and $\mu^+(r)$ respectively; these are nonincreasing and μ^- is left continuous, μ^+ is right continuous, with $\mu^-(r) = \mu^+(r)$ except possibly at countably many points.

COROLLARY 3.9. *Suppose U is a time coordinate distribution with $U(0) < 1/\lambda$. Then*

$$\mu^+(0) \leqq \liminf_{n\to\infty} \frac{N_n^s(\omega)}{n} \leqq \limsup_{n\to\infty} \frac{N_n^s(\omega)}{n} \leqq \mu^-(0) \quad \text{a.s.}$$

and

$$\mu^+(0) \leqq \liminf_{n\to\infty} \frac{N_n^b(\omega)}{n} \leqq \limsup_{n\to\infty} \frac{N_n^s(\omega)}{n} \leqq \mu^-(0) \quad \text{a.s.}$$

PROOF. The conclusion follows by the method of Theorem 4.1 of [10].

REMARK 3.10. As in [10], the results on route length provide bounds on the height process, and also extend to time coordinate distributions which allow negative time coordinates.

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