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A SHORT NOTE ON THE CONDITIONAL BOREL–CANTELLI LEMMA

BY LOUIS H. Y. CHEN

University of Singapore

In this note we give a very simple and elementary proof of a slightly more general form of the conditional Borel–Cantelli lemma.

In this note we give a very simple and elementary proof of a slightly more general form of the conditional Borel–Cantelli lemma (see Lévy (1937), Meyer (1972) and Freedman (1973)).

THEOREM. *Let $\{X_n: n \geqq 1\}$ be a sequence of nonnegative random variables defined on the probability space $(\Omega, \mathscr{F}, P)$ and $\{\mathscr{F}_n: n \geqq 0\}$ be a sequence of sub-σ-algebras of $\mathscr{F}$ (to which $\{X_n\}$ need not be adapted). Let $M_n = E(X_n \mid \mathscr{F}_{n-1})$ for $n \geqq 1$. If $\{\mathscr{F}_n: n \geqq 0\}$ is nondecreasing, then $\sum_{n=1}^\infty X_n < \infty$ a.s. on $\{\sum_{n=1}^\infty M_n < \infty\}$. Conversely, if $Y = \sup_n X_n/(1 + X_1 + \cdots + X_{n-1})$ is integrable and $\mathscr{F}_n \supset \mathscr{B}(X_1 + \cdots + X_n)$ for $n \geqq 1$, then $\sum_{n=1}^\infty M_n < \infty$ a.s. on $\{\sum_{n=1}^\infty X_n < \infty\}$.*

PROOF. Let $M_0 = 1$. The sum

$$\sum_{n=1}^\infty \frac{M_n}{(M_0 + M_1 + \cdots + M_{n-1})(M_0 + M_1 + \cdots + M_n)}$$

forms a telescoping series and is equal to $1 - (\sum_{n=0}^\infty M_n)^{-1}$. Therefore

$$\begin{aligned} 1 &\geqq E \sum_{n=1}^\infty \frac{M_n}{(M_0 + M_1 + \cdots + M_{n-1})(M_0 + M_1 + \cdots + M_n)} \\ &= E \sum_{n=1}^\infty \frac{X_n}{(M_0 + M_1 + \cdots + M_{n-1})(M_0 + M_1 + \cdots + M_n)} \\ &\geqq E\{(\textstyle\sum_{n=0}^\infty M_n)^{-2}(\sum_{n=1}^\infty X_n)\}\,. \end{aligned}$$

From this it follows that $\sum_{n=1}^\infty X_n$ converges a.s. on $\{\sum_{n=1}^\infty M_n < \infty\}$. To prove the converse, we set $X_0 = 1$ and obtain

$$\begin{aligned} &E\{(\textstyle\sum_{n=0}^\infty X_n)^{-2}(\sum_{n=1}^\infty M_n)\} \\ &\quad\leqq E \sum_{n=1}^\infty \frac{M_n}{(X_0 + X_1 + \cdots + X_{n-1})^2} \\ &\quad= E \sum_{n=1}^\infty \frac{X_n}{(X_0 + X_1 + \cdots + X_{n-1})^2} \\ &\quad= E\left\{\sum_{n=1}^\infty \frac{X_n}{(X_0 + X_1 + \cdots + X_{n-1})(X_0 + X_1 + \cdots + X_n)} \times \left[1 + \frac{X_n}{X_0 + X_1 + \cdots + X_{n-1}}\right]\right\} \\ &\quad\leqq E(1 + Y) < \infty\,. \end{aligned}$$

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This implies that $\sum_{n=1}^{\infty} M_n$ converges a.s. on $\{\sum_{n=1}^{\infty} X_n < \infty\}$ and proves the theorem.

The following curious result, which we state without proof, is an immediate consequence of the theorem.

COROLLARY. *Let* $\{X_n : n \geqq 1\}$ *be a sequence of nonnegative random variables defined on the probability space* $(\Omega, \mathscr{F}, P)$ *and adapted to the nondecreasing sequence of sub-*σ*-algebras* $\{\mathscr{F}_n : n \geqq 0\}$ *of* $\mathscr{F}$. *Let* $\mathscr{G}_0 \subset \mathscr{F}$ *be any* σ*-algebra, and for* $n \geqq 1$ *let* $\mathscr{G}_n = \mathscr{B}(X_1 + \cdots + X_n)$. *Suppose* $\sup_n X_n/(1 + X_1 + \cdots + X_{n-1})$ *is integrable. Then*

$$\{\sum_{n=1}^{\infty} E(X_n \mid \mathscr{F}_{n-1}) < \infty\} \subset_{\text{a.s.}} \{\sum_{n=1}^{\infty} E(X_n \mid \mathscr{G}_{n-1}) < \infty\}.$$

The direct part of the above theorem is the same as that in Meyer ((1972), Theorem 21, page 9) which is perhaps as general as it could be. On the other hand, the converse part improves on that in Meyer in the sense that the integrability of $\sup_n X_n$ is weakened to that of Y and $\{X_n\}$ need not be adapted to $\{\mathscr{F}_n\}$; for each $n \geqq 1$, $\mathscr{F}_n$ can be as small as $\mathscr{B}(X_1 + \cdots + X_n)$. Of course if the X_n are uniformly bounded and are adapted to $\{\mathscr{F}_n\}$, then the theorem reduces to a result in Freedman ((1973), page 912) which generalizes that of Lévy ((1973), Corollary 68, page 249). There is a variant of the converse part of the conditional Borel–Cantelli lemma in Freedman ((1973), Proposition 39, page 920) where the X_n are subject to a growth condition. Although it does not seem likely that the growth condition implies the integrability condition in the above theorem, this result of Freedman is an easy consequence of the conditional Borel–Cantelli lemma for uniformly bounded X_n. (This is how Freedman proved the results.) On the other hand, this result of Freedman can also be proved directly and very simply by taking the present approach.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SINGAPORE
SINGAPORE 10
REPUBLIC OF SINGAPORE