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# ON THE SPEED OF CONVERGENCE IN STRASSEN'S LAW OF THE ITERATED LOGARITHM

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Here there is derived a conditon on sequences $\varepsilon_n \downarrow 0$ which implies that $P[W(n\cdot)/(2n \log\log n)^{\frac{1}{2}} \notin K^{\varepsilon_n}$ i.o.$] = 0$, where $W$ is the Wiener process and $K$ is the compact set in Strassen's law of the iterated logarithm. A similar result for random walks is also given.

**1. Introduction.** Let $\{W(t, \omega) : t \geqq 0, \omega \in \Omega\}$ be a Brownian motion process defined on some probability space $(\Omega, \mathscr{A}, P)$. We assume that $W(t, \omega)$ is continuous in $t$ for each $\omega \in \Omega$. Let $C[0, 1]$ be the space of continuous real valued function on the interval $[0, 1]$, $\| \ \|$ the sup-norm on $C$ and $\mathscr{B}$ the class of Borel sets in $C[0, 1]$. For $n \in \mathbb{N}$ let $W_n(\omega) \in C[0, 1]$ be defined by $W_n(\omega)(t) = W(nt, \omega)/n^{\frac{1}{2}}$. It is well known that the $W_n$ all have the same distribution on $(C[0, 1], \mathscr{B})$. For $r > 0$ let $K_r = \{f \in C[0, 1] : f'$ exists and $\int_0^1 (f'(t))^2\, dt \leqq r^2\}$. $K_r$ is a compact set in $C[0, 1]$ (e.g., [2] page 282). Let $H = \bigcup_{r>0} K_r$. If $A \subset C[0, 1]$ and $\varepsilon > 0$ let $A^\varepsilon = \{f \in C[0, 1]$: exist $g \in A$ with $\|f - g\| < \varepsilon\}$. One half of Strassen's law of iterated logarithm [3] states that for each $\varepsilon > 0$

$$P(\limsup_{n\to\infty} \{W_n/(2\,\mathrm{LL}\, n)^{\frac{1}{2}} \notin K_1^{\varepsilon}\}) = 0 . \tag{1.1}$$

Here and elsewhere in this paper we set $\mathrm{L}\, x = \max(\log x, 1)$. (1.1) is sharpened in this paper in the followng way:

THEOREM 1. *If* $\alpha < \frac{1}{2}$ *and* $\varepsilon_n = (\mathrm{LL}\, n)^{-\alpha}$ *then*

$$P(\limsup_{n\to\infty} \{W_n/(2\,\mathrm{LL}\, n)^{\frac{1}{2}} \notin K_1^{\varepsilon_n}\}) = 0 . \tag{1.2}$$

The result should be seen as an attempt to prove "stronger" forms of infinite dimensional log log laws, that is, to give conditions on increasing sequences of sets $A_n \subset C[0, 1]$ such that $P(W_n \notin A_n$ infinitely often$) = 0$. But of course the above theorem is far away from providing a complete solution.

All proofs of infinite dimensional log log laws consist in approximating the random variables in question by elements of the compact set which appears in the theorem to be proven. The approximation used in the proof of Theorem 1 is the following: We take a lattice on the state space with suitable span and approximate $W_n/(2\,\mathrm{LL}\, n)^{\frac{1}{2}}$ by the linear interpolation between the time points where the process passes through a lattice point. One should compare this method with the approximation used by Strassen. He takes a grid on the time axis and interpolates the process between the points of the grid. One would think it possible to obtain a theorem like the above by simultaneously refining

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the grid with the growth of $n$. This is indeed the case but the result which can be obtained is less good. In fact with this method one can prove that (1.2) hold for $\varepsilon_n = (\mathrm{LL}\, n)^{-\alpha}$ for $\alpha < \frac{1}{3}$. Strassen's approximation is more adapted to the "Gaussian character" of the Wiener process where the former approximation uses the fact that Brownian motion is a Markov process.

With the use of Skorohod imbedding one can easily derive an invariance theorem. Let $\{X_n : n \in \mathbb{N}\}$ be independent identically distributed real random variables with $EX_i = 0$, $0 < \sigma^2 = \mathrm{Var}(X_i) < \infty$ and define $\{Y_n(t) : 0 \leqq t \leqq 1\}$ as the linear interpolation of the chain $Y_n(k/n) = n^{-\frac{1}{2}} \sum_{i=1}^k X_i$.

THEOREM 2. *If* $E|X_n|^{2+\delta} < \infty$ *for some* $\delta > 0$ *and* $\alpha$ *and* $\varepsilon_n$ *are as in Theorem* 1 *then*

$$P(\limsup_{n\to\infty} \{Y_n/(2\,\mathrm{LL}\, n)^{\frac{1}{2}} \notin K_1^{\varepsilon_n}\}) = 0\,. \tag{1.3}$$

**2. Proofs.** For $\delta > 0$ let $T_1^\delta = \inf\{t : |W(t)| = \delta\}$ and inductively

$$T_n^\delta = \inf\{t : |W(t + \textstyle\sum_{i=1}^{n-1} T_i^\delta) - W(\sum_{i=1}^{n-1} T_i^\delta)| = \delta\}\,.$$

By the strong Markov property the $T_i^\delta$ are independent and identically distributed. Using a standard transformation argument one sees that $T_i^\delta$ has the distribution of $\delta^2 T_i^1$. We will write $T_i$ for $T_i^1$. $T_i$ and $1/T_i$ have absolutely continuous distributions on the positive real numbers. Let $g$ be the density of $T_i$ and $h$ the density of $1/T_i$.

From well-known expressions for the distribution of the $T_i$ (see [1], page 330) and some elementary calculations one obtains

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} \textstyle\sum_{k=0}^{\infty} (-1)^k (2k+1) \exp(-(2k+1)^2/2x) \tag{2.1}$$

$$h(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} x^{-\frac{1}{2}} \textstyle\sum_{k=0}^{\infty} (-1)^k (2k+1) \exp(-(2k+1)^2 x/2) \tag{2.2}$$

for $x \geqq 0$.

LEMMA 1. *If* $t \leqq 1$ *and* $k \in \mathbb{N}$ *then*

$$P(\textstyle\sum_{i=1}^k T_i \leqq t) \leqq \left(\dfrac{t}{k}\right)^{-3k/2} \exp(-k^2/2t)\,. \tag{2.3}$$

PROOF. Let $\Delta_k = \{(x_1, \cdots, x_k) \in \mathbb{R}^k : 0 \leqq x_i \leqq 1, \sum_{i=1}^k x_i \leqq t\}$ and $\lambda^k$ be Lebesgue measure on $\mathbb{R}^k$. Then $P(\sum_{i=1}^k T_i \leqq t) = \int_{\Delta_k} g(x_1) \cdots g(x_k) \lambda^k(d(x_1, \cdots, x_k))$. If $x_i \leqq 1$ then $g(x_i) \leqq (2/\pi)^{\frac{1}{2}} x_i^{-\frac{3}{2}} \exp(-1/2x_i)$. So

$$P(\textstyle\sum_{i=1}^k T_i \leqq t) \leqq \left(\dfrac{2}{\pi}\right)^{k/2} \lambda^k(\Delta_k) \sup_{(x_1,\cdots,x_k)\in\Delta_k} g(x_1) \cdots g(x_k)$$

$$\leqq \left(\frac{t}{k}\right)^{-3k/2} \exp(-k^2/2t)\,.$$

LEMMA 2. *There exists a number* $c > 0$ *such that for all* $k \in \mathbb{N}$ *and* $d > 1$

$$P\left(\frac{1}{k} \textstyle\sum_{i=1}^k 1/T_i \geqq d\right) \leqq (2d^{\frac{1}{2}} + c)^k \exp\left(-\dfrac{k}{2}(d-1)\right)\,. \tag{2.4}$$

PROOF. Let $\tau = (d-1)/2d$ and $\mu$ be the distribution of $1/T_i - d$. For $x \geqq 1$ $h(x) \leqq (2/\pi)^{\frac{1}{2}} x^{-\frac{1}{2}} \exp(-x/2)$.

$$\begin{aligned}\rho &= E(\exp(\tau(1/T_i - d))) = \int_0^\infty \exp(\tau(x-d))h(x)\,dx \\ &\leqq \left(e^{\frac{1}{2}} + \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty e^{\tau x} x^{-\frac{1}{2}} e^{-x/2}\,dx\right) e^{-\tau d} \\ &= (e^{\frac{1}{2}} + 2/(1-2)^{\frac{1}{2}}) e^{-\tau d} \\ &= (e^{\frac{1}{2}} + 2d^{\frac{1}{2}}) e^{-(d-1)/2} < \infty\,.\end{aligned}$$

Then

$$\begin{aligned}P\left(\frac{1}{k}\sum_{i=1}^k (1/T_i) \geqq d\right) &= P(\sum_{i=1}^k (1/T_i - d) \geqq 0) \\ &= P(\exp(\tau(\sum_{i=1}^k (1/T_i - d))) \geqq 1) \\ &\leqq E(\exp(\tau \sum_{i=1}^k (1/T_i - d))) \\ &= \rho^k\,.\end{aligned}$$

The proof of the lemma is finished.

Let $\tau_k^\delta = \sum_{i=1}^k T_i^\delta$; $\tau_0^\delta = 0$. For $\delta > 0$ we define $\{W_\delta(t): 0 \leqq t \leqq 1\}$ as follows: Let

$$\begin{aligned}m(\delta) &= \max\{k : \tau_k^\delta \leqq 1\}\,, \\ W_\delta(\tau_k^\delta) &= W(\tau_k^\delta) \qquad \text{if} \quad k \leqq m(\delta)\,, \\ W_\delta(1) &= W(\tau_{m(\delta)}^\delta)\end{aligned}$$

and $W_\delta(t)$ linearly interpolated elsewhere. Clearly

$$\sup_{0 \leqq t \leqq 1} |W_\delta(t) - W(t)| \leqq 2\delta \tag{2.5}$$

$$W_\delta \in H \qquad \text{and} \qquad \int_0^1 (W_\delta'(t))^2\,dt = \sum_{i=1}^{m(\delta)} \frac{\delta^2}{T_i^\delta}\,. \tag{2.6}$$

PROOF OF THEOREM 1. Let $\rho > 0$, $\delta > 1$, $K \in \mathbb{N}$ such that $\rho^2/K > 1$. If $W$ is as above but restricted to $[0, 1]$, then

$$\begin{aligned}P(W \notin K_\rho^{2\delta}) &\leqq P(W_\delta \notin K_\rho) = P(\sum_{i=1}^{m(\delta)} \delta^2/T_i^\delta \geqq \rho^2) \\ &= P(\textstyle\bigcup_{k=1}^\infty (\{\sum_{i=1}^k \delta^2/T_i^\delta \geqq \rho^2\} \cap \{\sum_{i=1}^k T_i^\delta \leqq 1\})) \\ &\leqq \sum_{k=1}^\infty P(\sum_{i=1}^k 1/T_i \geqq \rho^2, \sum_{i=1}^k T_i \leqq 1/\delta^2) \\ &\leqq KP(\sum_{i=1}^K 1/T_i \geqq \rho^2) + \sum_{k=K+1}^\infty P(\sum_{i=1}^k T_i \leqq 1/\delta^2) \\ &\leqq K(2\rho/K^{\frac{1}{2}} + C)^K \exp\left(-\frac{K}{2}(\rho^2/K - 1)\right) \\ &\quad + \sum_{k=K+1}^\infty \left(\frac{1}{\delta^2 k}\right)^{-3k/2} \exp\left(-\frac{\delta^2 k^2}{2}\right)\end{aligned} \tag{2.7}$$

by Lemmas 1 and 2

$$= I_K^{(\rho)} + II_K^{(\delta)}\,, \quad \text{say.}$$

If now $s > 0$ and $\varepsilon(s) = s^{-\alpha'}$ for $2\alpha < \alpha' < 1$ ($\alpha$ as in the statement of the theorem) then

$$(2.8)\qquad \begin{aligned} P(W \notin sK_1^{\varepsilon(s)}) &\leqq P(W \notin K^{s\varepsilon(s)/2}_{s(1+\varepsilon(s)/2)}) \\ &\leqq I_{[s]}^{(s(1+s^{-\alpha'/2}))} + II_{[s]}^{(s^{1-\alpha'/4})} \end{aligned}$$

for $s$ sufficiently large, where $[s]$ as usual denotes the integer part of $s$.

Here we used the relation

$$(2.9)\qquad K_1^{\varepsilon} \supset K^{\varepsilon/2}_{1+\varepsilon/2} .$$

Indeed if $f \in K^{\varepsilon/2}_{1+\varepsilon/2}$ there exists $g \in K_{1+\varepsilon/2}$ with $\|f - g\| < \varepsilon/2$. Then $g/(1 + \varepsilon/2) \in K_1$ and $\|f - g/(1+\varepsilon/2)\| \leqq \varepsilon/2 + |1-(1+\varepsilon/2)^{-1}|\,\|g\| < \varepsilon$ because $\sup\{\|g\| : g \in K_{1+\varepsilon/2}\} = 1 + \varepsilon/2$. So (2.9) and then (2.8) follow. Now by some elementary calculations one obtains

$$(2.10)\qquad I_{[s]}^{(s(1+s^{-\alpha'/2}))} \leqq \exp(-\tfrac{1}{2}s^2(1 + s^{-\alpha'}(1 + o(1))))$$

$$(2.11)\qquad II_{[s]}^{(s^{1-\alpha'/4})} \leqq \exp(-s^{2+\delta})$$

for some $\delta > 0$ and $s$ sufficiently large, where $o(1)$ in (2.10) is understood to hold for $s \to \infty$.

For $m \in \mathbb{N}$ we take $n_m = [\exp(m/\mathrm{L}\, m)]$, $s_m = (2\,\mathrm{LL}\, n_m)^{\frac{1}{2}}$. Then using (2.8), (2.10) and (2.11) and the fact that $\sum_{m=1}^{\infty} (m/\mathrm{L}\, m)^{-(1+c(\mathrm{L}\, m)^{-\gamma})} < \infty$ for $c > 0$, $\gamma < 1$ one obtains

$$(2.12)\qquad \textstyle\sum_{m=1}^{\infty} P(W \notin s_m K_1^{\varepsilon(s_m)}) < \infty .$$

Now $W_n$ has the same distribution as $W$, so we obtain from (2.12)

$$P(\limsup_{m\to\infty} \{W_{n_m} \notin (2\,\mathrm{LL}\, n_m)^{\frac{1}{2}} K_1^{\varepsilon(s_m)}\}) = 0 .$$

For $\omega \in \Omega$ not belonging to this exceptional set, there is a $N(\omega) \in \mathbb{N}$ such that for $m \geqq N(\omega)$

$$W_{n_m} \in (2\,\mathrm{LL}\, n_m)^{\frac{1}{2}} K_1^{\varepsilon(s_m)} ,$$

that is, there exists $f_m \in K_1$ such that with the abbreviation $b(n) = (2n\,\mathrm{LL}\, n)^{\frac{1}{2}}$ $\sup_{0\leq t\leq 1} |W(n_m t)/b(n_m) - f_m(t)| \leqq (\mathrm{LL}\, n_m)^{-\alpha'/2}$. If $n_m \leqq n < n_{m+1}$ and $m \geqq N(\omega)$ let $g_n(t) = f_{m+1}(tn/n_{m+1})$. Then

$$\begin{aligned} \|W(n.)/b(n) - f_{m+1}\| &\leqq \|f_{m+1} - g_n\| + (b(n_{m+1})/b(n_m))\|W(n.)/b(n_{m+1}) - g_n\| \\ &\quad + ((b(n_{m+1})/b(n_m)) - 1)\|g_n\| \\ &\leqq (1 - n_m/n_{m+1})^{\frac{1}{2}} + (b(n_{m+1})/b(n_m))(\mathrm{LL}\, n_{m+1})^{-\alpha'/2} \\ &\quad + ((b(n_{m+1})/b(n_m)) - 1) \\ &= O((\mathrm{LL}\, n_{m+1})^{-\alpha'/2}) = O((\mathrm{LL}\, n)^{-\alpha'/2}) \leqq (\mathrm{LL}\, n)^{-\alpha} \end{aligned}$$

for $n$ sufficiently large, where we used the fact that for $f \in K_1$ $|f(t) - f(s)| \leqq (t - s)^{\frac{1}{2}}$. The theorem is proved.

PROOF OF THEOREM 2. Under the hypothesis of the theorem there exists a Brownian motion $W(t)$ and a sequence $\{X_i', i \in \mathbb{N}\}$ with the same distribution as the sequence $\{X_i : i \in \mathbb{N}\}$ and $0 < \rho < \frac{1}{2}$ such that

$$T = \sup_{t \geqq 0}\{t : |W(t) - Y'(t)| \geqq t^{\frac{1}{2}-\rho}\} < \infty \quad \text{w.p. } 1$$

where $Y'(k) = \sum_{i=1}^{k} X_i'$ and linearly interpolated elsewhere (see Theorem 4.6 of [4]). If $\omega$ is not in the exceptional set where $T = \infty$ we have for $n \geqq T(\omega)$

$$\sup_{0 \leqq t \leqq 1}\left|\frac{W(nt)}{n^{\frac{1}{2}}} - \frac{Y'(nt)}{n^{\frac{1}{2}}}\right| = \sup_{0 \leqq t \leqq n}\left|\frac{W(t)}{n^{\frac{1}{2}}} - \frac{Y'(t)}{n^{\frac{1}{2}}}\right|$$
$$\leqq \sup_{0 \leqq t \leqq T(\omega)} |W(t) - Y'(t)|/n^{\frac{1}{2}} + n^{-\rho} = O(n^{-\rho}) .$$

Theorem 2 follows from Theorem 1, the above relation and the fact that the sequence $\{Y_n(t); n \in \mathbb{N}\}$ has the same distribution as $\{Y'(nt)/n^{\frac{1}{2}} : n \in \mathbb{N}\}$.

**3. Concluding remarks.** It seems to be difficult to obtain lower class statements, e.g., to derive conditions on $\varepsilon_n \downarrow 0$ such that (with notation of Theorem 1) $P(W_n/(2 \text{ LL } n)^{\frac{1}{2}} \notin K_1^{\varepsilon_n}$ infinitely often$) = 1$. It would be interesting to know if $\varepsilon_n = (\text{LL } n)^{-\alpha}$ for $\alpha \geqq \frac{1}{2}$ belongs to that class. It is a fairly trivial consequence of the well-known integral tests for lower-class functions of sums of independent random variables that $\varepsilon_n = (\text{LL } n)^{-1}$ is lower class. Indeed if $W_n(1)/(2 \text{ LL } n)^{\frac{1}{2}} > 1 + \varepsilon_n$ then $W_n/(2 \text{ LL } n)^{\frac{1}{2}} \notin K_1^{\varepsilon} n$, but from the Kolmogorov–Petrovski–Erdös test it follows that $P(W_n(1) > (2 \text{ LL } n)^{\frac{1}{2}}(1 + 1/\text{LL } n) \text{ i.o.}) = 1$. On the other hand $P(W_n(1) > (2 \text{ LL } n)^{\frac{1}{2}}(1 + (\text{LL } n)^{-\alpha}) \text{ i.o.}) = 0$ for $\alpha < 1$. But of course $W_n$ may escape from $(2 \text{ LL } n)^{\frac{1}{2}} K_1^{\varepsilon_n}$ in another way than "through" $W_n(1)$. So there remains a gap for $\frac{1}{2} \leqq \alpha < 1$.

## REFERENCES

[1] FELLER, W. (1966). *An Introduction to Probability, Theory and Its Applications* **2**. Wiley, New York.

[2] STOUT, W. F. (1974). *Almost Sure Convergence*. Academic Press, New York.

[3] STRASSEN, V. (1964). An invariance principle for the law of iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 211–226.

[4] STRASSEN, V. (1965). Almost sure behavior of sums of independent random variables and martingales. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **II** 1 315–344.

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