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LIMIT THEOREMS FOR SEMI-MARKOV PROCESSES AND RENEWAL THEORY FOR MARKOV CHAINS

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With any Harris-recurrent Markov chain one can associate a sequence of random times at which the chain has the same distribution, and the chain can thereby be shown to be equivalent to one having a recurrence point. This idea makes available a regeneration scheme for such chains, which is exploited in this paper to prove the ergodic theorem for semi-Markov processes, and a renewal theorem for Markov chains on a general state space.

1. Introduction. The circle of ideas covering the limit theory for recurrent Markov chains, the renewal theorem, the theory of regenerative events and their relations, has received much attention in the probability literature. When there exists a single point in the state space of the Markov chain which is visited infinitely often, then much of the limit theory is simplified. One can then prove the ergodic theorem for the chain via the renewal process of inter-return times to this distinguished state. This idea also works nicely in the case of a semi-Markov process. (See, e.g., Çinlar [4], also Kesten [10] for other references.) Perhaps less well known is the converse direction, namely the proof of the renewal theorem from the ergodic theorem. Such a result was demonstrated recently by McDonald [12].

A useful technique which has more recently been applied to prove limit theorems for Markov chains is that of coupling a chain having an arbitrary initial distribution to another stationary one, thereby drawing conclusions about the limit behavior of the former. The use of this device to prove ergodic theorems goes back to Doeblin [5], and has recently been developed by Griffeath [7], [8]. (These papers contain further references.) The method can also be used to give simple proofs of the renewal theorem (Lindvall [11], and Athreya, McDonald and Ney [1]).

When the chain in question returns i.o. to a point x_0, with finite mean recurrence time, then the "coupling" is easily accomplished by showing that the stationary and the general processes will ultimately be at x_0 at the same time. Even when no such recurrence point exists a clever construction shows that the role of x_0 can effectively be played by a distinguished set, and a subsequence of the sequence of hitting times of that set, at which the two processes are coupled in such a way that their marginal distributions are not changed.

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This construction motivated us to observe an even simpler one, which enables us at certain random times to treat the distinguished set as a point, and directly bring to bear the tools of regeneration points and renewal theory. The purpose of this paper is to exploit this construction (introduced in Athreya and Ney [2]) to prove ergodic and renewal theorems for semi-Markov chains. Our hypotheses are in some ways better than presently known; and the proofs are strikingly simpler than current versions, some of which are quite technical.

The regeneration scheme for Markov chains developed in [2] is summarized in Section 2. The ergodic theorem for semi-Markov chains is proved in Section 3, and a general renewal theorem of H. Kesten [10] for functionals of Markov chains is in Section 4. Kesten's paper contains a good list of references to earlier results of the kind considered here. Particularly relevant are the papers of Orey [13], [14] and Jacod [9]. In Section 5 we observe that the processes considered in this paper are equivalent to processes having a recurrence point, which are constructed by adjoining a point to the state space and suitably modifying the transition function.

2. Regeneration for Markov chains. In [2] we introduced a new approach to the limit theory of Harris-recurrent Markov chains. In this section we summarize the results needed here. Details are in [2].

Let $\{X_n: n \geqq 0\}$ be a Markov chain on a measurable space $(S, \mathscr{S})$ with transition function $P(x, E)$. We shall say that (X_n) is $(A, \lambda, \varphi, n_0)$-*recurrent* if there exists a set $A \in \mathscr{S}$, a probability measure φ on A, a $\lambda > 0$ and an integer n_0 such that

$$(2.1)\qquad \begin{aligned} &P_x(X_n \in A \text{ for some } n \geqq 1) \equiv 1 \\ &P_x(X_{n_0} \in B) \geqq \lambda\varphi(B) \qquad \text{for all } x \text{ in } A \text{ and } B \subset A . \end{aligned}$$

This notion of recurrence is equivalent to the more standard definition of *Harris recurrence* but is more convenient for our purposes. We will limit ourselves here to the case $n_0 = 1$. In the rest of the paper we shall merely write "recurrent" when we mean $(A, \lambda, \varphi, 1)$-recurrent. The following is the key result.

REGENERATION LEMMA. *If $\{X_n\}$ is recurrent then there exists a random time N such that $P_x(N < \infty) = 1$ and $P_x(X_n \in B, N = n) = \varphi(A \cap B)P_x(N = n)$ for all B in $\mathscr{S}$, x in S, and nonnegative integers n.*

The idea of the proof is to define a transition function $Q(x, E) = (P(x, E) - p\varphi(E \cap A))/(1 - p)$ for $x \in A$. If $X_n = x \notin A$, distribute X_{n+1} over S according to $P(x, \cdot)$ just as before. However, if $X_n = x \in A$, then with probability $p < \lambda$ distribute it over A according to φ, and with probability $(1 - p)$ distribute it over the entire state space S according to $Q(x, \cdot)$. (Observe that due to the definition of Q, the transition probabilities for the chain remain unchanged.) Since A is visited i.o., and each time there is (independent) probability $p > 0$ that at the next step "A is entered according to φ", this event will ultimately occur at some time $N < \infty$ a.s. (See [2] for details.)

COROLLARY 2.1. *Let $\{X_n\}$ be recurrent. Then there exists a sequence of random times N_i, $i = 1, 2, \cdots$ such that X_{N_i} has distribution φ, and the random variables $\{N_{i+1} - N_i, i = 1, 2, \cdots\}$ are* i.i.d. *and independent of N_1.*

The regeneration lemma can be used to show the existence and uniqueness of an invariant measure for $\{X_n\}$. Define

$$\nu(E) = E_\varphi(\textstyle\sum_{i=0}^{N-1} \chi_E(X_i)) , \tag{2.2}$$

where N is the first regeneration time as in the lemma. (P_x, P_μ, E_x, E_μ denote probability measures and expectations when $X_0 = x$ or has distribution μ.)

THEOREM 2.1. *Let $\{X_n\}$ be recurrent, and $\nu(\cdot)$ be as in* (2.2). *Then $\nu(\cdot)$ is an invariant measure for $\{X_n\}$, and is unique up to a multiplicative constant. Furthermore $\nu(\cdot)$ is finite if and only if $E_\varphi N < \infty$.*

COROLLARY 2.2. *An invariant probability measure $\pi(\cdot)$ for $\{X_n\}$ exists if and only if $E_\varphi N < \infty$, and in that case*

$$\pi(E) = \nu(E)/\nu(S) .$$

Proofs are in [2].

3. Ergodic theorem for semi-Markov processes on general state spaces. Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a Markov chain on a measurable space $(S, \mathscr{S})$ with transition function $P(x, E)$. Conditioned on a realization $\{X_n = x_n : n \geqq 0\}$, one is given a sequence of independent, nonnegative random variables $\{L_n\}$, such that the distribution of L_n depends only on x_n. (The more general case when L_n also depends on x_{n+1} can be treated similarly by the methods of this paper if one is willing to add a minorization hypothesis of the form $P(X_1 \in E, L_0 \in F \mid X_0 = x) \geqq \lambda\varphi(E)G_x(F)$. We believe that this condition can in fact be removed, but this work is not yet complete.) Let $V_i = L_0 + \cdots + L_{i-1}$, $i \geqq 1$, $V_0 = 0$, and consider a continuous time process in which the chain waits in each state a random length of time, defined as follows:

$$W(t) = (X_n, t - V_n) \quad \text{if} \quad V_n \leqq t < V_{n+1} , \qquad n = 0, 1, \cdots . \tag{3.1}$$

If $W(t) \equiv (Z(t), A(t))$ then $\{Z(t); t \geqq 0\}$ is called a *semi-Markov process*, the X_i's *the states* of the process and the L_i's the *sojourn times*. From its construction it is clear that $\{W(t); t \geqq 0\}$ is a Markov process, and we will now prove an ergodic theorem for this process. Such a result was first proved by Orey [14] using operator theory. More recently Jacod [9] gave a proof using the techniques of space-time harmonic functions (see also Çinlar [3] for background and other references). Kesten [10] proved a renewal theorem for the case when the L_n's are not necessarily nonnegative by applying methods somewhat similar to Feller's [6] for the one-dimensional case. Orey [14] and Kesten [10] needed topological structure on $(S, \mathscr{S})$ and an invariant probability measure for $\{X_n\}$, whereas we will need neither of these. Application of the regenerative scheme of the previous section makes possible a simpler proof, and somewhat more transparent hypotheses.

Recall that $\{X_n : n \geqq 0\}$ is assumed to be $(A, \lambda, \varphi, 1)$-recurrent. The regeneration lemma (see Corollary 2.1) assures us that for any initial distribution of X_0, there exists an infinite sequence of random times N_i, such that X_{N_i} has distribution φ. This in turn, ensures that the process $\{W(t); t \geqq 0\}$ is well defined for all t and that no explosions can occur in finite time (see Çinlar [3]). We may also take $\{W(t); t \geqq 0\}$ to be a strong Markov process.

Let $f = S \times [0, \infty) \to R^+$ be a bounded, measurable function, such that $f(x, t)$ is continuous in t for each $x \in A$, and let

$$m(t) \equiv E_\varphi(f(W(t))) . \tag{3.2}$$

By the regeneration lemma it is clear that if we can show that $\lim_t m(t)$ exists, then the same limit obtains for $E_\mu f(W(t))$, for any distribution μ of $W(0)$. We will see that $m(t)$ satisfies a one-dimensional renewal equation and apply Feller's theorem ([6], page 363) to obtain the desired result.

Let

$$T = \textstyle\sum_{i=0}^{N-1} L_i , \tag{3.3}$$

where N is as in the regeneration lemma of Section 2. Then by the regeneration property of N and the strong Markov property of $\{W(t); t \geqq 0\}$, we have,

$$m(t) = a(t) + \textstyle\int_0^t m(t-u)\, dG(u) , \tag{3.4}$$

where $a(t) = E_\varphi(f(W(t); T > t))$, and $G(u) = P_\varphi(T \leqq u)$. If $E_\varphi(T) < \infty$, $G(\cdot)$ is nonlattice, and $a(\cdot)$ is directly Riemann integrable (d.r.i.), then we can conclude from Feller's theorem ([6], page 363) that

$$\lim_{t\to\infty} m(t) = \frac{\int_0^\infty a(t)\, dt}{E_\varphi(T)} . \tag{3.5}$$

It thus remains only to determine the hypotheses ensuring the above conditions and to identify the limit in (3.5). To that end note that by definition of $W(\cdot)$,

$$\begin{aligned} a(t) &= E_\varphi\{\textstyle\sum_0^{N-1} f(X_i, t - V_i); V_i \leqq t < V_{i+1}\} \\ &= \textstyle\sum_0^\infty E_\varphi\{f(X_i, t - V_i); V_i \leqq t < V_{i+1}, N > i\} . \end{aligned}$$

Hence

$$\begin{aligned} |a(t_0 + \varepsilon) - a(t_0)| \leqq \sup f \cdot P_\varphi\{t_0 < T \leqq t_0 + \varepsilon\} \\ + \textstyle\sum_{i=0}^\infty E_\varphi\{|f(X_i, t_0 + \varepsilon - V_i) - f(X_i, t_0 - V_i)|; \\ V_i \leqq t < V_{i+1}, N > i\} . \end{aligned} \tag{3.6}$$

If t_0 is a continuity point of $G(\cdot)$ then the first term on the right side of (3.6) $\to 0$ as $\varepsilon \to 0$; and the second term $\to 0$ by the continuity of $f(x, \cdot)$, and the dominated convergence theorem. Thus the set of discontinuities of $a(t)$ is countable, and $a(t)$ is R-integrable (in the ordinary sense) on any finite interval. Since furthermore $a(t) \leqq \sup f \cdot (1 - G(t))$, we see that $a(\cdot)$ is d.r.i.

To evaluate the integral note that

$$(3.7)\qquad \int_0^\infty E_\varphi(f(W(t)); T > t)\,dt = E_\varphi\left(\int_0^T f(W(t))\,dt\right) = E_\varphi\left(\sum_{i=0}^{N-1} \int_0^{L_i} f(X_i, u)\,du\right) = E_\varphi\left(\sum_{i=0}^{N-1} m_f(X_i)\right),$$

where

$$m_f(x) = E\left(\int_0^{L_0} f(x, u)\,du \mid X_0 = x\right).$$

If $\nu(A)$ is as defined in (2.2), then

$$(3.8)\qquad \int_0^\infty E_\varphi(f(W(t)); T > t)\,dt = \int_S m_f(x)\nu(dx) = \int_S \int_0^\infty f(x, u)P(L_0 > u \mid X_0 = x)\,du\nu(dx).$$

Setting $f \equiv 1$, (3.8) yields

$$(3.9)\qquad E_\varphi(T) = \int_S m_1(x)\nu(dx) = \int_S \int_0^\infty P(L_0 > u \mid X_0 = x)\,du\nu(dx).$$

Since $\nu(\cdot)$ is an invariant measure for $\{X_n\}$, $E_\varphi(T) < \infty$ if and only if $\int m_1(x)\lambda(dx) < \infty$ for any nontrivial invariant measure $\lambda(\cdot)$. Thus we have proved

THEOREM 3.1. (Orey–Jacod–Kesten). *Assume that*:

(i) $\{X_n\}$ *is recurrent*;
(ii) $\int_S m_1(x)\nu(dx) < \infty$;
(iii) $P_\varphi(T \leqq u)$ *is nonlattice.*

Then for any bounded measurable $f: S \times [0, \infty) \to R$ *such that* $f(x, t)$ *is continuous in* t *for each* $x \in A$ *and any initial measure* μ *on* S

$$(3.10)\qquad \lim_{t\to\infty} E_\mu f(W(t)) = \frac{\int_S m_f(x)\nu(dx)}{\int_S m_1(x)\nu(dx)}.$$

REMARKS. 1. A more standard form of the above limit is

$$\frac{\int_S \int_0^\infty f(x, u)P(L_0 > u \mid X_0 = x)\,du\nu(dx)}{\int_S \int_0^\infty P(L_0 > u \mid X_0 = x)\,du\nu(dx)}.$$

This is clear since $m_f(x) = \int_0^\infty f(x, u)P(L_0 > u \mid X_0 = x)\,du$.

2. As observed in Section 2 we note that under recurrence, $\nu(\cdot)$ is a nontrivial invariant measure and is unique up to a multiplicative constant. Thus in the above theorem $\nu(\cdot)$ can be replaced by any nontrivial invariant measure $\lambda(\cdot)$.

3. The result can easily be extended to functions f such that for all $x \in A$, $f(x, t)$ is discontinuous in t on at most a fixed (independent of x) countable set C.

4. Kesten's renewal theorem for Markov chains. The situation here remains the same as in Section 3, except that the requirement that the L_n's be nonnegative is dropped. Let $g(x, t): S \times R \to R^+$ be bounded, and continuous in t for each $x \in A$. Kesten's theorem [10] is concerned with the limiting behavior of

$$(4.1)\qquad m(x, t) \equiv E_x\left(\sum_0^\infty g(X_k, t - V_k)\right)$$

where $V_0 = 0$, $V_k = \sum_{i=0}^{k-1} L_i$, $k = 1, 2, \cdots$. Let $N_0 = 0$, and $N_1, N_2, \cdots$ be the sequence of regeneration times for the chain $\{X_n\}$. Then,

$$m(x, t) = E_x(\textstyle\sum_0^{N_1-1} g(X_k, t - V_k)) + E_x(\sum_{j=1}^{\infty} \sum_{k=N_j}^{N_{j+1}-1} g(X_k, t - V_k)) . \tag{4.2}$$

Since the N_i's are regeneration times we get

$$m(x, t) = E_x(\textstyle\sum_0^{N_1-1} g(X_k, t - V_k)) + E_x(\sum_0^{\infty} K(t - V_{N_1} - S_k)) , \tag{4.3}$$

where

$$K(t) = E_\varphi(\textstyle\sum_0^{N_1-1} g(X_k, T - V_k)) , \qquad S_0 = 0 , \qquad S_k = \sum_{j=1}^{k} (V_{N_{j+1}} - V_{N_j}) .$$

Notice that the random variables

$$\{V_{N_{j+1}} - V_{N_j}, j = 1, 2, \cdots\}$$

are i.i.d. and independent of V_{N_1}.

If $\{S_k\}$ is a nonlattice random walk with finite, nonzero mean and if $K(\cdot)$ is directly Riemann integrable, then by applying the *one*-dimensional key renewal theorem (see Remark 4 below) to the random walk $\{S_k\}$, we could conclude that $E(\sum_0^\infty K(t-S_k))$ and hence $E(\sum_0^\infty K(t-V_{N_1}-S_k))$ converge to $(E(S_1))^{-1} \int_{-\infty}^{+\infty} K(u)\,du$.

The nonlattice and moment requirements can be stated as

(i) $P_\varphi(T \leqq u)$ is nonlattice in u, $(T = \sum_{i=0}^{N_1-1} L_i)$;
(ii) $E_\varphi(|T|) < \infty$;
(iii) $E_\varphi(T) > 0$.

Sufficient conditions for these are:

(i)′ Same as (i);
(ii)′ $\int_A E(|L_0| \mid X_0 = x)\nu(dx) < \infty$;
(iii)′ $\int E(L_0 \mid X_0 = x)\nu(dx) > 0$, where $\nu(E) \equiv \sum_{i=0}^{N_1-1} \chi_E(X_i))$ as in Section 2.

Let us examine what is needed for the direct Riemann integrability of $K(\cdot)$. Observe first that from its definition

$$K(t) = \textstyle\sum_{i=0}^{\infty} E_\varphi\{g(X_i, t - V_i); N > i\} ,$$

and arguing as in the proof of Theorem 3.1, the continuity hypotheses on g implies that $K(\cdot)$ is continuous, and thus R-integrable on $[0, t_0]$ for $t_0 < \infty$. Furthermore for any $h > 0$

$$\begin{aligned} \textstyle\sum_{n=-\infty}^{+\infty} \sup_{nh \leqq t \leqq (n+1)h} K(t) &\leqq E_\varphi(\textstyle\sum_0^{N_1-1} \sum_{n=-\infty}^{+\infty} \sup_{n2h \leqq t < (n+1)2h} g(X_k, t)) \\ &= \textstyle\int_R (\sum_{n=-\infty}^{+\infty} \sup_{n2h \leqq t < (n+1)2h} g(x, t))\nu(dx) . \end{aligned} \tag{4.4}$$

Thus, $K(\cdot)$ is directly Riemann integrable if $g(x, \cdot)$ is continuous for $x \in A$, and for some $h > 0$

$$\textstyle\int_S (\sum_{n=-\infty}^{+\infty} \sup_{nh \leqq t < (n+1)h} g(x, t)\nu(dx) < \infty . \tag{4.5}$$

Finally we identify the limit by observing that if $g \geqq 0$, then

$$\begin{aligned} \textstyle\int_R K(t)\,dt &= E_\varphi(\textstyle\sum_0^{N_1-1} \int_R g(X_k, t - V_k)\,dt) \\ &= E_\varphi(\textstyle\sum_0^{N_1-1} \int_R g(X_k, t)\,dt) = \int_S (\int_R g(x, t)\,dt)\nu(dx) . \end{aligned}$$

Applying the above discussion to the positive and negative parts of a measurable function g we have the following

THEOREM 4.1. (Kesten [10]). *Assume that*:

(i) $\{X_n : n \geqq 0\}$ *is recurrent*;

(ii) $\int_S E(|L_0| \mid X_0 = x)\nu(dx) < \infty$, $\int_S E(L_0 \mid X_0 = x)\nu(dx) > 0$, $P_\varphi(\sum_{i=0}^{N_1-1} L_i \leqq u)$ *is nonlattice in* u;

(iii) $g(x, t): S \times R \to R$ *is bounded, measurable, continuous in* t *for* $x \in A$, *and satisfies* (4.5).

Then,

(a) $E_\varphi(\sum_0^\infty g(X_k, t - V_k)) \to \int_S (\int_R g(x, t)\, dt)\nu(dx) / \int_S E(L_0 \mid X_0 = x)\nu(dx)$,

(b) $E_x(\sum_0^\infty g(X_k, t - V_k))$ *converges to the same limit as above, provided*

$$E_x(\textstyle\sum_0^{N_1-1} g(X_k, t - V_k)) \to 0 \quad \text{as} \quad t \to \infty . \tag{4.6}$$

A set of sufficient conditions for (4.6) is given by the following

PROPOSITION 4.1. *Let* $g: S \times R \to R$ *and* $\bar{g}(x) \equiv \sup_t |g(x, t)|$ *be measurable and satisfy*

(i) $\int \bar{g}(x)\nu(dx) < \infty$, *and*

(ii) $g(x, t) \to 0$ *as* $t \to \infty$ a.e. ν.

Then,

$$E_x(\textstyle\sum_0^{N-1} g(X_k, t - V_k)) \to 0 \quad \text{as} \quad t \to \infty \quad \text{a.e.} \quad \nu .$$

PROOF. Let $\psi(y) \equiv E_y(\sum_0^{N-1} \bar{g}(X_k))$. By (i)

$$\begin{aligned} \infty > \int \bar{g}(x)\nu(dx) &= E_\varphi(\textstyle\sum_0^{N-1} \bar{g}(X_i)) \\ &\geqq E_\varphi(\textstyle\sum_0^{N-1} \bar{g}(X_i); N > r) \geqq E_\varphi(\textstyle\sum_{i=r}^{N-1} \bar{g}(X_i); N > r) \\ &= \int_S \psi(y) P_\varphi(X_r \in dy, N > r) = \int_S \psi(y)\nu_r(dy),\ \nu_r(E) = P_\varphi(X_r \in E, N > r) . \end{aligned}$$

Thus, $\psi(y) < \infty$ a.e. $\nu_r(\cdot)$ for each r and hence a.e. $\nu(\cdot)$. Now note that if $\nu(F) = 0$ then, because $\nu(\cdot)$ is invariant for $P(\cdot)$ and satisfies $\nu(F) = \int_S P^k(y, F)\nu(dy)$ for each k,

$$\nu\{y : P_y(X_k \in F \text{ for some } k \geqq 1) > 0\} = 0 .$$

Taking $F^c = \{y : \psi(y) < \infty, g(y, t) \to 0 \text{ as } t \to \infty\}$, we see that for y in F^c

$$\textstyle\sum_0^{N-1} g(X_k, t - V_k) \to 0 \quad \text{a.e.} \quad P_y ;$$

and $|\sum_0^{N-1} g(X_k, t - V_k)| \leqq \sum_0^{N-1} \bar{g}(X_k)$, which is integrable with respect to P_y. By the dominated convergence theorem, $E_y(\sum_0^{N-1} g(X_k, t - V_k)) \to 0$ for all y in F^c.

REMARKS. 1. If $g(\cdot)$ is of the form $\chi_A(x)\chi_I(t)$ where $\nu(A) < \infty$, and I is a bounded interval then the hypothesis (iii) of the theorem as well as those of the above proposition are automatically satisfied, thus yielding *Blackwell's theorem* *for Markov chains*: Under hypotheses (i) and (ii) of Theorem 4.1,

$$E_x\{\# \text{ visits to } A \times (t - I) \text{ by } (X_n, V_n)_0^\infty\} \to c|I|\nu(A) \quad \text{a.e.} \quad \nu(\cdot),$$

where

$$c^{-1} = \int_S E(L_0 \mid X_0 = x)\nu(dx).$$

A special case of the above, when L_i's are nonnegative, is in Jacod [9], in a slightly different form.

2. As in Section 3, we may replace $\nu(\cdot)$ in Theorem 4.1 and Proposition 4.1 by any nontrivial invariant measure $\lambda(\cdot)$.

3. As in Remark 4 of Section 3, $g(x, \cdot)$ can be allowed to have discontinuities on a fixed (independent of x) countable set.

4. For completeness we state and sketch a proof of the key renewal theorem for the two sided random walk on the line, in the form in which it is used above.

THEOREM 4.2. *Let $\{S_n : n = 0, 1, 2, \cdots\}$ be a nonlattice random walk with finite positive mean. Let $K: R \to R$ be a directly Riemann integrable function. Then,*

$$M(t) \equiv E \sum_0^\infty K(t - S_n) \to \frac{1}{E(S_1)} \int_{-\infty}^{+\infty} K(u)\, du.$$

PROOF. Assume without loss of generality that $K(\cdot)$ is nonnegative. Then,

$$M(t) = E(\sum_{j=0}^\infty \sum_{M_j}^{M_{j+1}-1} K(t - S_n))$$

where $M_0 = 0, M_1, M_2, \cdots$ are successive ladder epochs for $\{S_n\}$ i.e., $M_1 = \inf\{n : n > 0, S_n > 0\}$,

$$M_{k+1} = \inf\{n : n > M_k, S_n > S_{M_k}\} \quad k = 1, 2, \cdots.$$

Let $f(t) \equiv E(\sum_{n=0}^{M_1-1} K(t - S_n))$. The (ordinary) R-integrability of K implies that its set of discontinuities in $[0, t_0]$, for any $t_0 < \infty$, is a set of measure zero. This implies the same property for f, and hence its R-integrability in $[0, t_0]$. Furthermore, since $EM_1 < \infty$ and $K(\cdot)$ is d.r.i.

$$\sum_{-\infty}^{+\infty} h \sup_{nh \leq t < (n+1)h} f(t) \leqq (\sum_{-\infty}^{+\infty} h \sup_{n2h \leq t < (n+1)2h} K(t))E(M_1) < \infty,$$

and thus $f(\cdot)$ is d.r.i.

We can write $M(t) = E \sum_{j=0}^\infty f(t - S_{M_j}) = \int_0^\infty f(t - y)U(dy)$, where $U(E) = E(\sum_0^\infty \chi_E(S_{M_j}))$. Since $S_{M_0} = 0$, $S_{M_j} = \sum_{r=1}^j Z_r$, where $Z_r = S_{M_r} - S_{M_{r-1}}$, $r = 1, 2, \cdots$ are i.i.d. nonnegative nonlattice random variables, we see by Blackwell's renewal theorem for *nonnegative* random variables that

$$U(t, t + h] \to \frac{h}{E(Z_1)}.$$

Now arguing as in Feller [7]

$$\int_0^\infty f(t - y)U(dy) \to \frac{1}{E(Z_1)} \int_{-\infty}^{+\infty} f(u)\, du.$$

But,

$$\int_{-\infty}^{+\infty} f(u)\,du = E\sum_{j=0}^{M_1-1}\int_{-\infty}^{\infty} K(t-S_n)\,dt = E(N)\int_{-\infty}^{+\infty} K(t)\,dt$$

and $E(Z_1) = E(N)E(S_1)$, implying the theorem.

5. Equivalence to processes with recurrence points. In [2] we showed that given any (A, λ, φ)-recurrent chain $\{X_n\}$ with stationary measure π, and any bounded measurable $f\colon S \to R$, one can adjoin a point Δ to S, extend f to $\tilde{f}$ on $S \cup \Delta$, and define a Markov chain $\{\tilde{X}_n\}$ with transition function $\tilde{P}$ on $S \cup \Delta$, and stationary measure $\tilde{\pi}(\cdot)$ such that

$$E_\mu f(X_n) = E_\mu \tilde{f}(\tilde{X}_n)$$

for any initial distribution μ on S, and

$$E_\pi f(X_n) = E_{\tilde{\pi}} \tilde{f}(\tilde{X}_n)\,.$$

Similarly for the semi-Markov process $X(t)$, and bounded, measurable $f\colon S \times R^+ \to R$, one can extend f to $\tilde{f}\colon (S \cup \Delta) \times R^+ \to R$ by defining

$$f(\Delta, u) = \int_A f(x, u)\varphi(dx)\,,$$

and define a corresponding process $\tilde{X}(t)$ on $S \cup \Delta$ for which

$$E_\mu f(W(t)) = E_\mu \tilde{f}(\tilde{W}(t))\,.$$

An analogous construction for the Kesten process yields

$$E_x \sum g(X_n, t - S_n) = E_x \sum \tilde{g}(\tilde{X}_n, t - \tilde{S}_n)\,, \qquad x \in S$$

with $\tilde{X}_n$ having a recurrent point.

These equivalences can be proved from the same renewal equations as appeared in the proofs in Sections 3 and 4 above. They tell us, in effect, that for the semi-Markov processes under consideration, *it is no loss of generality to assume that the underlying Markov chain has a recurrence point.*

ADDENDUM. Between the times of submission and revision of this paper we have learned of concurrent work by E. Nummelin along the lines of [2] and the present paper, contained in reports of the Helsinki University of Technology, and currently also submitted for publication. He demonstrates and utilizes the existence of an atom whose role is similar to our regeneration sets, but his approach and results differ considerably in detail from ours.

REFERENCES

[1] ATHREYA, K. B., MCDONALD, D. and NEY, P. (1977). Coupling and the renewal theorem. To appear in *Amer. Math. Monthly*.

[2] ATHREYA, K. B. and NEY, P. (1977). A new approach to the limit theory of recurrent Markov chains. To appear in *Trans. Amer. Math. Soc.*

[3] ÇINLAR, E. (1969). On semi-Markov processes on arbitrary spaces. *Proc. Cambridge Philos. Soc.* **66** 381–392.

[4] ÇINLAR, E. (1975). *Introduction to Stochastic Processes.* Prentice Hall, New Jersey.

[5] DOEBLIN, W. (1940). Éléments d'une théorie générale des Chaines Simples constantes de Markoff. *Ann. Sci. École. Norm. Sup.* Paris III **57** 61–111.
[6] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, 2nd ed., Vol. 2. Wiley, N.Y.
[7] GRIFFEATH, D. (1976). Partial coupling and loss of memory for Markov chains. *Ann. Probability* **4** 850–858.
[8] GRIFFEATH, D. (1976). *Coupling Methods for Markov Processes.* Ph.D. Thesis, Cornell University, Ithaca, N.Y., (To appear in *Advances in Math.*).
[9] JACOD, J. (1971). Théorème de renouvellement et classification pour les chaines semi-Markoviennes. *Ann. Inst. H. Poincaré Sect. B.* **7** 83–129.
[10] KESTEN, H. (1974). Renewal theory for Markov chains. *Ann. Probability* **3** 355–387.
[11] LINDVALL, T. (1977). A probabilistic proof of Blackwell's renewal theorem. *Ann. Probability* **5** 482–485.
[12] MCDONALD, D. (1975). Renewal theorem and Markov chains. *Ann. Inst. H. Poincaré Sect. B.* Vol. XI **2** 187–197.
[13] OREY, S. (1959). Recurrent Markov chains. *Pacific J. Math.* **9** 805–827.
[14] OREY, S. (1961). Change of time scale for Markov processes. *Trans. Amer. Math. Soc.* **99** 384–390.

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