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ON GENERATORS OF SUBORDINATE SEMIGROUPS

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Let X be a standard Markov process with semigroup (P_t). We show how to compute the infinitesimal generators (weak and strong) of the semigroup $Q_t f(x) = E^x\{m_t f(X_t)\}$ with $m_t = \exp(-\tau_t)$ and τ_t a right continuous, increasing strong additive functional; the computation is in terms of the infinitesimal operators of (P_t) and the Lévy system of the joint process (X, τ).

1. Introduction. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t \theta_t, P^x)$ be a standard Markov process with state space $(E, \mathcal{E})$, where E is locally compact and second countable and $\mathcal{E}$ denotes its Borel sets (see Chapter 1 of [1] for definitions). Let $m_t = (m_t)_{t \geq 0}$ be a multiplicative functional (Chapter III, [1]) which we assume to be of the form

$$m_t = \exp - \tau_t \tag{1.1}$$

where τ_t is a right continuous positive strong additive functional (Chapter IV, [1]) which we shall assume to be quasileft continuous, i.e., $\lim \tau_{T_n} = \tau_T$ almost surely for any increasing sequence $\{T_n\}$ of stopping times relative to $\{\mathcal{M}_t\}$ with limit T. For simplicity we assume $\tau_0 = 0$ a.s. which implies that for all x, $P^x\{m_0 = 1\} = 1$.

We distinguish a cementery point Δ, which we assume isolated in E, put $\zeta = \inf\{t : X_t = \Delta\}$ and we assume that $\tau_{\zeta-} = \tau_\zeta = \tau_t$ for any $t > \zeta$.

Then $(X, \tau) = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \tau_t, \theta_t, P^x)$ is a Markov additive process according to [2]. We will denote by $\mathcal{K}$, $\mathcal{K}_t$, $\mathcal{H}$ and $\mathcal{H}_t$ the usual completions of $\sigma(X_s : s \geq 0)$, $\sigma(X_s; s \leq t)$, $\sigma(X_s, \tau_s: s \geq 0)$ and $\sigma(X_s : s \leq t)$ respectively, with respect to the family $\{P^\mu : \mu \text{ finite measure on } (E, \mathcal{E})\}$. Certainly $\mathcal{K}_t \subset \mathcal{H}_t \subset \mathcal{M}_t$ but they need not be equal.

The structure of τ_t can be described relatively to the regular version of the conditional probability $P^x\{\ |\mathcal{K}\}$ on $\mathcal{M}$ (the existence of which is 2.20 in [2]), which moreover is independent of x, as follows: τ_t can be written as

$$\tau_t = \tau_t^c + \tau_t^d + \tau_t^f \tag{1.2}$$

where (see [3]) τ^c is an increasing additive functional of X, τ^d is a pure jump, increasing, additive functional which is continuous in probability relative to $P^x\{\ |\mathcal{K}\}$, and τ^f is a pure jump increasing additive functional whose jumps coincide with those of X.

The Lévy system of (X, τ) is defined to be a pair (H, L), where H is a continuous increasing additive process of X and L a kernel from $(ExR_+, \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+))$ into $(E, \mathcal{E})$ (i.e., sending a function $f \in (\mathcal{E} \otimes (\mathbb{R}_+))_+$ into a function in $\mathcal{E}_+$), and such that for any $f \in (\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+))_+$ and any previsible process $(Z_t)_{t \geq 0}$ (see [5])

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the following identity holds for all $x \in E - \{\Delta\}$.

$$E^x\{\Sigma_{s>0} Z_s f(X_s - X_s, \tau_s - \tau_s -) I_{\{X_s - \neq X_s \text{ or } \tau_s \neq \tau_s\}}$$
$$= E^x \int_0^\infty Z_s \, dH_s \int_{\Phi \times \mathbf{R}_+} L(X_s, dy, du) f(X_s, y, u).$$

The existence of Lévy systems for Markov additive processes has been recently proved in great generality by Maisonneuve in [8].

We shall make the following assumption about the kernel L.

ASSUMPTION A.1. Let $f \in b\mathcal{E}$, the mapping $x \to \int_{E \times \mathbf{R}_+} L(x, dy, du) f(y)(1 - e^{-u})$ is bounded and finely continuous.

For the definitions of fine continuity, nearly Borel sets, and any other objects or concepts we deal with and do not define explicitly, the reader should look in [1] or in [9].

For Section 2 we shall make the following assumptions about τ^c, the continuous part of τ, and the Lévy system (H, L) of (X, τ).

ASSUMPTION A.2. We assume that $\tau_t^c = \int_0^t a(X_s)\, ds$ with $x \to a(x)$ being finely continuous and bounded.

ASSUMPTION A.3. We assume that $H_t \equiv t$.

In Section 2, working under assumptions A.1, A.2 and A.3, we compare the infinitesimal (resp. weak infinitesimal) generators and the characteristic operators of the semigroups (P_t) and (Q_t), where $P_t f(x) = E^x\{f(X_t)\}$ and $Q_t f(x) = E^x\{e^{-\tau_t} f(X_t)\}$ for $t \geq 0$, $x \in E$ and $f \in b$.

If we denote by A, $\tilde{A}$, $\mathcal{A}$ respectively the (strong) infinitesimal generator, the weak infinitesimal generator, and the characteristic operator of (P_t); and by A', $\tilde{A}'$, $\mathcal{A}'$ the same objects for (Q_t); and by $\mathfrak{D}_A$, $\mathfrak{D}_{A'}$, etc., their domains; then we show that $\mathfrak{D}_A = \mathfrak{D}_{A'}$, $\mathfrak{D}_{\tilde{A}} = \mathfrak{D}_{\tilde{A}'}$, $\mathfrak{D}_{\mathcal{A}} \subset \mathfrak{D}_{\mathcal{A}'}$ and that, for example, for $f \in \mathfrak{D}_A$

$$A'f(x) = Af(x) - a(x)f(x) - \int_{E x R_+} L(x, dy, du) f(y)(1 - e^{-u}).$$

In Section 3 we show how to remove hypotheses A.2 and A.3, i.e., that we no longer need to assume that $H_t \equiv t$ and that $\tau_t^c = \int_0^t a(X_s)\, ds$.

Let us now briefly recall the notion of subordinate process. It is proved in Chapter III of [1] that if X is a strong Markov process, and m_t is a strong multiplicative functional, one can construct a new strong Markov process $\hat{X}$ (see 3.11, Chapter III, [1]) on a new sample space, etc. such that $\hat{E}^x f(\hat{X}_t) = E^x\{m_t f(X_t)\} = Q_t f(x)$ for $f \in b$. Moreover the trajectories of $\hat{X}$ are right continuous if those of X are so, and from the formula right after (3.5) (same reference) it follows that for any stopping time T, $\hat{E} f(\hat{X}_T) = E^x\{m_T f(X_T)\}$.

We should also mention that it follows from the work of Sharpe [10] and of Kunita-Watanabe [7] that if m_t is such that for $t < \zeta$, $0 < m_t \leq 1$, and m_t is adapted to the $\mathcal{K}_t$'s, then (1.1) with $\tau^d \equiv 0$ in (1.2) is indeed the general case.

2. The infinitesimal generator of Q_t. All the statements about weak infinitesimal generators, their domains, etc., that are mentioned in this section and not explicitly referenced are taken from or are obvious modifications of statements in Chapters I, II and V in Dynkin [6].

We will define

$$\begin{aligned} \mathfrak{B}_0 &= \{f \in \mathfrak{B};\ \|P_t f - f\| \to 0 \quad \text{as} \quad t\downarrow 0\}, \\ \tilde{\mathfrak{B}}_0 &= \{f \in \mathfrak{B} : P_t f(x) \to f(x)\ \forall x \quad \text{as} \quad t\downarrow 0\}, \\ \mathcal{C}_0 &= \{\text{bounded finely continuous functions}\}, \\ \mathcal{C} &= \{\text{bounded continuous functions}\}. \end{aligned} \tag{2.1}$$

Then in Section 5 of Chapter V of [6] it is shown that $\mathfrak{B}_0 \subset \mathcal{C}_0 \subset \tilde{\mathfrak{B}}_0 \subset \mathfrak{B}$, and that every $f \in \tilde{\mathfrak{B}}_0$ is nearly Borel.

Analogously to (2.1) we define

$$\begin{aligned} \mathfrak{B}_0' &= \{f \in \mathfrak{B} : \|Q_t f - f\| \to 0 \quad \text{as} \quad t\downarrow 0\} \\ \tilde{\mathfrak{B}}_0' &= \{f \in \mathfrak{B} : Q_t f(x) \to f(x)\ \forall x \quad \text{as} \quad t\downarrow 0\} \end{aligned} \tag{2.2}$$

Since we are assuming that $m_0 = 1$, from $Q_t f(x) = P_t f(x) + E^x\{(m_t - 1)f(X_t)\}$, from the right continuity of m_t and the dominated convergence theorem it follows that $\mathfrak{B}_0 = \mathfrak{B}_0'$ and $\tilde{\mathfrak{B}}_0 = \tilde{\mathfrak{B}}_0'$ and therefore the comments after (2.1) hold for the subprocess $\hat{X}$ as well.

Let us now put, for $\alpha > 0$, $\mathfrak{D}_A = U^\alpha \mathfrak{B}_0$, $\mathfrak{D}_{\tilde{A}} = U^\alpha \tilde{\mathfrak{B}}_0$, $\mathfrak{D}_{A'} = V^\alpha B_0$ and $\mathfrak{D}_{\tilde{A}'} = V^\alpha \tilde{B}_0$; where as usual $U^\alpha = \int_0^\infty e^{-\alpha t} P_t\, dt$ and $V^\alpha = \int_0^\infty e^{-\alpha t} Q_t\, dt$, and also the left-hand side of each defining identity is independent of α. It is also easy to see that $P_t \mathfrak{B}_0 \subset \mathfrak{B}_0$, $P_t \tilde{\mathfrak{B}}_0 \subset \tilde{\mathfrak{B}}_0$, $Q_t \mathfrak{B}_0 \subset \mathfrak{B}_0$ and $Q_t \tilde{\mathfrak{B}}_0 \subset \tilde{\mathfrak{B}}_0$. Also, $U^\alpha \mathfrak{B}_0 \subset \mathfrak{B}_0$, $U^\alpha \tilde{\mathfrak{B}}_0 \subset \tilde{\mathfrak{B}}_0$, $V^\alpha \mathfrak{B}_0 \subset \mathfrak{B}_0 \subset \mathfrak{B}_0$, $V^\alpha \tilde{\mathfrak{B}}_0 \subset \tilde{\mathfrak{B}}_0$, and that $\alpha U^\alpha f$ and $\alpha V^\alpha f$ both tend to f uniformly (pointwise) if $f \in \mathfrak{B}_0$ (if $f \in \tilde{\mathfrak{B}}_0$). It can also be seen that U^α and V^α are both 1 : 1 on $\mathfrak{B}_0$ and $\tilde{\mathfrak{B}}_0$, and that $\mathfrak{D}_A \subset \mathfrak{D}_{\tilde{A}} \subset \mathfrak{B}_0 \subset \tilde{\mathfrak{B}}_0$ with $\mathfrak{D}_A$ (resp. $\mathfrak{D}_{\tilde{A}}$) dense in $\mathfrak{B}_0$ (resp. $\tilde{\mathfrak{B}}_0$). (Analogous relationships hold for A' and $\tilde{A}'$.)

Let $\mathcal{O}$ and $\mathcal{O}_0$ be the original and fine topologies on E respectively. In Sections 3 and 5 of Chapter V of [6] are defined the characteristic operators $\mathfrak{A}$ ($\mathfrak{A}_0$) relative to $\mathcal{O}$ and $\mathcal{O}_0$ respectively and the following results are of interest to us:

(i) If $f \in \mathfrak{D}_A$ then f and Af are finely continuous.
(ii) If $f \in \mathfrak{D}_{\tilde{A}}$ and Af is continuous then $\mathfrak{A}f(x) = \tilde{A}f(x)$.
(iii) If $f \in \mathfrak{D}_{\tilde{A}}$ and Af is finely continuous then $\mathfrak{A}_0 f(x) = \tilde{A}f(x)$.

The way to compute A, $\tilde{A}$, A' and $\tilde{A}'$ from P_t and Q_t is through

$$\begin{aligned} \mathfrak{D}_A &= \left\{ f \in \mathfrak{B}_0 : \frac{1}{t}(P_t f - f) \text{ converges uniformly to } g \in \mathfrak{B}_0 \text{ as } t\downarrow 0 \right\} \\ \mathfrak{D}_{\tilde{A}} &= \left\{ f \in \tilde{\mathfrak{B}}_0 : \frac{1}{t}(P_t f - f) \text{ converges boundedly to } g \in \tilde{\mathfrak{B}}_0 \text{ as } t\downarrow 0 \right\} \\ \mathfrak{D}_{A'} &= \left\{ f \in \mathfrak{B}_0 : \frac{1}{t}(Q_t f - f) \text{ converges uniformly to } g \in \mathfrak{B}_0 \text{ as } t\downarrow 0 \right\} \\ \mathfrak{D}_{\tilde{A}'} &= \left\{ f \in \tilde{\mathfrak{B}}_0 : \frac{1}{t}(Q_t f - f) \text{ converges boundedly to } g \in \tilde{\mathfrak{B}}_0 \text{ as } t\downarrow 0 \right\}. \end{aligned} \tag{2.3}$$

As usual we have $(\alpha - A)U^\alpha f = f$ for $f \in \mathfrak{B}_0$ and $U^\alpha(\alpha - A)g = g$ for $g \in \mathfrak{D}_A$; $(\alpha - \tilde{A})U^\alpha f = f$ for $f \in \tilde{\mathfrak{B}}_0$ and $U^\alpha(\alpha - \tilde{A})g = g$ for $g \in \mathfrak{D}_{\tilde{A}}$; and analogous relations for A', $\tilde{A}'$ and V^α. Since $U^\alpha : \mathfrak{B}_0 \to \mathfrak{D}_A$ (resp. $U^\alpha : \tilde{\mathfrak{B}}_0 \to \mathfrak{D}_{\tilde{A}}$) and $V^\alpha : \mathfrak{B}_0 \to \mathfrak{D}_{A'}$ (resp. $V^\alpha : \tilde{\mathfrak{B}}_0 \to \mathfrak{D}_{\tilde{A}'}$) are bijections with inverses $(\alpha - A)$ (resp. $(\alpha - \tilde{A})$) and $(\alpha - A')$ (resp. $(\alpha - \tilde{A}')$), then there is an obvious bijection from $\mathfrak{D}_{A'}$ (resp. from $\mathfrak{D}_{\tilde{A}}$ to $\mathfrak{D}_{\tilde{A}'}$). The questions are: how big is $\mathfrak{D}_A \cap \mathfrak{D}_{A'}$?; what is the relationship between A and A' on $\mathfrak{D}_A \cap \mathfrak{D}_{A'}$? (and analogous questions for $\tilde{A}$, $\tilde{A}'$).

For $f \in \mathfrak{D}_A$ (resp. $\mathfrak{D}_{\tilde{A}}$) it is easy to verify that

$$f(X_t) - f(X_0) - \int_0^t Af(X_s)\, ds = M_t \tag{2.4}$$

(and analogously with A replaced by $\tilde{A}$) is a local and locally square integrable martingale with respect to $(\Omega, \mathfrak{M}_t, P^x)$ (see Chapter IV [8]), and therefore that $f(X_t) = f(X_0) + \int_0^t Af(X_s)\, ds + M_t$ is a semimartingale. By the way, if $(E, \mathcal{E}) = (\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n))$ and if the coordinate mappings are in $\mathfrak{D}_A$ (resp. $\mathfrak{D}_{\tilde{A}}$), then X is an $\mathbb{R}^n$-valued semimartingale. We can now state our main

(2.5) THEOREM. *Let (P_t), (Q_t) be as above, and let $f \in \mathfrak{D}_A$. Assume also that* A.1, A.2 *and* A.3 *of Section* 1 *hold. Then $f \in \mathfrak{D}_{A'}$ and for every $x \in E$*

$$A'f(x) = Af(x) - a(x)f(x) - \int_{E\times R_+} L(x, dy, du) f(x)(1 - e^{-u}). \tag{2.6}$$

PROOF. The key ingredient comes in from the application of the change of variables formula (21, Chapter IV, [9]) to $m_t = \exp(-\tau_t)$ to obtain

$$\begin{aligned} m_t &= m_0 - \int_0^t m_{s-}\, d\tau_s + \Sigma_{0<s\leq t}\{(m_s - m_{s-}) + m_{s-}\Delta\tau_s\} \\ &= m_0 - \int_0^t a(X_s)\, ds - \Sigma_{0<s\leq t} e^{-\tau(s-)}(1 - e^{-\Delta\tau_s}). \end{aligned} \tag{2.7}$$

From the comment right after (2.4) and taking into account that the trajectories of m_t are of finite variation, then (from 23.2, Chapter IV, [9]) it follows that

$$d(m_t f(X_t)) = m_{t-}\, df(X_t) + f(X_t)\, dm_t. \tag{2.8}$$

Let us compute $\alpha V^\alpha f(x)$. From (2.7), (2.8) and our assumption on τ_0 it follows that

$$\begin{aligned} \alpha V^\alpha f(x) &= E^x \int_0^\infty e^{-\tau_t} f(X_t)\, d(-e^{-\alpha t}) = f(x) + E^x \int_0^\infty e^{-\alpha t} d(e^{-\tau_t} f(X_t)) \\ \alpha V^\alpha f(x) &= f(x) + E^x \int_0^\infty e^{-\alpha t} e^{-\tau(t-)}\, df(X_t) + E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dm_t \\ &= f(x) + E^x \int_0^\infty e^{-\alpha t} e^{-\tau_t} Af(X_t)\, dt - E^x \int_0^\infty e^{-\alpha t} f(X_t) a(X_t)\, dt \\ &\quad - E^x \Sigma_{s>0} e^{-\alpha s} e^{-\tau_s} f(X_s)(1 - e^{-\Delta\tau_s}) \\ &= f(x) + E^x \int_0^\infty e^{-\alpha t} e^{-\tau_t} Af(X_t)\, dt - E^x \int_0^\infty e^{-\alpha} a(X_t) f(X_t)\, dt \\ &\quad - E^x \int_0^\infty e^{-\alpha s} e^{-\tau s} d(s \wedge \zeta) \int_{E\times R_+} L(X_s, dy, du) f(y)(1 - e^{-u}) \end{aligned}$$

where in the fourth step we used the fact that $E^x \int_0^\infty e^{-\alpha t} e^{-\tau_{t-}}\, dM_t = 0$ since $\int_0^t e^{-\alpha s} e^{-\tau_{s-}}\, dM_s$ is a martingale which is zero at $t = 0$ according to Theorem 20, Chapter IV, [9]; and for the fourth step we used the definition of Lévy system given in Section 1. Also, since f is bounded and by our assumption on the Lévy system, $x \to \int_{E\times\mathbb{R}_+} L(x, dy, du) f(y)(1 - e^{-u})$ is bounded.

We have proved that

$$V^\alpha(\alpha - A - a(\cdot) + L(\cdot)) f(x) = f(x) \tag{2.9}$$

where $a(\cdot)f(x) = a(x)f(x)$ and $L(\cdot)f(x) = \int_{E\times R_+} L(x, dy, du)f(y)(1 - e^{-u})$.

From (2.8), and a similar computation, one obtains

(2.10)
$$Q_t f(x) = E^x\{m_t f(X_t)\} = f(x) - E^x\int_0^t m_s a(X_s)f(X_s)f(X_s)\, ds$$
$$+ E^x\int_0^t m_s Af(X_s)\, ds - E^x\int_0 m_s d(s \wedge \zeta)\int_{E\times R_+} L(X_s, dy, du)f(y)(1 - e^{-u});$$

then taking into account our regularity assumptions, from (2.10) it follows that

(2.11) $$A'f(x) = Af(x) - a(x)f(x) - \int_{E\times R_+} L(x, dy, du)f(y)(1 - e^{-u})$$

and (2.9) can be rewritten as $V^\alpha(\alpha - A')f(x) = f(x)\ \forall x \in E$.

(2.12) COROLLARY. *Under the assumptions of Theorem* (2.5) *it follows that* $\mathfrak{D}_A = \mathfrak{D}_{A'}$.

PROOF. From Theorem (2.5) it follows that $\mathfrak{D}_A \subset \mathfrak{D}_{A'}$ and there is a bijection between the two sets.

REMARK. To know whether A and A' determine the processes X and $\hat{X}$ (up to equivalence, at least) one should know whether it is possible to extend (P_t) and (Q_t) from $\mathfrak{B}_0$ to transition semigroups on $b\mathcal{E}$ in such a way that $Q_t f \leqslant P_t f$, $\forall t$, $\forall f \in \mathcal{E}_+$ holds. Then one would have to prove that the process with transition semigroup (Q_t) can be made equivalent to a process obtained by subordination from the process associated to (P_t) with respect to a multiplicative functional equivalent to m.

The proof of the next theorem can be carried out in the same way as the proof of (2.5).

(2.13) THEOREM. *Let* (P_t) *and* (Q_t) *be as above, and assume that hypotheses* A.1, A.2 *and* A.3 *hold. Let* $f \in \mathfrak{D}_{\tilde{A}}$, *then* $f \in \mathfrak{D}_{\tilde{A}'}$ *and* $\mathfrak{D}_{\tilde{A}} = \mathfrak{D}_{\tilde{A}'}$.

PROOF. The fact that $\mathfrak{D}_{\tilde{A}} = \mathfrak{D}_{\tilde{A}'}$ follows from $\mathfrak{D}_{\tilde{A}} \subset \mathfrak{D}_{\tilde{A}'}$, since there is a bijection between $\mathfrak{D}_{\tilde{A}}$ and $\mathfrak{D}_{\tilde{A}'}$.

In order to prove that $\mathfrak{D}_{\tilde{A}} \subset \mathfrak{D}_{\tilde{A}'}$ we take $h \in \mathfrak{D}_{\tilde{A}}$ and write it as $h = U^\alpha f$ for some $\alpha > 0$ and some $f \in \tilde{B}_0$, and prove that $\lim_{t\downarrow 0}\{Q_t U^\alpha f(x) - U^\alpha f(x)\}t^{-1}$ exists for every $x \in E$. All we need to note is that $U^\alpha f$ being in $\mathfrak{D}_{\tilde{A}}$ implies that $U^\alpha f(X_t)$ is a semimartingale and that, as in the proof of Theorem 2.5,

$$\frac{1}{t}(Q_t U^\alpha f(x) - U^\alpha f(x)) = \frac{1}{t}E^x\int_0^t d(e^{-\tau_s}U^\alpha f(X_s))$$
$$= \frac{1}{t}E^x\int_0^t U^\alpha f(X_s)de^{-\tau_s} + \frac{1}{t}\int_0^t e^{-\tau(s-)}\, dU^\alpha f(X_s)$$
$$= \frac{1}{t}E^x\int_0^t U^\alpha f(X_s)a(X_s)\, ds$$
$$- \frac{1}{t}E^x\int_0^t e^{-\tau_s}d(s \wedge \zeta)\int_{E\times R_+} L(X_s, dy, du)\, U^\alpha f(y)(1 - e^{-u})$$
$$+ \frac{1}{t}E^x\int_0^t e^{-\tau_s}\tilde{A}U^\alpha f(X_s).$$

Taking limits as $t\downarrow 0$, we see that convergence takes place as desired to

$$\tilde{A}'U^{\alpha}f(x) = \tilde{A}'h(x) = \tilde{A}h(x) - a(x)h(x) - \int_{E\times R_+} L(x, dy, du)h(y)(1 - e^{-u}). \tag{2.14}$$

REMARK. (2.14) gives an explicit way of computing $\tilde{A}'$ in terms of $\tilde{A}$ and (a, L).

We mentioned in Section 1 that the subordinate process $\hat{X}$ was a right continuous strong Markov process, and therefore one can define its characteristic operators $\mathcal{C}'$ and $\mathcal{C}_0'$ relative to the $\mathcal{O}$ and $\mathcal{O}_0$ topologies on E, and we have:

(2.14) THEOREM. *Let* (Q_t), (P_t) *be as above and assume that* A.1, A.2 *and* A.3 *hold. Let* $f \in \mathfrak{D}_{\mathcal{C}}$, *then* $f \in \mathfrak{D}_{\mathcal{C}'}$ *and*

$$\mathcal{C}'f(x) = \mathcal{C}f(x) - a(x)f(x) - \int_{ExR_+} L(x, dy, du)(1 - e^{-u}) \tag{2.15}$$

for every x.

REMARK. The same theorem holds true for $\mathcal{C}$ and $\mathcal{C}'$ replaced by $\mathcal{C}_0$ and $\mathcal{C}_0'$. Note also that if we allow $\mathcal{C}$ not to be defined at every $x \in E$, then $\mathcal{C}'$ would only be defined where $\mathcal{C}$ is.

PROOF. It follows as in the previous theorems from

$$e^{-\tau(T)}f(X_T) = \int_0^T f(X_s)d(e^{-\tau_s}) + \int_0^T e^{-\tau(s-)}\,df(X_s)$$

for any stopping time T. Note that for $x \neq \Delta$, and putting $T = T(\mathcal{U}) = \inf\{t > 0 : X_t \notin \mathcal{U}\}$, $\hat{T} = \inf\{t : \hat{X}_t \notin \mathcal{U}\}$ we have $\hat{E}^x\{f(\hat{X}_{\hat{T}})\} = E^x\{e^{-\tau(T)}f(X_T)\}$ and

$$\hat{E}^x\{\hat{T}\} = E^x\int_0^T e^{-\tau_t}\,dt$$

and it is not hard to verify that

$$E^x\left\{\int_0^{T(\mathcal{U})} e^{-\tau_t}\,dt\right\}/E^x\{T(\mathcal{U})\} \to 1$$

when we take any sequence of neighborhoods $\mathcal{U}$ decreasing to x such that $T(\mathcal{U})\downarrow 0$.

The results of this section generalize some of the results of Section 4, Chapter IX of [6].

3. Reduction to the simple case.

As we said before, in this section we prove that in order to compute the characteristic operator of (Q_t), our assumptions about τ_t^c and the Lévy system of (X, τ) are good enough.

In this section we will be following [4] quite closely and we shall assume that $m_t = \exp - \tau_t$ where τ_t is a quasileft continuous strong additive functional such that

$$\tau_t = \tau_t^c + \tau_t^d + \tau_t^f$$

where τ^c, the continuous part of τ, is a continuous additive functional of X; τ^d is a pure jump increasing additive process which is continuous in probability with respect to $P^x\{\ |\mathcal{K}\}$; and τ^f is a pure jump increasing additive process whose jumps coincide with those of X.

Let (H', L') denote a Lévy system for (X, τ), with H' being a continuous additive functional of X and L' a transition kernel such that for all positive $f \in \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+)$, all $x \in E$ and $t > 0$,

$$E^x\{\Sigma_{0<s\leqslant t} f(X_{s-}, X_s, \tau_s - \tau_{s-}) I_{\{X_{s-}\neq X_s \text{ or } \tau_{s-}\neq \tau_s\}} = E^x \int_0^t dH'_s \int_{E x \mathbb{R}_+} L'(X_s, dy, du) f \qquad (X_s, y, u)$$

Again as in [4] we put

$$H_t = H'_t + \tau_t^c + t. \tag{3.2}$$

Then H_t is a strictly increasing continuous additive functional of X. There exists positive, $\mathcal{E}$-measurable functions $a(x)$ and $h(x)$ (Chapter V, [1]) such that

$$\tau_t^c = \int_0^t a(X_s)\, dH_s, \qquad H'_t = \int_0^t h(X_s)\, dH_s. \tag{3.3}$$

We shall also define

$$L(x, ., .) = h(x) L'(x, ., .) \tag{3.4}$$

$$L^d(x, B) = L(x, \{x\}, B), \qquad L^j(x, A, B) = L(x, A - x, B) \tag{3.5}$$

$$K(x, A) = L^j(x, A, R_+), \qquad F(x, y; B) = \frac{L^x(x, dy, B)}{K(x, dy)} \tag{3.6}$$

where L^d describes the jumps of τ^d given X, and F describes the jumps of τ^j given that X has had a jump (see [4]).

From (3.4), (3.3) and (3.2) it follows that

$$E^x\{\Sigma_{s\leqslant t} f(X_{s-}, X_s, \tau_s - \tau_{s-}) I_{\{X_{s-}\neq X_s \text{ or } \tau_{s-}\neq\tau_s\}}\} = E^x \int_0^t dH_s \int_{E x R_+} (X_s, dy, du) f(X_s, y, u)$$

for all $x \in E$, $t > 0$ and positive $f \in \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+)$.

Let us now define $\sigma_t = \inf\{s : H_s > t\}$. Since H_s is continuous and strictly increasing, it follows that σ_t is continuous, strictly increasing and $\sigma_\infty = \lim_{t\to\infty} \sigma_t = \infty$. If we define $Y_t = X_{\sigma_t}$, $S_t = \tau_{\sigma_t}$, $\tau = \mathfrak{M}$, $\tau_t = \mathfrak{M}_{\sigma_t}$, $\hat{\theta}_t = \theta_{\sigma_t}$, then we can restate Proposition (2.35) of [3] as:

(3.7) PROPOSITION. *Let (X, τ) be a Markov additive process having a Lévy system (H, L) with strictly increasing H. Let (σ_t) and $(Y_t, S_t)_{t\geqslant 0}$ as above. Then $(Y, S) = (\Omega, \tau, \tau_t, Y_t, S_t, \hat{\theta}_t, P^x)$ is a Markov additive process with Lévy system $(\hat{H}, L)$, where $\hat{H}_t = t \wedge \hat{\zeta}$ and $\hat{\zeta} = H_\zeta$.*

We also have the following two results taken from [1].

(3.8) PROPOSITION. *Let X be a standard Markov process; then Y (the time change of X relative to σ_t) is a standard Markov process with lifetime $\hat{\zeta} = H_\zeta$.*

This is exercise (2.11) in Section 2, Chapter V of [1].

(3.9) PROPOSITION. *Let X and Y be as above. Then X and Y have the same hitting distributions.*

This corresponds to the comments at the beginning of Section 5, Chapter V of [1]. Recall that this means that if $D \in \mathcal{E}$ and $T = \inf\{t > 0 : X_t \in D\}$, $T' = \inf\{t > 0 : Y_t \in D\}$ and $f \in \mathcal{E}$, then for all $x \in E$, $E^x f(X_T) = \hat{E}^x f(Y_{T'})$.

In Section 1 we denoted by $\hat{X}_t$ the canonical subprocess corresponding to $e^{-\tau_t}$. Let us now denote by $\hat{Y}_t$ the process $\hat{X}_{\sigma_t}$, and state

(3.10) LEMMA. *The process $\hat{Y}_t$ is the canonical subprocess (of Y) corresponding to $S_t = \tau_{\sigma_t}$.*

PROOF. Let $f \in b\mathcal{E}$, then $\hat{E}^x f(\hat{X}_\sigma) = E^x\{e^{-\tau_{\sigma_t}} f(X_{\sigma_t})\}$ for all $x \in E$.

From $T' = \sigma_T$ for $T = \inf\{t > 0 : X_t \in D\}$ and $T' = \inf\{t > 0 : Y_t = X_{\sigma_t} \in D\}$, and $\hat{T}' = \sigma_{\hat{T}}$ for $\hat{T} = \inf\{t > 0 : \hat{X}_t \in D\}$ and $\hat{T}' = \inf\{t > 0 : \hat{Y}_t = \hat{X}_{\sigma_t} \in D\}$, the following proposition follows easily (see comment at the end of Remark 5.9, Section 3, Chapter V of [6]).

(3.11) PROPOSITION. *The processes X and Y both have the same characteristic operator. The same holds true for the processes $\hat{X}$ and $\hat{Y}$.*

Now, to end up this section, recall that $\mathcal{D}_A \subset \mathcal{D}_{\mathcal{Q}} \subset \mathcal{D}_{\tilde{A}}$ and $\mathcal{D}_{A'} \subset \mathcal{D}_{\mathcal{Q}'} \subset \mathcal{D}_{\tilde{A}'}$. Therefore as far as A and A' (resp. $\mathcal{Q}$ and $\mathcal{Q}'$) goes, there is no loss of generality in working under the assumptions of Section 2. As far as $\tilde{A}$ and $\tilde{A}'$ goes, we do not lose anything if we work with functions f such that $\tilde{A}f$ is continuous (resp. finely continuous) for then $\tilde{A}f(x) = \mathcal{Q}f(x)$ (resp. $\tilde{A}f(x) = \mathcal{Q}_0 f(x)$) and $\tilde{A}'f(x) = \mathcal{Q}'f(x)$ (resp. $\tilde{A}'f(x) = \mathcal{Q}_0 f(x)$) for all $x \in E$.

4. A generalization of the Feynman-Kac formula. Suppose we have a standard process $\tilde{X} = (\tilde{\Omega}, \mathcal{G}, \mathcal{G}_t, \tilde{X}_t, \tilde{\theta}_t, \tilde{P}^x)$ on $(E, \mathcal{E})$ with infinitesimal generator A defined on $\mathcal{D}_A$ and dense in $\mathcal{B}_0$. Suppose moreover that we are given a finely continuous function $a(x)$ and a kernel $L(x, A, B)$ from $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}_+)$ into $\mathcal{E}$ such that $x \to L(x, A, B)$ is finely continuous, $\forall A \in \mathcal{E}$, $\forall B \in \mathcal{B}(R_+)$.

Assume that from these ingredients one can construct a Markov additive process $(X, \tau) = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \tau_t, \theta_t, P^x)$, such that $X = (\Omega, \mathcal{M}, \mathcal{M}_t, \theta_t, X_t, P^x)$ is a standard process, equivalent to $\tilde{X}$, and that τ is an increasing strong additive functional such that a and L are as in Section 1, then we have

(4.1) THEOREM. *With the notations introduced above, if $f \in \mathcal{D}_A$, then the solution $\mathcal{U}(t, x)$ of*

$$\frac{\partial \mathcal{U}}{\partial t}(t, x) = A\mathcal{U}(t, x) - a(x)\mathcal{U}(t, x) - \int_{E\times R} L(x, dy, du)\mathcal{U}(t, x)(1 - e^{-u})$$

with $\mathcal{U}(0, x) = f(x)$ is given by $\mathcal{U}(t, x) = E^x\{e^{-\tau_t} f(X_t)\}$.

But we are leaving the point as it is for the time being.

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