The Annals of Probability
1979, Vol. 7, No. 1, 179–183

CONDITIONAL EXPECTATION AND ORDERING

BY EP DE JONGE

Katholieke Universiteit, Nijmegen

Let $(\Omega, \mathcal{Q}, \mu)$ be a probability space and let L be an ideal in $M(\Omega, \mathcal{Q}, \mu)$ containing χ_Ω. A one-one correspondence between the class of "order closed" linear subspaces of L and the sub σ-algebras of $\mathcal{Q}$ is proved. Furthermore, if $T : L \to M(\Omega, \mathcal{Q}, \mu)$ is a strictly positive order continuous projectionlike linear map then T is shown to be a conditional expectation $E_\nu(. \,|\mathcal{Q}_0)$. It follows that if $T : L \to M(\Omega, \mathcal{Q}, \mu)$ is a positive expectation invariant projectionlike linear map, then even $T = E_\mu(. \,|\mathcal{Q}_0)$.

1. Introduction. In [3] we generalized the concept of conditional expectation to the Riesz space case. We are aware that many mathematicians working in the field of probability theory are not familiar with Riesz space theory. Therefore, we present in this paper the measure theoretical analogues of two of the main results of [3], provided with measure theoretical proofs. Our first main result is a one-one correspondence between sub σ-algebras and certain subspaces and our second main result is a representation theorem for conditional expectations. The main difference between our results and the results known so far is that we do not work with norms (although applications to, for instance, L_p-spaces are immediate) but that our results are based on the natural partial ordering in function spaces.

Throughout this paper $(\Omega, \mathcal{Q}, \mu)$ will be a fixed probability space. By $M(\Omega, \mathcal{Q}, \mu)$ we shall denote the collection of all measurable real-valued functions on Ω, where μ-a.e. equal functions are identified. For $f, g \in M(\Omega, \mathcal{Q}, \mu)$ we write $f \leqslant g$ whenever $f(x) \leqslant g(x)$ for μ-almost every $x \in \Omega$. The notations $f \vee g$ and $f \wedge g$ are with respect to this partial ordering. If K is a subset of $M(\Omega, \mathcal{Q}, \mu)$, then K^+ will denote the collection of all $f \in K$ for which $f \geqslant 0$. Furthermore, the characteristic function of a set A will be denoted by χ_A. The theory of $L_p(\Omega, \mathcal{Q}, \mu)$-spaces will be assumed to be known $(1 \leqslant p \leqslant \infty)$. Finally, for an element $f \in L_1(\Omega, \mathcal{Q}, \mu)$, the integral with respect to μ will be denoted by $E_\mu(f)$.

2. Measurable subspaces and σ-algebras. In this section, let L be an ideal in $M(\Omega, \mathcal{Q}, \mu)$ containing the element χ_Ω. Hence, L is a linear subspace of $M(\Omega, \mathcal{Q}, \mu)$ such that $f \in L$, $g \in M(\Omega, \mathcal{Q}, \mu)$ and $|g| \leqslant |f|$ implies $g \in L$. The condition $\chi_\Omega \in L$ is a smoothness condition rather than an essential one. We observe that $f, g \in L$ implies $f \vee g, f \wedge g, |f| \in L$ and that $L_\infty(\Omega, \mathcal{Q}, \mu) \subset L$. The following definition gives the order analogue of measurable subspace as defined in [2].

DEFINITION 2.1. A linear subspace L_0 of L is said to be a measurable subspace of L if

(i) $f, g \in L_0$ implies $f \vee g \in L_0$,

Received February 28, 1977.
AMS 1970 *subject classifications.* Primary 60A05; Secondary 46E30, 47B55.
Key words and phrases. Conditional expectation, measurable subspace.

(ii) $f_1, f_2, \cdots \in L_0^+$ and $f_n \uparrow f(\mu\text{-a.e.})$ in L implies $f \in L_0$,
(iii) $\chi_\Omega \in L_0$.

The following theorem (which is a corollary of Theorem 3.5 of [3]) gives a characterization of measurable subspaces similarly as in the work of [2].

THEOREM 2.2. *Let L_0 be a linear subspace of L. Then the following assertions are equivalent.*

(a) *L_0 is a measurable subspace of L.*
(b) *There exists an up to μ-null sets unique sub σ-algebra $\mathcal{Q}_0$ of $\mathcal{Q}$ such that $L_0 = L \cap M(\Omega, \mathcal{Q}_0, \mu)$.*

PROOF. (i) (b) $\Rightarrow$ (a) is obvious. (ii) (a) $\Rightarrow$ (b). Define $\mathcal{Q}_0 = \{A \in \mathcal{Q} : \chi_A \in L_0\}$. Then $\mathcal{Q}_0$ is a sub σ-algebra of $\mathcal{Q}$. This can be seen as follows. It is clear that $\chi_\Omega \in L_0$, so $\Omega \in \mathcal{Q}_0$. Furthermore, if $A \in \mathcal{Q}_0$, then

$$\chi_{A^c} = \chi_\Omega - \chi_A \in L_0,$$

so $A^c \in \mathcal{Q}_0$. Finally, if $A_1, A_2, \cdots \in \mathcal{Q}_0$ are given and if $A = \cup_{n=1}^\infty A_n$, then

$$\chi_A = \lim_{k\to\infty} \chi_{\cup_{n=1}^k A_n} = \lim_{k\to\infty} \bigvee_{n=1}^k \chi_{A_n} \in L_0,$$

so $A \in \mathcal{Q}_0$. Next we show that $L_0 = L \cap M(\Omega, \mathcal{Q}_0, \mu)$ holds. Note already that $L \cap M(\Omega, \mathcal{Q}_0, \mu) \subset L_0$ is clear. For the converse direction, let $f \in L_0^+$ be given. For all $\alpha \in \mathbb{Q}^+$, set $f_\alpha = (f - \alpha\chi_\Omega) \vee 0$. Then $f_\alpha \in L_0^+$. Furthermore, for $\alpha \in \mathbb{Q}^+$, let

$$A_\alpha = \{x \in \Omega : f_\alpha(x) \neq 0\} \in \mathcal{Q}$$

(A_α is defined up to a μ-null set). Since

$$\chi_{A_\alpha} = \sup\{\chi_{A_\alpha} \wedge nf_\alpha : n = 1, 2, \cdots\}$$

and since

$$\chi_{A_\alpha} \wedge nf_\alpha = \chi_\Omega \wedge nf_\alpha \in L_0$$

for all n, it follows that $\chi_{A_\alpha} \in L_0$, so $A_\alpha \in \mathcal{Q}_0$. It is clear that

$$f = \sup\{\alpha\chi_{A_\alpha} : \alpha \in \mathbb{Q}^+\},$$

so $f \in M(\Omega, \mathcal{Q}_0, \mu)$. The uniqueness of $\mathcal{Q}_0$ is clear. Thus the theorem has been proved.

We observe that $\mathcal{Q}_0$ in the preceding theorem can also be defined by setting $\mathcal{Q}_0$ as the smallest sub σ-algebra of $\mathcal{Q}$ for which all $f \in L_0$ are measurable.

3. A representation theorem. In this section let L be an ideal in $L_1(\Omega, \mathcal{Q}, \mu)$ such that $\chi_\Omega \in L$. Furthermore, unless stated otherwise, T will denote a linear map from L into $M(\Omega, \mathcal{Q}, \mu)$ satisfying

(T1) If $f \in L^+, f \not\equiv 0$, then $Tf \geqslant 0, Tf \not\equiv 0$;
(T2) $f_n \downarrow 0$ in L^+ implies $Tf_n \downarrow 0$ in $M(\Omega, \mathcal{Q}, \mu)$;
(T3) $T(\chi_\Omega) = \chi_\Omega$;
(T4) $T^2 = T$ on $L \cap T^{-1}(L)$.

Before proving our second main result we derive some auxiliary lemmas.

LEMMA 3.1. *$T(L) \cap L_\infty(\Omega, \mathcal{A}, \mu)$ is a measurable subspace of $L_\infty(\Omega, \mathcal{A}, \mu)$.*

PROOF. For brevity, write $L_0 = T(L) \cap L_\infty(\Omega, \mathcal{A}, \mu)$. It is clear that L_0 is a linear subspace of $L_\infty(\Omega, \mathcal{A}, \mu)$ and that $\chi_\Omega \in L_0$. Furthermore if $f_1, f_2, \cdots$ is a sequence in L_0 such that $f_n \uparrow f$ in $L_\infty(\Omega, \mathcal{A}, \mu)$, then $Tf_n = f_n$ by T4 and $Tf_n \uparrow Tf$ by T2. Hence

$$Tf = \sup Tf_n = \sup f_n = f,$$

so $f \in T(L)$. This shows that $f \in L_0$. Finally, let $f, g \in L_0$ be given. There exists a constant $C \geqslant 0$ such that

$$0 \leqslant f \vee 0 \leqslant f + C\chi_\Omega.$$

Hence, since $L_\infty(\Omega, \mathcal{A}, \mu) \subset$ L,

$$0 \leqslant T(f \vee 0) \leqslant Tf + CT(\chi_\Omega) = f + C\chi_\Omega \in L_\infty(\Omega, \mathcal{A}, \mu),$$

so $T(f \vee 0) \in L$. Moreover

$$T(f \vee 0) \geqslant Tf \vee T0 = f \vee 0$$

and $T(T(f \vee 0) - f \vee 0) = T(f \vee 0) - T(f \vee 0) = 0$, so $T(f \vee 0) = f \vee 0$ has to hold by T1. This shows that $f \vee 0 \in L_0$. Thus

$$f \vee g = ((f - g) \vee 0) + g \in L_0.$$

LEMMA 3.2. (a) *There exists an up to μ-null sets unique sub σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$ such that $L_\infty(\Omega, \mathcal{A}_0, \mu) = T(L) \cap L_\infty(\Omega, \mathcal{A}, \mu)$.*

(b) *For all $f \in L_\infty(\Omega, \mathcal{A}, \mu)$ we have $Tf \in L_\infty(\Omega, \mathcal{A}_0, \mu)$.*
(c) *For all $A \in \mathcal{A}_0$ and for all $f \in L_\infty(\Omega, \mathcal{A}, \mu)$ we have*

$$\chi_A Tf = T(\chi_A f).$$

PROOF. (a) is clear from Lemma 3.1 and Theorem 2.2.

(b) If $f \in L_\infty(\Omega, \mathcal{A}, \mu)$, then there exist $a, b \in \mathbb{R}$ such that

$$a\chi_\Omega \leqslant f \leqslant b\chi_\Omega.$$

Hence

$$a\chi_\Omega = aT(\chi_\Omega) \leqslant Tf \leqslant bT\chi_\Omega = b\chi_\Omega,$$

so $Tf \in L_\infty(\Omega, \mathcal{A}, \mu)$ and by (a) also $Tf \in L_\infty(\Omega, \mathcal{A}_0, \mu)$.
(c) We may assume that $0 \leqslant f \leqslant \chi_\Omega$. Now, let $A \in \mathcal{A}_0$ be given. Then

$$Tf = T(\chi_A f) + T(\chi_{A^c} f); \qquad Tf = \chi_A Tf + \chi_{A^c} Tf.$$

Furthermore, $\chi_{A^c} \in T^{-1}(L) \cap L$, so $T(\chi_{A^c}) = \chi_{A^c}$ and

$$0 \leqslant T(\chi_{A^c} f) \leqslant T(\chi_{A^c}) = \chi_{A^c}.$$

This shows that $\chi_A T(\chi_{A^c} f) = 0$ which implies

$$\chi_A Tf = T(\chi_A f).$$

Next, we state and prove our second main result.

THEOREM 3.3. *Let T satisfy* T1–T4. *Then there exists a unique (up to μ-null sets) sub σ-algebra $\mathcal{Q}_0$ of $\mathcal{Q}$ and there exists a probability measure ν on $\mathcal{Q}$ such that*

(i) $Tf = E_\nu(f|\mathcal{Q}_0)$ *for all* $f \in L$;
(ii) $\mu = \nu$ *on* $\mathcal{Q}_0$;
(iii) $\nu \ll \mu$.

PROOF. Let $\mathcal{Q}_0$ be the σ-algebra of Lemma 3.2(a). For all $A \in \mathcal{Q}$ set

$$\nu(A) = E_\mu(T(\chi_A)).$$

Then ν is clearly a probability measure satisfying (ii) and (iii). In order to verify (i) it suffices to show that $T\chi_A = E_\nu(\chi_A|\mathcal{Q}_0)$ for all $A \in \mathcal{Q}$ (by T2). To this end, let $A \in \mathcal{Q}$ and $B \in \mathcal{Q}_0$ be given. Since $T(\chi_A) \in L^+_\infty(\Omega, \mathcal{Q}_0, \mu)$ and since $\mu = \nu$ on $\mathcal{Q}_0$ it follows that

$$E_\nu(\chi_B T(\chi_A)) = E_\mu(\chi_B T(\chi_A)) = E_\mu(T(\chi_{B\cap A})) = \nu(B \cap A),$$

by Lemma 3.2(b), (c). This is the desired result.

REMARK. (i) In [3] we worked with directed (not necessarily countable) sets instead of sequences. However, since L is super Dedekind complete in the present case it follows that the results are the same.

(ii) In many cases condition T2 of Theorem 3.3 can be deleted. We present two examples. For more detailed results we refer the reader to [3].

COROLLARY 3.4. *Let $L = L_p(\Omega, \mathcal{Q}, \mu)$ $(1 \leqslant p < \infty)$ and assume that $T : L \to L$ is a linear map satisfying* T1, T3 *and* T4. *Then $Tf = E_\nu(f|\mathcal{Q}_0)$ for all $f \in L$, where ν and $\mathcal{Q}_0$ are as in Theorem* 3.3.

PROOF. We have to verify that T2 holds. To this end, let $f_n \downarrow 0$ in L and assume that $Tf_n \geqslant g \geqslant 0$ for all n, where $g \in L$ is such that $g \not\equiv 0$. Next, let $h \in L_q^+(\Omega, \mathcal{Q}, \mu)$ $(p^{-1} + q^{-1} = 1)$ be such that $E_\mu(hg) > 0$ and define

$$\varphi(f) = E_\mu(hTf)$$

for all $f \in L$. Then φ is a positive linear functional on L, so φ is continuous by Proposition II.5.5 of [6]. Hence, there exists an $h_0 \in L_q^+(\Omega, \mathcal{Q}, \mu)$ such that $\varphi(f) = E_\mu(h_0 f)$. Hence

$$0 = \lim \varphi(f_n) = \lim E_\mu(hTf_n) \geqslant E_\mu(hg) > 0,$$

which is the desired contradiction.

Finally, we present necessary and sufficient conditions so that $\nu = \mu$ on $\mathcal{Q}$ has to hold in Theorem 3.3.

COROLLARY 3.5. *Let $T : L \to M(\Omega, \mathcal{Q}, \mu)$ be a linear map satisfying*

(T1)′ *If $f \in L^+$, then $Tf \in M^+(\Omega, \mathcal{Q}, \mu)$*;
(T3) $T(\chi_\Omega) = \chi_\Omega$,
(T4) $T^2 = T$ *on* $L \cap T^{-1}(L)$;
(T5) $E_\mu(f) = E_\mu(Tf)$ *for all* $f \in L$.

Then there exists an up to μ-null sets unique sub σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$ such that $Tf = E_\mu(f|\mathcal{A}_0)$ for all $f \in L$. Conversely, if $T : L \to M(\Omega, \mathcal{A}, \mu)$ is a linear map such that $Tf = E_\mu(f|\mathcal{A}_0)$ for some sub σ-algebra $\mathcal{A}_0$ of $\mathcal{A}$, then T satisfies T1′, T3, T4 *and* T5 (*and hence also* T1 *and* T2).

PROOF. First observe that T1′ together with T5 imply T1. Next, let $f_n \downarrow 0$ in L. Then $E_\mu(f_n)\downarrow 0$, so $E_\mu(Tf_n)\downarrow 0$. This implies that $Tf_n \downarrow 0$ has to hold. Thus T satisfies T2, so we are in the situation of Theorem 3.3. Now, the construction of ν in Theorem 3.3 together with T5 shows that $\nu = \mu$ has to hold on $\mathcal{A}$. Finally, the converse is clear from the well known properties of the conditional expectation.

4. Concluding remarks. Already many results concerning characterizations of conditional expectations have been published. A review of the results up to 1965 can be found in Pfanzagl's introduction of [4]. Besides these results we mention [1] Lemma 4, [4] Theorems 1, 2, 3 and [5] Theorem 1. All results mentioned in the introduction of [4] as well as the results of [1] deal with operators from an L_p-space into an L_p-space, while in [5] these results are slightly generalized to Orlicz spaces. This makes our results completely different since any knowledge of norms is superfluous in Theorem 3.3 (and hence also in the Corollaries 3.4 and 3.5). On the other hand, Corollary 3.4 shows that many L_p-space results are easy consequences of Theorem 3.3. Another difference between our results and almost all other results obtained so far is that the measure μ is only used to define the equivalence classes of functions. It follows that in Theorem 3.3 μ can be replaced by any equivalent measure μ' provided that $L \subset L_1(\mu')$ still holds. This makes Theorem 3.3 also different from the results of [4]. Finally, we observe that Corollary 3.5 is also immediate from [4] Theorem 3.

REFERENCES

[1] ANDO, T. (1966). Contractive projections in L_p spaces. *Pac. J. Math.* **17** 391–405.
[2] BAHADUR, R. R. (1955). Measurable subspaces and subalgebras. *Proc. Amer. Math. Soc.* **14** 298–304.
[3] JONGE, E. DE. Conditional expectation in Riesz space setting. Unpublished manuscript.
[4] PFANZAGL, J. (1967). Characterizations of conditional expectations. *Ann. Math. Statist.* **38** 415–421.
[5] RAO, M. M. (1965). Conditional expectations and closed projections. *Proc. Kon. Ned. Akad. Wet.* **A68** 100–112.
[6] SCHAEFER, H. H. (1974). *Banach Lattices and Positive Operators.* Springer, Berlin-Heidelberg-New York.

MATHEMATISCH INSTITUUT,
KATHOLIEKE UNIVERSITEIT,
TOERNOOIVELD,
NIJMEGEN,
THE NETHERLANDS.