The Annals of Probability
1979, Vol. 7, No. 2, 379–383

MULTIVALUED STATE COMPONENT SYSTEMS[1]

BY SHELDON M. ROSS
University of California, Berkeley

Consider a system that is composed of n components, each of which is operating at some performance level. We suppose that there exists a nondecreasing function ϕ such that $\phi(x_1, \cdots, x_n)$ denotes the performance level of the system when the ith component's performance level is x_i, $i = 1, \cdots, n$. We allow both x_i and $\phi(x_1, \cdots, x_n)$ to be arbitrary nonnegative numbers and extend many of the important results of the usual binary model to this more general framework. In particular, we obtain a fundamental inequality for $E[\phi(X_1, \cdots, X_n)]$ when ϕ is binary, which can, among other things, be used to generate a host of inequalities concerning increasing failure rate average distributions including, as a special case, the convolution and system closure theorem. We also define the concept of an increasing failure rate average stochastic process and prove the analog of the closure theorem; and then also do the same for new better than used stochastic processes.

0. Introduction and summary. Consider a system that is composed of n components each of which is operating at some performance level. We suppose that there exists a nondecreasing function ϕ, called the structure function, such that $\phi(x_1, \cdots, x_n)$ denotes the performance level of the system when the ith component's performance level is x_i, $i = 1, \cdots, n$.

Whereas almost all previous work has assumed that both x_i and $\phi(x_1, \cdots, x_n)$ were binary variables we shall allow both to be arbitrary nonnegative numbers. In the next few sections, we extend many of the important results of the usual binary model to this more general framework. In particular, we obtain, in Section 1, a fundamental inequality for $E[\phi(X_1, \cdots, X_n)]$ when ϕ is binary which can, among other things, be used to generate a host of inequalities concerning increasing failure rate average distributions including as special cases the convolution and the system closure theorem. In Section 2, we define the concept of an increasing failure rate average stochastic process and prove the analog of the closure theorem; and in Section 3 we do the same for new better than used stochastic processes.

1. The structure function. Suppose now the performance level of component i is a random variable X_i having distribution $\bar{F}_i$ where $\bar{F}_i(x) = P\{X_i > x\}$, and suppose that the X_i are independent. We define the function $r(\bar{F}_1, \cdots, \bar{F}_n)$ by

$$r(\bar{F}_1, \cdots, \bar{F}_n) = E[\phi(X_1, \cdots, X_n)]$$

Received May 9, 1977.
[1]This research has been partially supported by the Air Force Office of Scientific Research (AFSC), USAF, under Grant AFOSR-77-3213 and the Office of Naval Research under Contract N00014-77-C-0299 with the University of California.
AMS 1970 *subject classifications.* Primary 60K10; secondary 62N05.
Key words and phrases. Multivalued, closure theorem, stochastic process, increasing failure rate average.

and call r the reliability function of the system. It immediately follows from the monotonicity of ϕ that

PROPOSITION 1. *If $\bar{F}_i$ and $\bar{G}_i$ are distributions such that $\bar{F}_i(x) \geqslant \bar{G}_i(x)$ for all x, then*

$$r(\bar{F}_1, \cdots, \bar{F}_n) \geqslant r(\bar{G}_1, \cdots, \bar{G}_n).$$

We shall need the following lemma which is a slight variation of a lemma used by Block and Savits [2].

LEMMA 1. *Let $r(s)$ be a nonnegative nondecreasing function of s, $s \geqslant 0$ and let $\bar{G}$ be a distribution function with $\bar{G}(0) = 1$. Then, for $0 < \alpha \leqslant 1$,*

$$\int_0^\infty (r(s))^\alpha d(1 - \bar{G}^\alpha(s)) \geqslant \left[\int_0^\infty r(s) d(1 - \bar{G}(s))\right]^\alpha.$$

PROOF. Lemma 4.1 (page 217) of Ross [3] (also given as Lemma 2.3 (page 84) of Barlow and Proschan [1]) generalizes to give that for $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$, $y_i \geqslant 0$, $\Sigma_i^n y_i > 0$

$$(\Sigma_1^n r(x_i) y_i)^\alpha \leqslant \Sigma_1^n (r(x_i))^\alpha \left[(\Sigma_{k=i}^n y_k)^\alpha - (\Sigma_{k=i+1}^n y_k)^\alpha\right].$$

From this, the conclusion follows from a standard limiting argument (as in [2]). □

The following theorem is of fundamental importance.

THEOREM 1. *If ϕ is a binary function, then*

$$(*) \qquad r(\bar{F}_1^\alpha, \cdots, \bar{F}_n^\alpha) \geqslant \left[r(\bar{F}_1, \cdots, \bar{F}_n)\right]^\alpha$$

for all $0 \leqslant \alpha \leqslant 1$.

PROOF. The proof is by induction. When $n = 1$, it follows from the monotonicity of ϕ that it must be of the form

$$\begin{aligned} \phi(x) &= 1, \quad x > c \\ &= 0, \quad x \leqslant c \end{aligned}$$

for some c. Hence, $E[\phi(X_1)] = \bar{F}_1(c)$, and so both sides of the inequality (*) are equal. So assume (*) for all binary structures of $n - 1$ components, and consider the n component case. Conditioning on X_n yields

$$(1) \qquad r(\bar{F}_1^\alpha, \cdots, \bar{F}_n^\alpha) = \int r_s(\bar{F}_1^\alpha, \cdots, \bar{F}_{n-1}^\alpha) d(1 - \bar{F}_n^\alpha(s))$$

where

$$r_s(\bar{F}_1^\alpha, \cdots, \bar{F}_{n-1}^\alpha) = E[\phi(X_1, \cdots, X_{n-1}, s)]$$

with X_i having distribution $\bar{F}_i^\alpha$. By the induction hypothesis, we see that

$$r_s(\bar{F}_1^\alpha, \cdots, \bar{F}_{n-1}^\alpha) \geqslant \left[r_s(\bar{F}_1, \cdots, \bar{F}_{n-1})\right]^\alpha$$

and so from (1)

$$r(\bar{F}_1^\alpha, \cdots, \bar{F}_n^\alpha) \geqslant \int \left(r_s\left[\bar{F}_1, \cdots, \bar{F}_{n-1}\right]\right)^\alpha d(1 - \bar{F}_n^\alpha(s)).$$

As it follows from the monotonicity of ϕ that r_s is nondecreasing in s, we can apply Lemma 1 to the above to obtain that

$$r(\bar{F}_1^\alpha, \cdots, \bar{F}_n^\alpha) \geqslant \left[\int_0^\infty r_s(\bar{F}_1, \cdots, \bar{F}_{n-1}) d(1 - \bar{F}_n(s))\right]^\alpha = \left(r(\bar{F}_1, \cdots, \bar{F}_n)\right)^\alpha. \quad \square$$

DEFINITION. The distribution function $\bar{F}$, with $\bar{F}(0) = 1$, is said to be an increasing failure rate average distribution if $\bar{F}(\alpha x) \geqslant \bar{F}^\alpha(x)$, for all $0 \leqslant \alpha < 1$, $x \geqslant 0$.

COROLLARY 1. *If $X_1, \cdots, X_n$ are independent random variables, each having an increasing failure rate average distribution, then*
(i) *for all nondecreasing binary functions ϕ,*

$$E[\phi(X_1/\alpha, \cdots, X_n/\alpha)] \geqslant (E[\phi(X_1, \cdots, X_n)])^\alpha \qquad \textit{for} \quad 0 < \alpha < 1;$$

(ii) *for all nondecreasing functions $t(\mathbf{x}) = t(x_1, \cdots, x_n)$ such that $t(\mathbf{x}) \geqslant \alpha t(\mathbf{x}/\alpha)$ for $0 < \alpha < 1$, $t(X_1, \cdots, X_n)$ has an increasing failure rate average distribution.*

PROOF. Part (i) follows immediately from Proposition 1 and Theorem 1. Part (ii) follows from part (i) by letting

$$\begin{aligned} \phi(\mathbf{x}) &= 1 \qquad \text{if} \quad t(\mathbf{x}) > a \\ &= 0 \qquad \text{otherwise} \quad . \end{aligned} \qquad \square$$

REMARKS. The increasing failure rate average convolution and system closure theorems are both special cases of Corollary 1 (ii). Corollary 1 (i) is an n-variate version of Lemma 2.1 of Block and Savits [2].

2. The generalized increasing failure rate average closure theorem. In this section, we suppose that the component performance levels vary with time and we let $X_i(t)$ denote the level of component i at time t. Thus, for instance, $\phi(\mathbf{X}(t)) = \phi(X_1(t), \cdots, X_n(t))$ denotes the systems performance level at time t.

DEFINITION. The real-valued stochastic process $\{X(t), t \geqslant 0\}$ is said to be an increasing failure rate average process if T_a is an increasing failure rate average random variable for every a, where

$$T_a = \inf\{t : X(t) \leqslant a\}$$

is the first time the process reaches or goes below a.

THEOREM 2. *If $\{X_i(t)\}$, $i = 1, \cdots, n$, are nonincreasing independent increasing failure rate average processes, then $\{\phi(\mathbf{X}(t))\}$ is also increasing failure rate average whenever ϕ is nondecreasing.*

PROOF. Let $\bar{F}_{i,s}(x) = P\{X_i(s) > x\}$, and suppose first that ϕ is a binary function. Let T denote the first time t that $\phi(\mathbf{X}(t)) = 0$. Now

$$\begin{aligned} (2) \qquad P\{T > \alpha t\} &= P\{\phi(\mathbf{X}(\alpha t)) = 1\} \text{ by monotonicity} \\ &= E[\phi(\mathbf{X}(\alpha t))] \\ &= r(\bar{F}_{1,\alpha t}, \cdots, \bar{F}_{n,\alpha t}). \end{aligned}$$

Now

$$\bar{F}_{i,\alpha t}(b) = P\{X_i(\alpha t) > b\}$$
$$= P\{T_{i,b} > \alpha t\}$$

where $T_{i,b}$ denotes the first time that $X_i(t)$ hits or goes below b. Hence, from the hypothesis on $\{X_i(t)\}$, we see that

$$P\{T_{i,b} > \alpha t\} \geqslant (P\{T_{i,b} > t\})^\alpha$$
$$= \bar{F}^\alpha_{i,t}(b).$$

Thus,

$$\bar{F}_{i,\alpha t}(b) \geqslant \bar{F}^\alpha_{i,t}(b)$$

and so from (2) and Proposition 1

$$P\{T > \alpha t\} \geqslant r\left(\bar{F}^\alpha_{1,t}, \cdots, \bar{F}^\alpha_{n,t}\right)$$
$$\geqslant \left(r\left(\bar{F}_{1,t}, \cdots, \bar{F}_{n,t}\right)\right)^\alpha \text{ by Theorem 1}$$
$$= (P\{T > t\})^\alpha$$

which proves the result when ϕ is binary. For an arbitrary nondecreasing ϕ, we can show that the time to go below b has an increasing failure rate average distribution by using the result in the binary case on the binary function defined by

$$\phi_b(\mathbf{x}) = 1 \quad \text{if} \quad \phi(\mathbf{x}) > b$$
$$= 0 \quad \text{if} \quad \phi(\mathbf{x}) \leqslant b. \qquad \square$$

3. A new better than used closure theorem. We start with a definition.

DEFINITION. The nonincreasing stochastic process $\{X(t), t \geqslant 0\}$ is said to be a new better than used process if, with probability 1,

$$P\{T_a > s + t | X(u), 0 \leqslant u \leqslant s\} \leqslant P\{T_a > t\}$$

for all $s, t, a \geqslant 0$, where T_a denotes the first time the process hits or goes below a.

THEOREM 3. *If the component processes are independent new better than used processes, then* $\{\phi(\mathbf{X}(t))\}$ *is a new better than used process.*

PROOF. Suppose first that ϕ is binary and let T denote the first time the process $\phi(\mathbf{X}(t))$ hits 0. Now consider

$$P\{T > s + t | X_i(u), 0 \leqslant u \leqslant s, i = 1, \cdots, n\}$$
$$= E[\phi(\mathbf{X}(s + t)) | X_i(u), 0 \leqslant u \leqslant s, i = 1, \cdots, n].$$

Now it follows from the definition of a new better than used process that the conditional distribution of $X_i(s + t)$, given $X_i(u)$, $0 \leqslant u \leqslant s$, is stochastically smaller than the distribution of $X_i(t)$. Hence, from Proposition 1, we see that

$$E[\phi(\mathbf{X}(s + t)) | X_i(u), 0 \leqslant u \leqslant s, t = 1, \cdots, n] \leqslant E[\phi(\mathbf{X}(t))] = P\{T > t\}$$

which proves the result when ϕ is binary. As before, we can reduce the nonbinary case to the above by defining

$$\phi_a(\mathbf{x}) = 1 \quad \text{if} \quad \phi(\mathbf{x}) > a$$
$$= 0 \quad \text{if} \quad \phi(\mathbf{x}) \leqslant a \qquad \square$$

REFERENCES

[1] BARLOW, R. and PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart and Winston, New York.

[2] BLOCK, H. and SAVITS, T. (1976). The IFRA closure problem. *Ann. Probability* **4** 1030–1033.

[3] ROSS, S. (1972). *Introduction to Probability Models*. Academic Press, New York.

OPERATIONS RESEARCH CENTER
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720