The Annals of Probability
1979, Vol. 7, No. 2, 377-378

AN ALTERNATE PROOF OF A THEOREM OF KESTEN CONCERNING MARKOV RANDOM FIELDS

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Let S be a countable set, Q a strictly positive matrix on $S \times S$, $\mathscr{G}(Q)$ the set of one-dimensional Markov random fields taking values in S determined by Q. In this paper a short proof of Kesten's sufficient condition for $\mathscr{G}(Q) = \phi$ is presented.

The purpose of this note is to present a short proof of a theorem of Kesten (Theorem 2 in [3]) about one-dimensional Markov random fields. We will use the terminology and notation of [3], and will rely on two additional results about random fields. The first of these, Equation 2.1 in [2], asserts that $\mathscr{G}(Q)$ can be obtained by taking convex combinations of elements of $\mathscr{G}_e(Q)$, the set of extreme points of $\mathscr{G}(Q)$. The second, Theorem 6 in [4], shows that each $\mu \in \mathscr{G}_e(Q)$ is determined by a pair of sequences of strictly positive functions on S, $\{l_n(\cdot)\}_{n\in\mathbb{Z}}$ and $\{r_n(\cdot)\}_{n\in\mathbb{Z}}$ which satisfy:

(1a) $$l_n Q(y) \equiv \sum_{x\in S} l_n(x)Q(x, y) = l_{n+1}(y) \qquad n \in \mathbb{Z},\ y \in S$$

(1b) $$Qr_{n+1}(x) \equiv \sum_{y\in S} Q(x, y)r_{n+1}(y) = r_n(x) \qquad n \in \mathbb{Z},\ x \in S$$

(1c) $$l_n \cdot r_n \equiv \sum_{x\in S} l_n(x)r_n(x) = 1 \qquad n \in \mathbb{Z}$$

(1d) $$\mu\{\omega(n) = x_0, \omega(n+1) = x_1, \cdots, \omega(n+k) = x_k\} = l_n(x_0)Q(x_0, x_1) \cdots Q(x_{k-1}, x_k)r_{n+k}(x_k), \qquad n \in \mathbb{Z},\ k \in \mathbb{Z}^+,\ x_i \in S.$$

THEOREM (Kesten). *If there exists $\delta > 0$ and $m \geqq 1$ such that*

(2) $$\sum_{n=1}^{m} Q^n(x, x) > \delta, \qquad x \in S,$$

and Q is not equivalent to a positive recurrent stochastic matrix, then $\mathscr{G}(Q) = \phi$.

REMARK 1. The proof shows, as does the original, that $\mathscr{G}(Q) = \{$the stationary Markov chain$\}$ if (2) is satisfied and Q *is* equivalent to a positive recurrent stochastic matrix.

PROOF OF THEOREM. As explained in [3], it suffices to take $m = 1$. Suppose $\mathscr{G}(Q) \neq \phi$. By the integral representation theorem in [2] we may assume there exists $\mu \in \mathscr{G}_e(Q)$, which by Spitzer's theorem must be determined by a pair l_n, r_n as in (1). From (2) we obtain the following:

$$l_{n+1}(x) > \delta l_n(x), \qquad r_n(x) > \frac{1}{\delta} r_{n+1}(x), \qquad c_n \equiv l_{n-1} \cdot r_n < 1/\delta, \qquad n \in \mathbb{Z},\ x \in S.$$

Received October 3, 1977.
AMS 1970 *subject classifications.* Primary 60J10, 60K35.
Key words and phrases. Markov random field, entrance law.

In fact, c_n is independent of n, since $c_{n+1} = l_n \cdot r_{n+1} = l_{n-1} Q \cdot r_{n+1} = l_{n-1} \cdot Q r_{n+1} = c_n$. Let $c = c_n$.

Set $\tilde{l}_n(x) = c^{-1} l_{n-1}(x)$. It follows that $\tilde{l}_n$ is strictly positive, $\tilde{l}_n Q = \tilde{l}_{n+1}$, and $\tilde{l}_n \cdot r_n = 1$. The pair $\tilde{l}_n$, r_n determine an element $\tilde{\mu} \in \mathscr{G}(Q)$ via the recipe in (1d). Now set $\tilde{\tilde{l}}_n(x) = (l_n(x) - \delta c \tilde{l}_n(x))/(1 - \delta c)$. Hence $\tilde{\tilde{l}}_n$, r_n determine an element $\tilde{\tilde{\mu}} \in \mathscr{G}(Q)$.

Unravelling this, we see $l_n = \delta c \tilde{l}_n + (1 - \delta c)\tilde{\tilde{l}}_n$, or, $\mu = \delta c \tilde{\mu} + (1 - \delta c)\tilde{\tilde{\mu}}$. Since $0 < \delta c < 1$, and μ is an extreme point, $\mu = \tilde{\mu} = \tilde{\tilde{\mu}}$, or $l_n = \tilde{l}_n = \tilde{\tilde{l}}_n$. This implies $l_n = c^{-1} l_{n-1}$ or $l_{n-1} Q = c^{-1} l_{n-1}$. Setting $l_0 = \pi$ gives $l_n = c^{-n}\pi$, $n \in \mathbb{Z}$.

This process is repeated on the r_n side leaving l_n untouched (set $\tilde{r}_n = c^{-1} r_{n+1}$, $\tilde{\tilde{r}}_n = (r_n - \delta c \tilde{r}_n)/(1 - \delta c)$, etc.). The convexity argument gives $Q r_n = c^{-1} r_n$, and setting $f = r_0$ gives $r_n = c^n f$, $n \in \mathbb{Z}$. The right hand side of (1d) now becomes

$$\pi(x_0) Q(x_0, x_1) \cdots Q(x_{k-1}, x_k) f(x_k) c^k ,$$

an expression independent of n. This means μ is translation invariant, contradicting Theorem 1 of [3]. Hence $\mathscr{G}_e(Q) = \phi$ and $\mathscr{G}(Q) = \phi$. □

REMARK 2. A similar (but shorter) argument shows that the set $\mathscr{L}(Q)$ of entrance laws for Q (see [1]) is empty if Q is an irreducible stochastic matrix, not positive recurrent, which satisfies (2).

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