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INFINITELY DIVISIBLE DISTRIBUTIONS WITH UNIMODAL LÉVY SPECTRAL FUNCTIONS

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The class of infinitely divisible characteristic functions which have unimodal Lévy spectral functions is determined. It is shown that membership in this class is related to solutions of the equations $\phi(u) = \phi^r(ru)\phi_r(u)$, where $r \in (0, 1)$ and ϕ and ϕ_r are characteristic functions. We point out how elements of this class can serve as limit laws as well as some connections between this class and the class of self-decomposable characteristic functions.

1. Introduction. Suppose $\phi(u)$ is an infinitely divisible characteristic function on the real line, R. It is well known that $\phi(u)$ never vanishes for $u \in R$ and the logarithm of $\phi(u)$ may be represented according to Lévy's formula as $\ln \phi(u) = i\gamma u - u^2\sigma^2 + \int_R (e^{iux} - 1 - iux/(1 + x^2))\, dM(x)$. The barred integral sign means that the integration is taken over $R \setminus \{0\}$. The function M occurring in this representation is referred to as the Lévy spectral function of ϕ. M is defined on $(-\infty, 0) \cup (0, +\infty)$ and is nondecreasing on each of the half lines. If, in addition, M is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$, then M is said to be unimodal.

The following result on unimodal Lévy spectral functions was recently presented in Alf-O'Connor (1977).

LEMMA 1. *Let $\phi(u)$ be an infinitely divisible characteristic function with Lévy spectral function, M. Then M is unimodal if and only if there exists an infinitely divisible characteristic function $f(u)$ such that* $\ln \phi(u) = \int_0^1 \ln f(ut)\, dt$ *for all $u \in R$.*

Let U be the class of infinitely divisible characteristic functions whose Lévy spectral functions are unimodal. Lemma 1, above, states that elements of U can be expressed as mixtures of logarithms of infinitely divisible characteristic functions. The main objective of this article is to further describe U. In this analysis of U, it is often necessary to identify those functions which are logarithms of infinitely divisible characteristic functions. Johansen (1966) has given a necessary and sufficient condition and his result is stated below.

LEMMA 2. *Let h be a continuous, complex-valued function on R. Then* $h(u) = \ln \phi(u)$ *for some infinitely divisible characteristic function ϕ if and only if h satisfies* (i) $h(0) = 0$, (ii) $h(u) = \overline{h(-u)}$ *for all* $u \in R$ *and* (iii) *if* $u_1, \cdots, u_n \in R$ *and* $\alpha_1, \cdots, \alpha_n$ *are arbitrary complex numbers satisfying* $\sum_{j=1}^n \alpha_j = 0$, *then* $\sum_{j=1}^n \sum_{k=1}^n h(u_j - u_k)\alpha_j \bar{\alpha}_k \geq 0$.

2. Main results. We begin by showing that U consists entirely of infinitely divisible characteristic functions which satisfy a certain functional equation.

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THEOREM 1. *Let $\phi(u)$ be a characteristic function without real zeros. Then ϕ is infinitely divisible and the Lévy spectral function, M, of ϕ is unimodal if and only if the following condition (*) is satisfied, where:*

> (*) *$\phi'(u)$ exists on $R \setminus \{0\}$ and $\lim_{u\to 0} u\phi'(u) = 0$ and for each r, $0 < r < 1$, there is a characteristic function ϕ_r such that $\phi(u) = \phi^r(ru)\phi_r(u)$ for all $u \in R$.*

Moreover, in this case, ϕ_r is infinitely divisible.

PROOF. First assume that ϕ is infinitely divisible and that the Lévy spectral function M is unimodal. According to Lemma 1, we may choose an infinitely divisible characteristic function $f(u)$ so that $\ln \phi(u) = \int_0^1 \ln f(ut)\, dt = u^{-1}\int_0^u \ln f(t)\, dt$. Clearly $\phi(u)$ is C^1 on $R \setminus \{0\}$, and $\lim_{u\to 0} u\phi'(u) = 0$. Let $0 < r < 1$ be given. In view of Lemma 2, $\int_r^1 \ln f(ut)\, dt$ is a logarithm of an infinitely divisible characteristic function. So by letting $\phi_r(u) = \exp(\int_r^1 \ln f(ut)\, dt)$, condition (*) is satisfied with ϕ_r infinitely divisible.

Conversely suppose condition (*) is in effect. For each $k = 1, 2 \cdots$ define the characteristic function $\phi_k(u)$ by the rule

$$\phi_k(u) = \frac{\phi(ku/k-1)}{\phi^{(k-1)/k}(u)}. \tag{1}$$

Observe that $\phi_k(u) \neq 0$ for all $u \in R$ and k. For $u \neq 0$, $\ln \phi_k^k(u) = \ln \phi(ku/k-1) + u(\ln \phi(ku/k-1) - \ln \phi(u))/u/(k-1)$. As $k \to +\infty$, the right-hand side of this equation converges to $\ln \phi(u) + u\phi'(u)/\phi(u)$; whence it follows that $\lim_{k\to+\infty} \phi_k^k(u) = \phi(u)\exp(u\phi'(u)/\phi(u))$. Set $f(0) = 1$ and for $u \neq 0$, set $f(u) = \phi(u) \exp(u\phi'(u)/\phi(u))$. It follows from the continuity theorem and a result of Feller (1966), page 534, that $f(u)$ is an infinitely divisible characteristic function. But this last equation may be rewritten $\ln \phi(u) = u^{-1}\int_0^u \ln f(t)\, dt$ for $u \in R$. In view of Lemma 1, the Lévy spectral function of ϕ must be unimodal and a repetition of the arguments used in the necessary part of this proof show that the characteristic function $\phi_r(u)$ occurring in (*) must be infinitely divisible. This completes the proof of this theorem.

THEOREM 2. *Let $\phi(u)$ be a characteristic function such that $\phi(u) \neq 0$, and assume for each $r \in (0, 1)$ a characteristic function ϕ_r may be chosen so that $\phi(u) = \phi^r(ru)\phi_r(u)$ for all $u \in R$. Then if ϕ_r is infinitely divisible for each $r \in (0, 1)$, ϕ is infinitely divisible and the Lévy spectral function of ϕ is unimodal.*

PROOF. If ϕ_r is infinitely divisible for all $r \in (0, 1)$, then so is ϕ since $\phi(u) = \lim_{r\to 0+} \phi_r(u)$ for all $u \in R$.

Let M and M_r be the Lévy spectral functions of ϕ and ϕ_r respectively. Using the uniqueness property of the Lévy representation, it follows that $M(x) - rM(x/r) = M_r(x)$ for all $x \in R \setminus \{0\}$ and $r \in (0, 1)$. Thus for each $r \in (0, 1)$, $M(x) - rM(x/r)$ is an increasing function on $(-\infty, 0)$ and on $(0, +\infty)$. Let us now show that M must be convex on $(-\infty, 0)$.

Let $u < v < 0$. Then $M(u) - M(v) \leqslant rM(u/r) - rM(v/r)$ for all $r \in (0, 1)$. In particular with $r = v/u$ we have $M(u) - M(v) \leqslant r(M(u - (v - u)/r) - M(u))$ or $M(u) \leqslant (1/1 + r)M(v) + (r/1 + r)M(u - (v - u)/r)$. Hence M is convex on $(-\infty, 0)$. Similar reasoning will show that M must be concave on $(0, +\infty)$ and so M is a unimodal Lévy spectral function.

It may be worthwhile to point out how elements of U serve as limit laws. Consider the class of characteristic functions which are limits in distribution of the random variables $S_n = \Sigma_k X_{nk}$ subject to:

(i) the array $\{X_{nk} : n = 1, 2, \cdots ; k = 1, \cdots, n\}$ is a u.a.n. system of row-wise independent random variables;

(ii) there exists a sequence of characteristic functions $\{\phi_n(u)\}$ such that the characteristic function of X_{nk} is given by the formula $f_{nk}(u) = (\Pi_{j=k}^n \phi_j(u/n))^{1/n}$ for $n = 1, 2, \cdots ; k = 1, \cdots, n$ and $u \in R$.

According to the central limit theorem (Loéve (1963), page 309) all limit laws of S_n, provided assumption (i) is met, must be infinitely divisible. We shall now show that the additional condition (ii) is necessary and sufficient for the limit law to belong to U.

THEOREM 3. *Following the notation above, a characteristic function ϕ belongs to U if and only if ϕ is a limit law of S_n where both conditions* (i) *and* (ii) *are assumed.*

PROOF. First suppose ϕ is a limit law of S_n and choose ϕ_n and f_{nk} to satisfy conditions (i) and (ii). Then we have that ϕ is infinitely divisible and $\phi(u) = \lim_{n\to+\infty}\Pi_{k=1}^n f_{nk}(u) = \lim_{n\to+\infty}(\Pi_{k=1}^n \phi_k^k(u/n))^{1/n}$. Let $r \in (0, 1)$. Select a subsequence $\{m\}$ such that $m/n \to r$ as $n \to +\infty$. Then

$$\phi(u) = \lim_{n\to+\infty}\left(\Pi_{k=1}^m \phi_k^k((m/n)u/m)^{(1/m)(m/n)}\left(\Pi_{k=m+1}^n \phi_k^k(u/n)\right)^{1/n}\right.$$

The first product converges to $\phi^r(ru)$ and hence by the continuity theorem, the second product converges to a characteristic function—call it $\phi_r(u)$. Since $(\Pi_{r=m+1}^n \phi_k^k(u/n))^{1/n} = f_{n(m+1)}^{m+1}(u)\Pi_{k=m+2}^n f_{nk}(u)$, it follows from the central limit theorem that ϕ_r must be infinitely divisible. By Theorem 2, ϕ belongs to the class U.

Now assume that ϕ belongs to U. By Lemma 1, we may choose an infinitely divisible characteristic function $f(u)$ such that $\ln \phi(u) = \int_0^1 \ln f(ut)\, dt$. Define the sequence of characteristic functions $\{\phi_k : k = 1, 2, \cdots\}$ by the rule $\phi_k(u) = \phi(ku)/\phi^{(k-1)/k}((k-1)u)$. By virtue of Theorem 1, each ϕ_k is infinitely divisible and $\ln \phi_k(u/n) = (n/k)\int_{(k-1/n)}^{k/n} \ln f(ut)\, dt$. Set $f_{nk}(u) = (\Pi_{j=k}^n \phi_j(u/n))^{1/n}$ for $1 \leqslant k \leqslant n = 1, 2, \cdots$. Let $T > 0$ and $j_n \leqslant n$ be an arbitrary sequence of integers. It follows from the Toeplitz lemma that $\ln f_{nj_n}(u) = \Sigma_{j=j_n}^n (1/j)\int_{(j-1)/n}^{j/n} \ln f(ut)\, dt$ converges to 0 uniformly on $[-T, T]$. Hence $\{f_{nk} : 1 \leqslant k \leqslant n = 1, 2, \cdots\}$ is a u.a.n. system of characteristic functions. Since for any n, $\Pi_{k=1}^n f_{nk}(u) = \phi(u)$, the necessity of conditions (i) and (ii) is justified. This completes the proof of Theorem 3.

The above results present a solution to a problem posed by Medgyessy ((1977), page 36), who asks for a functional equation characterizing infinitely divisible distributions which have unimodal Lévy spectral functions.

The functional equation given in condition (*) is similar to the defining functional equation for self-decomposable characteristic functions. Recall that a characteristic function ϕ is said to be self-decomposable if for each $r \in (0, 1)$, there is a characteristic function ϕ_r such that $\phi(u) = \phi(ru)\phi_r(u)$. Let L be the class of self-decomposable characteristic functions. If $\phi \in L$, then ϕ must be infinitely divisible and its Lévy spectral function, M, has the property that $xM'(x)$ does not increase on $(-\infty, 0)$ and on $(0, +\infty)$. (See Lukacs (1970), page 164). Hence $M'(x)$ is nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, +\infty)$ and so M must be unimodal. Thus $L \subseteq U$ and from Theorem 1, all self-decomposable characteristic functions are C^1 on $R \setminus \{0\}$. We wish to further explore the relationship between the classes L and U.

THEOREM 4. *Suppose $\phi(u)$ is an infinitely divisible characteristic function which satisfies the equation $\phi(u) = \phi^r(ru)\phi_r(u)$ where $r \in (0, 1)$ and ϕ_r is a characteristic function. Further assume that $\int_0^1 v^{-1} \ln \phi(uv)\, dv$ is a continuous function of u. Define the function $f(u)$ by the rule*

$$f(u) = \phi(u)\exp\left(\int_0^1 v^{-1} \ln \phi(uv)\, dv\right). \tag{2}$$

Then f belongs to the class L and ϕ belongs to the class U.

PROOF. Because $\ln \phi(u)$ satisfies (i), (ii), and (iii), of Lemma 2, it is easy to see that $\int_0^1 v^{-1} \ln \phi(uv)\, dv$ does also. Hence $f(u)$, defined in (2), is an infinitely divisible characteristic function.

Using the partition $\{0, 1/n, 2/n, \cdots, n/n\}$ of $[0, 1]$, we have that

$$\int_0^1 v^{-1} \ln \phi(uv)\, dv = \lim_{n\to+\infty}\Sigma_{k=1}^n k^{-1} \ln \phi(ku/n).$$

Hence

$$\begin{aligned} f(u) &= \lim_{n\to+\infty}\phi(u)\Pi_{k=1}^n \phi^{1/k}(ku/n) \\ &= \lim_{n\to+\infty}\phi(u)\Pi_{k=1}^n \phi^{1/k}((k-1)u/n)\lim_{n\to+\infty}\phi^{1/n}(u) \\ &\quad \times \Pi_{k=2}^n \phi^{1/k(k-1)}((k-1)u/n). \end{aligned}$$

The second limit is identically 1 as can be seen by taking logarithms of the product term and using the Toeplitz lemma. So

$$\begin{aligned} f(u) &= \lim_{n\to+\infty}\phi(u)\Pi_{k=1}^n \phi^{1/k}((k-1)u/n) \\ &= \lim_{n\to+\infty}\Pi_{k=1}^n \phi(ku/n)/\phi^{(k-1)/k}((k-1)u/n). \end{aligned}$$

Let $r \in (0, 1)$ and let $m < n$ be a sequence of integers with $m/n \to r$. Then

$$\begin{aligned} f(u) = \lim_{n\to+\infty}&\Pi_{k=1}^m \phi((ku/m)m/n)/\phi^{(k-1)/k}(((k-1)u/m)m/n) \\ &\times \Pi_{k=m+1}^n \phi(ku/n)/\phi^{(k-1)/k}((k-1)u/n). \end{aligned}$$

The first product converges to $f(ru)$. The assumption on ϕ together with the continuity theorem insure that the second product converges to a characteristic function; hence $f(u) = f(ru)f_r(u)$ and thus $f \in L$.

In view of the above remarks, f is C^1 on $R \setminus \{0\}$, and since $\ln \phi(u) = \ln f(u) - \int_0^u v^{-1} \ln \phi(v)\, dv$, it follows that ϕ is C^1 on $R \setminus \{0\}$. By Theorem 1, ϕ belongs to the class U.

THEOREM 5. *Let ϕ be an infinitely divisible characteristic function with distribution function $F(x)$ and suppose $F(x) = 0$ if $x < 0$. Let $b \in R$, $\sigma^2 \geqslant 0$ and $\lambda > 0$ and set*

$$f_\lambda(u) = \exp\big(ibu - u^2\sigma^2 + i\lambda \textstyle\int_0^u \ln \phi(t)\, dt\big). \tag{3}$$

Then f_λ is an infinitely divisible characteristic function and its Lévy spectral function, H_λ, has support contained in $(0, +\infty)$ and satisfies $\int_1^{+\infty} x\, dH_\lambda(x) < +\infty$. If, in addition, the Lévy spectral function of ϕ is unimodal, then f_λ belongs to the class L.

Conversely, if f is any class L characteristic function whose Lévy spectral function, H, has support on $0, +\infty)$ and satisfies $\int_1^{+\infty} x\, dH(x) < +\infty$, f can be written as in (3) where $\phi \in U$.

PROOF. Let ϕ and F have the stated properties and let $a = \inf\{x : F(x) > 0\}$. Let M be the Lévy spectral function of ϕ. According to Theorem 11.2.2 of Lukacs (1970), the support of M is contained in $(0, +\infty)$, $\int_{0+}^1 x\, dM(x)$ is finite, and so

$$\ln \phi(u) = iau + \int_{0+}^{+\infty} (e^{iux} - 1)\, dM(x) \qquad \text{for all} \quad u \in R.$$

Let $b \in R$, $\sigma^2 \geqslant 0$ and $\lambda > 0$. Then

$$\begin{aligned} i\lambda \int_0^u \ln \phi(t)\, dt &= -\lambda a u^2/2 + \lambda \lim_{\varepsilon \to 0+} \int_\varepsilon^{+\infty} (e^{iux} - 1 - iux)\, dM(x)/x \\ &= ic\lambda u - \lambda a u^2/2 + \lambda \int_{0+}^{+\infty} (e^{iux} - 1 - iux/1 + x^2)\, dM(x)/x \end{aligned}$$

where $c = \int_{0+}^{\infty} x^2(1 + x^2)^{-1}\, dM(x)$. Thus, $f_\lambda(u) = \exp(ibu - u^2\sigma^2 + i\lambda \int_0^u \ln \phi(t)\, dt)$ is infinitely divisible and its Lévy spectral function $H_\lambda(x)$ is given by

$$H_\lambda(x) = 0 \quad \text{if} \quad x < 0 \qquad \text{and} \quad H_\lambda(x) = -\lambda \int_x^{+\infty} z^{-1}\, dM(z) \quad \text{if} \quad x > 0.$$

Clearly the support of H_λ is contained in $(0, +\infty)$, $\int_1^{+\infty} x\, dH_\lambda(x) < +\infty$, and if M is unimodal, then f_λ belongs to the class L.

Next suppose f belongs to the class L and $\ln f(u) = i\gamma u - u^2\sigma^2 + \int_{0+}^{+\infty} (e^{iux} - 1 - iux/1 + x^2)\, dH(x)$. Also, suppose $\int_1^{+\infty} x\, dH(x) < +\infty$. Let $h(x) = H'(x)$. Define $p(x) = xh(x)$ and let $M(x) = 0$ if $x < 0$ and $M(x) = -\int_x^{+\infty} p(u)\, du$ if $x > 0$. Since p is nonincreasing on $(0, +\infty)$, M is a unimodal Lévy spectral function with $\int_{0+}^1 x\, dM(x) < +\infty$. Define $\phi(u)$ by the rule

$$\phi(u) = \exp \int_{0+}^{+\infty} (e^{iux} - 1)\, dM(x).$$

Then ϕ is infinitely divisible and by using the above mentioned theorem of Lukacs (1970), its distribution function satisfies $F(x) = 0$ if $x < 0$. Repeating the arguments above we have

$$\ln f(u) = iu\big(\gamma - \textstyle\int_{0+}^{+\infty} x^2(1 + x^2)^{-1}\, dM(x)\big) - u^2\sigma^2 + i\int_0^u \ln \phi(t)\, dt.$$

This completes the proof of the theorem.

3. Final remarks.

1^0. Let $g(u)$ be any infinitely divisible characteristic function which has the property $\int_0^1 v^{-1} \ln g(uv)\, dv$ is a continuous function of u. By setting $f(u) = \exp(\int_0^1 v^{-1} \ln g(uv)\, dv)$, it is easy to see that for all $r \in (0, 1)$, $f(u)$ may be factored $f(u) = f(ru)f_r(u)$ for a characteristic function $f_r(u)$. If we let $\phi(u) = \exp(\int_0^1 \ln g(uv)\, dv)$, then ϕ belongs to U and ϕ and f are related as in (2).

2^0. Using the notation of Theorem 5, let $f_\lambda(u)$ be written as in (3) with $\sigma^2 = 0$. First note that f_λ has a normal component if and only if $a = \inf\{x : F(x) > 0\} > 0$. Also, as a consequence of Theorem 2 of Wolfe (1971), $\int_1^{+\infty} x\, dH_\lambda(x) < +\infty$ if and only if the distribution function of f_λ has a finite first moment μ. In this case, $\mu = if_\lambda'(0) = b$.

3^0. Let $\phi(u)$ be an arbitrary characteristic function with distribution function $F(x)$. F is said to be unimodal with vertex d if F is convex on $(-\infty, d)$ and concave on $(d, +\infty)$. For $\lambda > 0$, define $f_\lambda(u)$ by the rule $f_\lambda(u) = \exp(\lambda(\phi(u) - 1))$. Then f_λ is an infinitely divisible characteristic function and its Lévy spectral function M_λ is given by

$$\begin{aligned} M(x) &= \lambda F(x) && \text{if } x < 0, \\ &= \lambda(F(x) - 1) && \text{if } x > 0. \end{aligned}$$

Obviously M_λ is unimodal at 0 if and only if F is. However, if M_λ is unimodal at 0 one cannot infer the distribution function of f_λ is unimodal. The following example is taken from Wolfe (1970).

Let $\phi(u)$ be a characteristic function of a random variable with density $p(x) = e^{-x}$ if $x > 0$ and 0 otherwise. For each $\lambda > 0$, let $f_\lambda(u) = \exp(\lambda(\phi(u) - 1))$. Then f_λ belongs to U for all $\lambda > 0$, but the distribution function of f_λ is unimodal if and only if $\lambda \leqslant 2$.

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