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ON STATIONARY STRATEGIES FOR ABSOLUTELY CONTINUOUS HOUSES[1]

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Whether stationary families of strategies are uniformly adequate for a leavable, analytically measurable, nonnegative gambling problem whose optimal return function is everywhere finite is a question which remains open. It is, however, given an affirmative answer if, for example, the fortune space is Euclidean and all nontrivial, available gambles are absolutely continuous with respect to Lebesgue measure.

1. Introduction. Consonant with [6], a gambling problem is an ordered couple (Γ, u), where Γ is a gambling house and u is a real-valued function, both defined on the same set F. In [6], u was assumed to be bounded; here u may be unbounded, but it is assumed that u is nonnegative and that the optimal return function V is everywhere finite.

If, for each $f \in F$, $\bar{\sigma}(f)$ is a strategy, $\bar{\sigma}$ is a *plan*. If $\mathcal{P}$ is a set of plans available in Γ and if, for each $\varepsilon > 0$ and $f \in F$, $\exists \bar{\sigma} \in \mathcal{P}$ such that

$$u(\bar{\sigma}(f)) \geqslant (1 - \varepsilon) V(f), \tag{1.1}$$

then $\mathcal{P}$ is an *adequate* set of plans for (Γ, u). If, for each $\varepsilon > 0$, $\exists \bar{\sigma} \in \mathcal{P}$ such that (1.1) holds for all f, then $\mathcal{P}$ is *uniformly adequate* for (Γ, u).

A *Markov kernel* is a gamble-valued function γ defined on F. If $\gamma(f) \in \Gamma(f)$ for all f, then γ is a Γ-*selector* or a Γ-kernel. The plan γ^∞ that prescribes $\gamma(f)$ whenever the current fortune is f is *stationary*. The question raised in [6] as to whether stationary plans are uniformly adequate for leavable problems was settled in the negative by a surprising example of Ornstein [13]. (A later example [7] shows that stationary plans need not even be adequate.) It is natural, therefore, to ask whether stationary plans are uniformly adequate if the problem (Γ, u) is Borel measurable, but we have succeeded in answering this query only if Γ is Borel absolutely continuous.

A Borel, or even analytically, measurable house Γ is *Borel absolutely continuous* if, for some probability measure α, countably additive on the Borel subsets of F, every nontrivial gamble γ available in Γ assigns probability zero to every Borel subset of F to which α assigns probability zero. (If $\gamma \in \Gamma(f)$ and $\gamma \neq \delta(f)$, then γ is *nontrivial for* Γ *at* f.) The principal purpose of this paper is to show that if it is

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also supposed that Γ is Borel absolutely continuous, then stationary plans are indeed uniformly adequate.

2. A few conventions and some notation. Every gamble γ is assumed to be defined on the set of all nonnegative, extended-real-valued functions with domain F and to satisfy the usual conditions;

(a) $\gamma(u_1 + u_2) = \gamma u_1 + \gamma u_2$,

(b) $\gamma(tu) = t\gamma u$ for $t \geqslant 0$,

(c) $u_1 \leqslant u_2 \Rightarrow \gamma u_1 \leqslant \gamma u_2$,

(d) $\gamma c = c$ for all constants c.

By using the same argument given for [6], Theorem 2.8.1, one verifies that every nonnegative, extended-real-valued, finitary function g can be integrated in a unique way by every strategy σ so as to satisfy $\sigma c = c$ and

$$\sigma g = \int \sigma[f](gf)\, d\sigma_0(f). \tag{2.1}$$

Notation, such as $\sigma[f]$, used often in [6], will ordinarily not be explained here. From (2.1) follows the more general

$$\sigma g = \int \sigma[p_t](gp_t)\, d\sigma, \tag{2.2}$$

which holds for every stop rule t, as in [6], Equation 3.7.1. Here, $p_t(h)$ is the partial history $p = (f_1, \cdots, f_n)$ of $h = (f_1, f_2, \cdots)$ and $t(h) = n$. It is natural to regard $\sigma[p_t](gp_t)$ as the conditional σ-expectation of g given the past up to time t. The *time remaining after* p, say $t[p]$, is defined by

$$t[p](h) = t(ph) - n,$$

where $h \in H$, ph is the history which consists of p followed by h, and n is the number of coordinates of p. Notice that $t[p]$ is a stop rule if $t(ph) > n$; it is a nonpositive constant otherwise. As in [6], Section 2.5, $\sigma[p]$ denotes the conditional strategy given p. If π is the policy (σ, t), then the *conditional policy given* p is $\pi[p] = (\sigma[p], t[p])$. The utility $u(\pi[p]) = \int u(f_{t[p]})\, d\sigma[p]$ is well defined if $t[p]$ is a stop rule. If, on the other hand, $t[p]$ is nonpositive, set $u(\pi[p]) = u(f_k)$ where $p = (f_1, \cdots, f_n)$ and $t(f_1, \cdots, f_n, \cdots) = k \leqslant n$. Let s be a stop rule. Then the formula

$$u(\pi) = \int u(\pi[p_s])\, d\sigma \tag{2.3}$$

is a special case of (2.2) as well as an extension to stop rules of [6], Formula 2.10.2.

Many of the definitions and results in [6], which were established there for bounded u's, extend without difficulty to nonnegative u's, and will be used here without comment. Recall that, as defined in [6], the two optimal return functions for a gambling problem, V and U, satisfy $V \leqslant U$. In this paper, whenever it is considerably simpler to do so, the problem will be assumed to be a leavable one, in which event $V = U$. Nevertheless, when greater generality is to be hinted at or when the logic of an argument is clarified by doing so, "V" will often be used in lieu of, and sometimes in addition to, "U" even when $V = U$.

3. Preliminary lemmas. For $\beta \in \Gamma(f)$, let V_β be the supremum of $u(\sigma)$ over all σ available in Γ at f for which the initial gamble σ_0 is β.

LEMMA 3.1. $V_\beta = \beta V$.

PROOF. Apply [6], Corollary 3.3.4, to see that $V_\beta \leqslant \beta V$. For the reverse inequality, let $\varepsilon > 0$, let $\bar{\sigma}$ be a plan satisfying (1.1), and let σ be the strategy with initial gamble β and with $\sigma[f] = \bar{\sigma}(f)$ for all f. Then

$$V_\beta \geqslant u(\sigma) = \int u(\bar{\sigma}(f))\, d\beta(f) \geqslant (1 - \varepsilon)\beta V. \qquad \square$$

LEMMA 3.2. *Let γ be a Γ-selector. Then γ^∞ is optimal if and only if, for all f, both of these conditions hold*:

(a) $\gamma(f)V = V(f)$,

(b) $u(f) < V(f) \Rightarrow V(f) \leqslant \sup u(\gamma^\infty(f), t)$,

where the supremum is taken over all stop rules. Moreover, if γ^∞ is optimal, then

(c) $\gamma(f) = \delta(f) \Rightarrow u(f) = V(f)$.

PROOF. Suppose γ^∞ is optimal. Plainly, Lemma 3.1 implies (a), and

$$V(f) = u(\gamma^\infty(f)) = \limsup_t u(\gamma^\infty(f), t) \leqslant \sup_t u(\gamma^\infty(f), t), \tag{3.1}$$

so (b) holds. For the converse, suppose (a) and (b) hold for all f. Fix $\varepsilon > 0, f \in F$, and a stop rule s. It suffices to find a stop rule $t \geqslant s$ such that $u(\gamma^\infty(f), t) \geqslant V(f) - \varepsilon$. If $u(f') < V(f')$, then by (b), there is a stop rule $\bar{t}(f')$ such that

$$u(\gamma^\infty(f'), \bar{t}(f')) \geqslant V(f') - \varepsilon. \tag{3.2}$$

If $u(f') \geqslant V(f')$, let $\bar{t}(f') = 0$. Define t to be the composition of s with the family $\bar{t}$; that is,

$$t(h) = s(h) + \bar{t}(f_{s(h)})(f_{s(h)+1}, f_{s(h)+2}, \cdots) \tag{3.3}$$

for all h. Then

$$\begin{aligned} u(\gamma^\infty(f), t) &= \int u(\gamma^\infty(f_s), \bar{t}(f_s))\, d\gamma^\infty(f) \\ &\geqslant \int V(f_s)\, d\gamma^\infty(f) - \varepsilon \\ &= V(f) - \varepsilon. \end{aligned}$$

The first equality is an instance of (2.3); the inequality is by definition of $\bar{t}$; and the final equation holds for every stop rule s as can be seen using (a) and an induction on the structure of f_s.

The final assertion of the lemma is trivial to verify. $\square$

LEMMA 3.3. *If Γ' is a subhouse of Γ such that, for every f at which $u(f) < U(f)$, $\Gamma'(f)$ includes every $\gamma \in \Gamma(f)$ except possibly $\delta(f)$, then $U' = U$.*

PROOF. Obviously, $U' \leqslant U$. The reverse inequality will follow from [6], Theorem 2.12.1, once it is verified that U' is excessive for Γ. For the verification, fix f and $\gamma \in \Gamma(f)$. If $u(f) < U(f)$, then either $\gamma = \delta(f)$ or $\gamma \in \Gamma'(f)$. The desired inequality, $\gamma U' \leqslant U'(f)$, is obvious in the first case and a consequence of [6, Theorem 2.14.1] in the second. If $u(f) = U(f)$, then

$$\gamma U' \leqslant \gamma U \leqslant U(f) = u(f) \leqslant U'(f),$$

where the first inequality holds because $U' \leqslant U$; the second because of [6], Theorem 2.14.1; the equality by hypothesis; and the final inequality by definition of U'. ▯

LEMMA 3.4. *At any f at which $u(f) < U(f)$, there is a $\gamma \in \Gamma(f)$ distinct from $\delta(f)$.*

4. Stop-or-go houses. Throughout this section, Γ is a leavable, *stop-or-go house* which means that, for some gamble-valued function α defined on F, $\Gamma(f)$ is $\{\alpha(f), \delta(f)\}$. A stationary plan γ^∞ is *promising* if, for all f,

(a) $\gamma(f)V = V(f)$,

and

(b) $\gamma(f) = \delta(f) \Rightarrow u(f) = V(f)$.

PROPOSITION 4.1. *For a stationary plan to be everywhere optimal it is necessary and sufficient that it be promising.*

(For predecessors of, and for results closely related to, Proposition 4.1, consult [5], [6], [8], and [16].)

PROOF. The necessity is evident even without the help of Lemma 3.2. Suppose therefore that γ^∞ is promising, in which event condition (a) of Lemma 3.2 certainly holds. To see that condition (b) also holds, let $\Gamma'(f)$ be the one-gamble house $\{\gamma(f)\}$. If $u(f) < U(f)$, then $u(f) < V(f)$, for $U = V$ for leavable Γ. For such f, $\gamma(f) \neq \delta(f)$ implies that $\gamma(f) = \alpha(f)$. In sum if $u(f) < U(f)$, then $\Gamma'(f)$ contains $\alpha(f)$ so that the hypothesis of Lemma 3.3 is satisfied. So $U' = U$. If $u(f) < V(f)$, as is now plain, $u(f) < U'(f)$, and there must be, for each $\varepsilon > 0$, a policy π available in Γ' at f for which $u(\pi) > U'(f) - \varepsilon$. Equivalently, if $u(f) < V(f)$, $\sup u(\gamma^\infty(f), t) = U'(f)$, where the sup is taken over all stop rules t. Since $U' = U = V$, condition (b) of Lemma 3.2 holds. That lemma now yields the conclusion that γ^∞ is optimal for Γ. ▯

The problem of showing that optimal stationary plans exist has been reduced to showing that promising stationary plans exist. For showing that the latter exist, this simple lemma is useful.

LEMMA 4.1. *At any f at which $u(f) < V(f)$, $\alpha(f)$ is distinct from $\delta(f)$ and $\alpha(f)V = V(f)$. At any f at which $\alpha(f)V < V(f)$, $\delta(f) \in \Gamma(f)$ and $u(f) = V(f)$.*

PROOF. Suppose $u(f) < V(f)$. Then Lemma 3.4 applies to show that $\alpha(f)$ is distinct from $\delta(f)$. Moreover, since $u(f) < V(f)$, there must be for each $\varepsilon > 0$ an ε-optimal strategy available at f whose initial gamble is of $\alpha(f)$. As Lemma 3.1 now implies, $\alpha(f)V = V(f)$. Suppose $\alpha(f)V < V(f)$. Then by Lemma 3.1, there is available at f some γ other than $\alpha(f)$ which γ can be nothing but $\delta(f)$. That $u(f) = V(f)$ is the main content of the first sentence of Lemma 4.1. □

COROLLARY 4.1. *There exist everywhere-optimal stationary plans. In fact, there exist* Γ*-selectors* γ *with this property: at each* f *at which* $u(f) < V(f)$, $\gamma(f) = \alpha(f)$; *and at each* f *at which* $\alpha(f)V < V(f)$, $\gamma(f) = \delta(f)$; *for each such* γ, γ^∞ *is everywhere optimal. Moreover, there are no other everywhere-optimal stationary plans. If* $\gamma(f)$ *is* $\alpha(f)$ *or* $\delta(f)$ *according as* $u(f) < V(f)$ *or not, then* γ^∞ *is the optimal stationary plan for which the time until stagnation is a minimum for every history.*

PROOF. That there exist Γ-selectors γ with the stated property and that, for such γ, γ^∞ is promising is immediate from Lemma 4.1. That each such γ^∞ is everywhere-optimal is implied by Proposition 4.1, as is the assertion that there are no other everywhere-optimal stationary plans. The final assertion is now evident. □

5. There is a stationary family which yields at least $(1 - \varepsilon)U_n$. Let $U_0 = u$ and, for $k \geqslant 1$ and $f \in F$, let $U_k(f)$ be the most a gambler with initial fortune f can attain if play is allowed to continue up to time k but not beyond. By [6], Theorem 2.15.2, for $k \geqslant 0$ and $f \in F$,

$$U_{k+1}(f) = \sup\{\gamma U_k : \gamma \in \Gamma(f)\}, \tag{5.1}$$

which obviously implies that, for $0 < \beta < 1$, there exists a (Γ, β)*-sequence*, that is, a sequence of Γ-selectors $\gamma_1, \gamma_2, \cdots$ such that, for all $k \geqslant 1$ and all f,

$$\gamma_k(f)U_{k-1} \geqslant \beta^{\frac{1}{2}} U_k(f). \tag{5.2}$$

For each (Γ, β)-sequence, $\gamma_1, \gamma_2, \cdots$, each $n \geqslant 1$, and each f, if $k = k(f) = k(f, n)$ is the least nonnegative integer which satisfies

$$\beta^k U_k(f) = \max_{0 \leqslant j \leqslant n} \beta^j U_j(f), \tag{5.3}$$

and if

$$\begin{aligned} \gamma(f) &= \gamma_{k(f)}(f) && \text{if } k(f) \geqslant 1, \\ &= \delta(f) && \text{if } k(f) = 0, \end{aligned} \tag{5.4}$$

then the Γ-selector γ is called a (Γ, β, n)*-selector*. Here is a generalization of [15], Theorem 1.2.

PROPOSITION 5.1. *For each* $\varepsilon > 0$ *and* $n \geqslant 1$, *there is a* Γ*-selector* γ *such that*

$$u(\gamma^\infty(f)) \geqslant (1 - \varepsilon)U_n(f) \qquad \text{for all } f. \tag{5.5}$$

Indeed, for each β, n *and each* (Γ, β, n)*-selector* γ,

$$u(\gamma^\infty(f)) \geqslant \beta^n U_n(f). \tag{5.6}$$

PROOF. Fix β and n, let $k = k(f)$ satisfy (5.3), define γ as in (5.4), let $W(f)$ be the right-hand side of (5.3), let $\alpha = \beta^{-\frac{1}{2}}$, and, for any f for which $k(f) \geqslant 1$, calculate thus:

$$\begin{aligned}(5.7)\qquad \gamma(f)W &\geqslant \gamma(f)\{\beta^{k-1}U_{k-1}\}\\ &= \beta^{k-1}\gamma_k(f)U_{k-1}\\ &\geqslant \alpha\beta^k U_k(f)\\ &= \alpha W(f).\end{aligned}$$

For any f at which $k(f) = 0$, $\gamma(f)W = \delta(f)W = W(f)$.

Fix f and let $\sigma = \gamma^\infty(f)$. The process $W(f), W(f_1), \cdots$ is, by the previous paragraph, expectation increasing under σ and, by [6], Corollary 3.3.4, $W(\sigma) \geqslant W(\sigma, t) \geqslant W(\sigma, s) \geqslant W(f)$ for all stop rules s, t with $t \geqslant s$. Since $W \geqslant \beta^n U_n$, for (5.6), it suffices to show $u(\sigma) \geqslant W(\sigma)$. This is obviously true if $u(f) = W(f)$. So assume $u(f) < W(f)$, or equivalently, that $\gamma(f) \neq \delta(f)$.

Let $h = (f_1, f_2, \cdots)$ and let $t_0(h)$ be the first k such that $u(f_k) = W(f_k)$, where it is understood that $t_0(h)$ is $+\infty$ if there is no such k.

For each stop rule t, write $W(\sigma, t) = a_t + b_t$ where $a_t = \int_{t\geqslant t_0} W(f_t)\, d\sigma$ and $b_t = \int_{t<t_0} W(f_t)\, d\sigma$. Then

$$a_t = \int_{t\geqslant t_0} W(f_{t_0})\, d\sigma = \int_{t\geqslant t_0} u(f_{t_0})\, d\sigma = \int_{t\geqslant t_0} u(f_t)\, d\sigma \leqslant u(\sigma, t).$$

The first and third equalities hold because σ stagnates at time t_0 [6], Theorem 3.4.3, the second equality holds by the definition of t_0, and the inequality holds because $u \geqslant 0$. It now suffices to show $b_t \to 0$ since, in that case,

$$(5.8)\qquad u(\sigma) = \limsup_t u(\sigma, t) \geqslant \limsup_t a_t = \limsup_t W(\sigma, t) = W(\sigma).$$

To each stop rule t, associate $\hat{t}$ given by

$$\begin{aligned}\hat{t}(h) &= t(h) &&\text{if}\quad t(h) \geqslant t_0(h),\\ &= t(h) + 1 &&\text{if}\quad t(h) < t_0(h).\end{aligned}$$

Then,

$$\begin{aligned}(5.9)\qquad W(\sigma, \hat{t}) &= \int W(\sigma[p_t], \hat{t}[p_t])\, d\sigma\\ &= \int_{t\geqslant t_0} W(f_t)\, d\sigma + \int_{t<t_0} \gamma(f_t)W\, d\sigma\\ &\geqslant a_t + \int_{t<t_0} \alpha W(f_t)\, d\sigma\\ &= a_t + \alpha b_t.\end{aligned}$$

The first equality is an instance of (2.3) and the inequality is by (5.7).

For $\varepsilon > 0$ choose a stop rule s such that $W(\sigma, s) > W(\sigma) - \varepsilon$. (The choice is possible because $W(\sigma) \leqslant U(\sigma) \leqslant U(f) = V(f) < +\infty$. The first inequality holds because $W \leqslant U$, the second by [4], Corollary 3.3.4, and the standing hypothesis that V is finite.) Let t be a stop rule no less than s and calculate thus.

$$(5.10)\qquad a_t + b_t = W(\sigma, t) \geqslant W(\sigma, s) > W(\sigma) - \varepsilon \geqslant W(\sigma, \hat{t}) - \varepsilon \geqslant a_t + \alpha b_t - \varepsilon.$$

So $b_t \leqslant \varepsilon/(\alpha - 1)$, which proves that b_t converges to zero. ☐

As contrasts with Proposition 5.1, there may be no stationary family which yields as much as $U_n - \varepsilon$ even when $n = 2$:

EXAMPLE 5.1 (a modification of an example of Blackwell in [3]). Let F be the set of integers; let $u(n) = 0$ for $n \geqslant 0$, $u(n) = 2^{-n} - 1$ for $n < 0$; $\Gamma(n) = \{\delta(n)\}$ for $n \leqslant 0$, $\Gamma(n) = \{\delta(n), \frac{1}{2}(\delta(n+1) + \delta(0)), \delta(-n)\}$ for $n > 0$. Then $U_2(n) = 2^n - 1/2$ for $n > 0$. But, if γ is a Γ-selector, then either $\gamma(n) = \frac{1}{2}(\delta(n+1) + \delta(0))$ and $u(\gamma^\infty(n)) = 0$ for all $n > 0$ or there is a positive n with $\gamma(n) = \delta(n)$ or $\delta(-n)$ in which case $u(\gamma^\infty(n)) \leqslant 2^n - 1 = U_2(n) - 1/2$.

6. Analytically measurable gambling problems. This section shows that measurable stationary plans are adequate for leavable, analytic problems. As is consonant with Blackwell, Freedman, and Orkin's paper [4], a gambling problem (Γ, u) is *analytic* if Γ is analytic and u is semi-analytic. Analytic problems include the measurable problems defined and studied by Strauch [14], and, a fortiori, the continuous gambling problems studied in [6].

Recall that a separable metric space X is *analytic* if there is a continuous function from the set of irrationals in the unit interval onto X (Kuratowski [11]). Let $\mathfrak{B}(X)$ and $\mathfrak{A}(X)$ denote the sigma-field of Borel subsets of X and the sigma-field generated by the analytic subsets of X respectively. An extended real-valued function g whose domain is X is called *semianalytic* if it is nonnegative and, for all real numbers c, the set of x for which $g(x) > c$ is analytic. For a discussion of these concepts see [4] or [11], Section 39, XI.

If X is analytic and $\mathfrak{P}(X)$, the set of countably additive probability measures defined on $\mathfrak{B}(X)$, is equipped with the weak-star topology, then $\mathfrak{P}(X)$ too is analytic [4], Lemma 25.

A house Γ is *analytic* if F is analytic and the set $\{(f, \gamma) : \gamma \in \Gamma(f)\}$ is an analytic subset of $F \times \mathfrak{P}(F)$. (Here each gamble γ is identified with its restriction to $\mathfrak{B}(F)$ and the integral γg of a semianalytic function g is the conventional, countably additive one.)

LEMMA 6.1 (Lemma 1(3) of Meyer and Traki [9]). *Let u be semianalytic on F. Then the mapping $\gamma \to \gamma u$ from $\mathfrak{P}(F)$ to the extended real numbers is semianalytic.*

Define the operator Γ^* by

$$(6.1) \qquad (\Gamma^* u)(f) = \sup\{\gamma u : \gamma \in \Gamma(f)\}, \qquad f \in F.$$

As was noted in [6], Theorem 2.15.2, if Γ is leavable, then for all positive n, $\Gamma^* U_n = U_{n+1}$. With the convention that $U_0 = u$, the equality holds for $n = 0$ too.

Throughout the remainder of this paper assume that (Γ, u) is a leavable, analytic problem.

LEMMA 6.2. *$\Gamma^* u$ and, consequently, each U_n is semianalytic. Furthermore $U_n \uparrow U$ as $n \uparrow \infty$, so U, too, is semianalytic.*

PROOF. For any real number c, the set of f such that $(\Gamma^* u)(f) > c$ is the projection on F of $\{(f, \gamma) : \gamma u > c, \gamma \in \Gamma(f)\}$, which set is analytic by Lemma 6.1

and the hypothesis that Γ is analytic. Hence, its projection is also analytic. Consequently, $\Gamma^* u = U_1$ is semianalytic.

Induction and the comment following (6.1) show that each U_n is semianalytic. By [6], Theorem 2.15.5g, $U_n \uparrow U$. So U, too, is semianalytic. □

Demonstrations of the existence of measurable strategies require the measurable choice of gambles, which makes this selection lemma useful.

LEMMA 6.3. *Let π be the projection of the product $X \times Y$ of the two analytic sets X and Y onto X; $\cdots \supset A_{-1} \supset A_0 \supset A_1 \cdots$ be a doubly infinite sequence of analytic subsets of $X \times Y$; $B_i = \pi(A_i)$ for all i; and let $B_\infty = \cap B_i$. Then, for each integer k, there is an analytically measurable mapping, s, of the union of the B_i into Y which satisfies these two conditions*:

(i) *$x \in B_i - B_{i+1}$ implies $(x, s(x)) \in A_i$ for all i,*

(ii) *$x \in B_\infty$ implies $(x, s(x)) \in A_k$.*

(*That s is analytically measurable means $s^{-1}(D) \in \mathcal{Q}(X)$ for each $D \in \mathcal{B}(Y)$.*)

PROOF. According to a selection theorem of Mackey and von Neumann [4], Proposition 15, for each i including $i = \infty$, there exists an $\mathcal{Q}(X)$-measurable mapping $s_i : B_i \to Y$ such that, for all $x \in B_i$, $(x, s_i(s))$ is an element of A_i or A_k according as i is finite or not. If $x \in B_\infty$, let $s(x) = s_\infty(x)$. If $x \in UB_i - B_\infty$, let $s(x) = s_i(x)$ where i is the unique integer such that $x \in B_i$ and $x \notin B_{i+1}$. That s satisfies (i) and (ii) is easily verified.

The above lemma, as well as its proof, were abstracted out of Blackwell, Freedman and Orkin [4].

A selector γ for Γ is *Borel* (*analytic*) if it is a Borel measurable (analytically-measurable) function from F to $\mathcal{P}(F)$, where in each case, $\mathcal{P}(F)$ is equipped with the sigma-field of its Borel subsets. It can happen that there is no Borel selector for a nonleavable, Borel measurable house [14]. There are, however, analytic selectors, not only for such houses, but for all analytic houses as is immediate from the Mackey-von Neumann selection theorem, which theorem is the important special case of Lemma 6.3 in which the A_i do not vary with i.

LEMMA 6.4. *For each $\varepsilon > 0$, there is an analytic Γ-selector γ such that*

$$\gamma(f)u \geqslant (1 - \varepsilon)U_1(f) \quad \text{for all} \quad f, \tag{6.2}$$

and

$$\gamma(f) = \delta(f) \Rightarrow u(f) = U_1(f). \tag{6.3}$$

(The assumption, otherwise in force, that Γ is leavable, is not needed for this lemma.)

PROOF. Choose a positive δ such that

$$(1 + \delta)^{-1} > 1 - \varepsilon.$$

For each integer n, let A_n be the set of all (f, γ) such that γ is available at f, γ is different from $\delta(f)$, and γu exceeds $(1 + \delta)^n$. It is easily verified, with the aid of Lemma 6.1, that each A_n is analytic. Hence, with X and Y replaced by F and $\mathcal{P}(F)$, the hypothesis of Lemma 6.3 is satisfied. Let s be the map which Lemma 6.3 delivers and let γ be s on the domain of s which is the union of the B_n in the notation of Lemma 6.3 and which contains the set $[U_1 > u]$. Define γ to agree with δ off the domain of s. It is straightforward to verify (6.2) and (6.3). ▯

For predecessors of Lemma 6.4, see [4], No. 43 and [6], Section 16.

COROLLARY 6.1. *For each* $\varepsilon > 0$ *and* $\alpha \in \mathcal{P}(F)$ *there is a Borel* Γ*-selector* γ *such that* (6.2) *holds except for a set of* f*'s which has* α*-probability zero.*

PROOF. By Lemma 6.4, there is an analytic selector γ' which makes (6.2) true when γ is replaced by γ'. Choose a Borel, Markov kernel β such that the set of f for which $\gamma(f)$ is different from $\beta(f)$, call it A_0, has α-probability zero. Then choose a Borel subset A of F such that $A_0 \subset A$ and $\alpha(A) = 0$. Define $\gamma(f) = \beta(f)$ if $f \notin A$ and $\gamma(f) = \delta(f)$ if $f \in A$. ▯

A stationary family γ^∞ is *Borel* (*analytic*) if the selector γ is Borel (analytic).

PROPOSITION 6.1. *For each positive integer* n *and* $\varepsilon > 0$, *there is an analytic* Γ*-selector* γ *such that*

$$u(\gamma^\infty(f)) \geqslant (1 - \varepsilon)U_n(f) \tag{6.4}$$

for all $f \in F$.

PROOF. Choose β such that $0 < \beta < 1$ and $\beta^n > 1 - \varepsilon$. By Lemma 6.4, there is, for $1 \leqslant k \leqslant n$, an analytic Γ-selector γ_k such that $\gamma_k(f)U_{k-1} > \beta U_k(f)$ for all f. Proposition 5.1 now applies. ▯

Since $U_n \to U$, it follows from Proposition 6.1 that analytic, stationary plans are adequate. In fact, Borel stationary plans are adequate as Proposition 7.1 below implies.

No assertion about the measurability of the left-hand side of (6.4) is made here. Indeed, we do not know whether $u(\gamma^\infty(\cdot))$ is analytically measurable even if γ is Borel measurable, unless u is bounded [17], Theorem 2.

7. Borel stationary plans are almost uniformly adequate. There is a notion of the adequacy of a set $\mathcal{P}$ of plans which is intermediate in strength between ordinary adequacy and uniform adequacy. Namely, a set $\mathcal{P}$ of plans available in Γ is *almost uniformly adequate* if, for each $\varepsilon > 0$ and each measure α countably additive on the Borel subsets of F, there is a $\bar{\sigma} \in \mathcal{P}$ such that the set of f for which (1.1) fails to hold has measure zero under α. Of course, if F is finite or denumerable, almost uniform adequacy is the same as uniform adequacy.

PROPOSITION 7.1. *Borel stationary plans are almost uniformly adequate for* Γ.

This proposition has predecessors in [1], [2], [6], [13] and [15]. Because the present proposition treats unbounded, albeit nonnegative, utilities and because it covers analytic problems rather than Borel problems only, the proof given here differs from that of its predecessors. But the reader will easily discern the underlying similarity of the arguments and, in particular, our debt to Ornstein [13], who was the first to settle the problem of stationarity for a large class of countably additive problems (Γ, u) based on a denumerable fortune space F. The result which corresponds to Proposition 7.1 in the case of positive dynamic programming was stated by Frid [10, Theorem 1], but his proof has an error. (The sets G and H defined in Lemma 3 of [10] need not be Borel.)

A leavable house Γ' defined on the analytic set F is *(Borel) countably parametrized* if it is the union of (the graphs of) a countable number of Borel measurable Markov kernels. A house Γ' is a *subhouse* of Γ, written $\Gamma' \subseteq \Gamma$, if, for each f, $\Gamma'(f) \subseteq \Gamma(f)$. A house Γ' is a *union* of houses Γ_n if, for each f, $\Gamma'(f)$ is the set-theoretic union of $\Gamma_n(f)$.

For each $\alpha \in \mathcal{P}(f)$ and each Borel Markov kernel γ, let $\alpha\gamma$ be that element of $\mathcal{P}(F)$ defined by

$$(\alpha\gamma)(A) = \int \gamma(f)(A)\, d\alpha(f) \tag{7.1}$$

for $A \in \mathcal{B}(F)$. The trivial fact that a subset A of F which has probability one under the completion of $\alpha\gamma$ also has probability one under the completion of $\gamma(f)$ for α-almost all f is used twice in the proof of this lemma.

LEMMA 7.1. *For each $\alpha \in \mathcal{P}(F)$, there is a countably parametrized subhouse Γ' of Γ and a Borel measurable, nonnegative function u' on F such that*

(i) *$u' \leq u$ everywhere, and*

(ii) *$U' \geq U$ α-almost everywhere,*

where U' is the optimal return function for (Γ', u').

PROOF. The weaker lemma obtained by replacing (ii) with the weaker condition

$$U' \geq (1 - \varepsilon) U_n \qquad \alpha - \text{a.s.} \tag{7.2}$$

will be proved first. To this end, choose λ in (0, 1) such that $\lambda^n > (1 - \varepsilon)$. By (5.1) and Corollary 6.1, there is a Borel Γ-selector γ_1 such that

$$\gamma_1(f) U_{n-1} \geq \lambda U_n(f) \qquad \alpha - \text{a.s.} \tag{7.3}$$

Use the notation of (7.1), set $\alpha_1 = \alpha\gamma_1$, and again call on Corollary 6.1 to obtain another Borel Γ-selector γ_2 with

$$\gamma_2(f) U_{n-2} \geq \lambda U_{n-1}(f) \qquad \alpha_1 - \text{a.s.}$$

Continue thus to define inductively Borel Γ-selectors $\gamma_1, \cdots, \gamma_n$ and measures $\alpha_0 = \alpha, \alpha_1, \cdots, \alpha_n$ so that, for $1 \leq k \leq n$, $\alpha_k = \alpha_{k-1}\gamma_k$ and

$$\gamma_k(f) U_{n-k} \geq \lambda U_{n-k+1}(f) \qquad \alpha_{k-1} - \text{a.s.} \tag{7.4}$$

Let $\Gamma'(f) = \{\gamma_1(f), \cdots, \gamma_n(f), \delta(f)\}$. Since u is measurable with respect to the completion of α_n, there is a Borel $u' \geq 0$ which satisfies (i) and which agrees with u on a set of α_n-probability one. So $\gamma_n(f)[u' = u] = 1$ α_{n-1} − a.s. and, hence,

$$\gamma_n(f) u' \geq \gamma_n(f) u \qquad \alpha_{n-1} - \text{a.s.} \tag{7.5}$$

Let U_j' be the optimal j-day return for the problem (Γ', u'). As will now be shown, for $1 \leqslant j \leqslant n$,

$$U_j' \geqslant \lambda^j U_j \qquad \alpha_{n-j} \text{ a.s.} \tag{7.6}$$

To verify (7.6) for $j = 1$, calculate thus:

$$\begin{aligned} U_1'(f) &\geqslant \gamma_n(f)u' \\ &\geqslant \gamma_n(f)u \qquad \alpha_{n-1} \text{ a.s.} \\ &\geqslant \lambda U_1(f) \qquad \alpha_{n-1} \text{ a.s.} \end{aligned}$$

The first inequality is by definition of U_1', the second is (7.5), and the third is the instance of (7.4) in which $k = n$. Suppose (7.6) holds for an integer j which is at most $n - 1$. That is, the set of f_1 such that

$$U_j'(f_1) \geqslant \lambda^j U_j(f_1)$$

has $(\alpha_{n-j-1}\gamma_{n-j})$-probability one. So, for α_{n-j-1}-almost all f, the same set has $\gamma_{n-j}(f)$-probability one. Now calculate:

$$\begin{aligned} U_{j+1}'(f) &\geqslant \gamma_{n-j}(f)U_j' \\ &\geqslant \lambda^j \gamma_{n-j}(f)U_j \qquad \alpha_{n-j-1} \text{ a.s.} \\ &\geqslant \lambda^{j+1}U_{j+1}(f) \qquad \alpha_{n-j-1} \text{ a.s.} \end{aligned}$$

The first inequality is by (5.1) and the third is by (7.4). The inequality (7.6) is now established for $j + 1$ and, by induction, for $1 \leqslant j \leqslant n$. Inequality (7.2) follows from (7.6).

Thus, for $n = 1, 2, \cdots$, there is a countably parametrized house $\Gamma_n \subseteq \Gamma$ and a nonnegative Borel utility function $u_n \leqslant u$ such that, if R_n is the return function for Γ_n, then $R_n \geqslant (1 - 1/n)U_n$ α-almost surely. Let Γ' be the union of the Γ_n, and $u' = \sup u_n$. Obviously, (i) holds and, since $U = \sup U_n$, (ii) is easily verified. ▯

LEMMA 7.2. *Suppose Γ is countably parametrized, u is nonnegative Borel, and ε is a positive number. Then the functions $U_1, U_2, \cdots$ and U are Borel; for each positive integer n, there is a Borel Γ-selector γ such that $u(\gamma^\infty(\cdot))$ is Borel measurable,*

$$u(\gamma^\infty(f)) \geqslant (1 - \varepsilon/2)U_n(f) \qquad \text{for all } f, \tag{7.7}$$

and, for each $\alpha \in \mathcal{P}(F)$ and all sufficiently large n,

$$u(\gamma^\infty(f)) \geqslant (1 - \varepsilon)U(f) \qquad \text{with } \alpha\text{-probability at least } 1 - \varepsilon. \tag{7.8}$$

PROOF. For a proof of the first assertion, use Formula 5.1 and Lemma 6.2 or [15], Theorem 4.1. Next choose β in (0, 1) such that $\beta^n > 1 - \varepsilon/2$. Let $\gamma_1', \gamma_2', \cdots$ be the Borel Markov kernels comprising Γ. For each k and f, let $\gamma_k(f)$ be the first element of the sequence $\gamma_1'(f), \gamma_2'(f), \cdots$ satisfying (5.2). Then $\gamma_1, \gamma_2, \cdots$ are Borel measurable and constitute a (Γ, β) sequence as defined in Section 5. That the corresponding Γ-selector γ satisfies (7.7) is evident in view of Proposition 5.1. Since $U_n \uparrow U$ (Lemma 6.2), (7.8) holds for all sufficiently large n.

The proof would be complete if $u(\gamma^{\infty}(\cdot))$ could be shown to be Borel measurable. Whether it is or not, we do not know and, for present purposes, need not know, for, as will now be shown, there is a Borel Γ-selector λ such that $u(\lambda^{\infty}(f)) \geqslant u(\gamma^{\infty}(f))$ for all f and $u(\lambda^{\infty}(\cdot))$ is Borel. Consider the Borel gambling problem (Γ', u) where $\Gamma'(f) = \{\delta(f), \gamma(f)\}$ for all f. Then U' is Borel measurable. So, if $\lambda = \gamma$ or δ according as $u < U'$, or $u = U'$, then λ, too, is Borel measurable. Of course, $U'(f) \geqslant u(\gamma^{\infty}(f))$ and by Corollary 4.1, $u(\lambda^{\infty}(f))$ is $U'(f)$ for all f. So $u(\lambda^{\infty}(\cdot))$ is Boreal measurable and is no less than $u(\gamma^{\infty}(\cdot))$. □

Incomplete stop rules, as defined in [6], are here called *stopping times*. A stop rule is simply a stopping time which has only finite values. As in [6], the partial history $(f_1, \cdots, f_n)$ of the history $h = (f_1, f_2, \cdots)$ is denoted by $p_n(h)$. Strategies σ and σ' *agree prior to a time* τ if $\sigma_0 = \sigma_0'$ and, for every h and n with $0 < n < \tau(h)$, $\sigma(p_n(h)) = \sigma'(p_n(h))$. If $\tau(h) < +\infty$, abbreviate $p_{\tau(h)}(h)$ to $p_\tau(h)$.

LEMMA 7.3. *If σ is available in Γ, σ and σ' agree prior to time τ, $\varepsilon \geqslant 0$, and $u(\sigma'[p_\tau(h)]) \geqslant (1-\varepsilon)V(f_\tau(h))$ whenever $\tau(h) < +\infty$, then $u(\sigma') \geqslant (1-\varepsilon)u(\sigma)$.*

PROOF. This lemma is one of the implications in [9], Lemma 3. □

Given σ and $\varepsilon > 0$, introduce $\tau = r(\sigma, \varepsilon)$ as the first time (if any) when σ is not conditionally ε-optimal; that is,

$$\tau(h) = r(\sigma, \varepsilon)(h) = \inf\{n : u(\sigma[p_n(h)]) < (1-\varepsilon)V(f_n)\}. \tag{7.9}$$

The infimum of an empty set of n's is, by convention, $+\infty$.

The next lemma states that a strategy which agrees with a very good strategy until that strategy is conditionally less than good is itself a good strategy.

LEMMA 7.4. *Let σ be a strategy available at f and let $\varepsilon > 0$. If σ' is any strategy which agrees with σ prior to time $\tau = r(\sigma, \varepsilon)$, then $u(\sigma') \geqslant u(\sigma) - \varepsilon^{-1}[V(f) - u(\sigma)]$. Therefore, if $u(\sigma) \geqslant (1-\varepsilon^2/2)V(f)$, then $u(\sigma') \geqslant (1-\varepsilon)V(f)$.*

PROOF. This lemma is part of [9], Lemma 4. □

In view of Lemma 7.1, it suffices to prove Proposition 7.1 under the additional assumption that Γ is countably parametrized and u is Borel, which assumption is in force for the remainder of this section.

To each stop-or-go house Σ, associate the house $\Gamma \circ \Sigma$ which is defined by

$$(\Gamma \circ \Sigma)(f) = \Sigma(f), \quad \text{if } \Sigma(f) \text{ contains two elements,}$$
$$= \Gamma(f), \quad \text{otherwise.}$$

Plainly, $\Sigma \subseteq \Gamma \Rightarrow \Sigma \subseteq \Gamma \circ \Sigma \subseteq \Gamma$ and $\Sigma \subseteq \Sigma' \Rightarrow \Gamma \circ \Sigma' \subseteq \Gamma \circ \Sigma$.

If λ is Borel measurable and $\Sigma(f) = \{\lambda(f), \delta(f)\}$ for all f, then Σ is Borel and is countably parametrized, as is $\Gamma \circ \Sigma$. So, by Lemma 7.2, the optimal return function W for Σ is Borel measurable, as is the optimal return function R for $\Gamma \circ \Sigma$.

LEMMA 7.5. *Suppose Σ is a leavable, Borel, stop-or-go subhouse of Γ, $\alpha \in \mathcal{P}(F)$, and $\varepsilon > 0$. Then there is a leavable, Borel, stop-or-go house Σ' such that*

(i) $\Sigma \subseteq \Sigma' \subseteq \Gamma$,

(ii) $\alpha[W' \geqslant (1 - \varepsilon)R] \geqslant 1 - \varepsilon$, *and*

(iii) $R' \geqslant (1 - \varepsilon)R$.

(Here, W' and R' are the optimal return functions for Σ' and $\Gamma \circ \Sigma'$, respectively.)

PROOF. By Lemma 7.2, there is a Borel $\Gamma \circ \Sigma$-selector γ such that

$$\alpha(S) \geqslant 1 - \varepsilon, \tag{7.10}$$

where $S = \{f : u(\gamma^\infty(f)) \geqslant (1 - \varepsilon^2/2)R(f)\}$. Let $T = \{f : u(\gamma^\infty(f)) \geqslant (1 - \varepsilon)R(f)\}$, and let Σ' be the smallest leavable, stop-or-go house which is at least as large as Σ and in which $\gamma(f)$ is available at each f in T. That is,

$$\begin{aligned} \Sigma'(f) &= \{\gamma(f), \delta(f)\}, \quad \text{if} \quad f \in T \quad \text{and} \quad \Sigma(f) = \{\delta(f)\}, \\ &= \Sigma(f), \quad \text{if} \quad \Sigma(f) \quad \text{contains two elements}, \\ &= \{\delta(f)\}, \quad \text{otherwise}. \end{aligned}$$

Obviously, Σ' satisfies (i). To check (ii), let λ be the Σ'-selector which equals γ on T and is δ on T^c. Then, for each $f \in S$, $\lambda^\infty(f)$ agrees with $\gamma^\infty(f)$ prior to the time of the first exit from T. So, by Lemma 7.4.

$$W'(f) \geqslant u(\lambda^\infty(f)) \geqslant (1 - \varepsilon)R(f)$$

for $f \in S$. Condition (ii) now follows from (7.10).

There remains to verify (iii). Since γ is a $\Gamma \circ \Sigma'$-selector, the inequality of (iii) certainly holds for $f \in T$. For $f \notin T$, let σ be any strategy available at f in $\Gamma \circ \Sigma$ and define σ' to be that strategy which agrees with σ prior to the time τ of first entrance into T and such that the conditional strategy $\sigma'[p_\tau(h)]$ is $\gamma^\infty(f_\tau(h))$ whenever $\tau(h) < \infty$. Then $R'(f) \geqslant u(\sigma') \geqslant (1 - \varepsilon)u(\sigma)$, where the first inequality holds because σ' is available in Γ' at f and the second is by Lemma 7.3. ▯

LEMMA 7.6. *Let $\alpha \in \mathcal{P}(F)$ and $\varepsilon > 0$. There is a sequence $\Sigma_0 \subseteq \Sigma_1 \subseteq \cdots$ of leavable, Borel stop-or-go subhouses of Γ whose optimal return functions $W_0, W_1, \cdots$ satisfy*

$$\alpha[W_n \geqslant (1 - \varepsilon)U]\uparrow 1. \tag{7.11}$$

PROOF. Let Σ_0 be the trivial house in which only $\delta(f)$ is available at f for every f. Then $\Gamma \circ \Sigma_0 = \Gamma$ and R_0, the return function for $\Gamma \circ \Sigma_0$, is U.

Suppose that Σ_n has been defined and that $0 < \varepsilon_n < 1$. Let R_n be the return function for $\Gamma \circ \Sigma_n$. Then, by the previous lemma, there exists Σ_{n+1} such that

$$\alpha[W_{n+1} \geqslant (1 - \varepsilon_n)R_n] \geqslant 1 - \varepsilon_n \tag{7.12}$$

and

$$R_{n+1} \geqslant (1 - \varepsilon_n)R_n. \tag{7.13}$$

For $n \geqslant 1$,

$$R_n \geqslant \left(\textstyle\prod_{i=1}^{n-1}(1 - \varepsilon_i)\right)R_0,$$

as follows from (7.13). Thus the event

$$\left[W_{n+1} \geqslant (\Pi_{i=1}^{n}(1 - \varepsilon_i))U \right]$$

includes the event occurring in the left-hand side of (7.12) and, therefore, has α-probability at least $1 - \varepsilon_n$. The proof is complete once the ε_n are chosen so that

$$\Pi_{n=1}^{\infty}(1 - \varepsilon_n) > 1 - \varepsilon. \qquad \square$$

To complete the proof of Proposition 7.1, let $\Sigma_1, \Sigma_2, \cdots$ be as in Lemma 7.6 and let Σ be the union of the Σ_n. Then Σ is a leavable, Borel, stop-or-go subhouse of Γ whose return function W is at least $(1 - \varepsilon)U$ α-almost surely. Suppose $\Sigma(f) = \{\lambda(f), \delta(f)\}$ and γ is that Σ-selector which agrees with δ on the set $[u = W]$ and with λ on the complementary set. Then γ is Borel measurable and, by Corollary 4.1, $u(\gamma^{\infty}(f)) = W(f)$ for all f. The proof of Proposition 7.1 is now complete.

8. Absolutely continuous houses. As defined in the introduction an analytic house Γ is *Borel absolutely continuous with respect to* $\alpha \in \mathcal{P}(F)$ if, for every $f \in F$ and $\gamma \in \Gamma(f)$, $\gamma \neq \delta(f)$ implies γ is Borel absolutely continuous with respect to α.

For this section the rather innocuous assumption is made that every available gamble γ is *regular* on the nonnegative functions which means that the integral of every nonnegative function is the supremum of the integrals of the bounded functions which it majorizes. Consequently, if two nonnegative functions agree on a set of γ-probability one, their γ-integrals are the same.

THEOREM 8.1. *If Γ is leavable and Borel absolutely continuous with respect to some $\alpha \in \mathcal{P}(F)$, U is everywhere finite, and $0 < \varepsilon < 1$, then there is an analytic Γ-selector γ such that*

$$u(\gamma^{\infty}(f)) \geqslant (1 - \varepsilon)U(f) \qquad \text{for all } f \in F.$$

A final lemma, which does not use the assumption of absolute continuity, is needed for the proof.

LEMMA 8.1. *There is an analytic Γ-selector γ_2 such that*
(i) $\gamma_2(f)U \geqslant (1 - \varepsilon_1)U(f)$ *for all* f,
(ii) $[\gamma_2 = \delta] \subseteq [u = U]$.

PROOF. Choose ε_2 so that $0 < \varepsilon_2 < 1$ and $(1 - \varepsilon_2)^2 > (1 - \varepsilon_1)$. By Lemma 6.4, there is, for each $n \geqslant 0$, an analytic Γ-selector λ_n such that

$$\lambda_n(f)U_n \geqslant (1 - \varepsilon_2)U_{n+1}(f) \qquad \text{for all } f, \tag{8.3}$$

and

$$\lambda_n(f) = \delta(f) \Rightarrow U_n(f) = U_{n+1}(f). \tag{8.4}$$

If $u(f) = U(f)$, define $\gamma_2(f)$ to be $\delta(f)$. If $u(f) < U(f)$, let $n(f)$ be the least n such that $U_{n+1}(f)$ exceeds the maximum of $u(f)$ with $(1 - \varepsilon_2)U(f)$, and define $\gamma_2(f)$ to be $\lambda_{n(f)}(f)$. It remains to verify (i) and (ii). Both are trivial if $u(f) = U(f)$, so suppose $u(f) < U(f)$ and set $m = n(f)$. By definition of m, $U_m(f) < U_{m+1}(f)$ and, hence, condition (ii) follows from (8.4). Condition (i) is a consequence of the following calculation: $\gamma_2(f)U \geqslant \gamma_2(f)U_m = \lambda_m(f)U_m \geqslant (1 - \varepsilon_2)U_{m+1}(f) \geqslant (1 - \varepsilon_2)^2 U(f) > (1 - \varepsilon_1)U(f)$. ▯

PROOF OF THEOREM 8.1. Choose ε_1 so that $0 < \varepsilon_1 < 1$ and $(1 - \varepsilon_1)^2 > 1 - \varepsilon$. By Proposition 7.1, there is a Borel selector γ_1 and a Borel subset S of $\{f : u(\gamma_1^\infty(f)) \geqslant (1 - \varepsilon_1)U(f)\}$ for which $\alpha(S) = 1$. Since Γ is absolutely continuous with respect to α, S has probability one under every gamble available at a fortune $f \in S$. Thus, for each $f \in S$, $f_n \in S$ for all n with $\gamma_1^\infty(f)$-probability one. If γ is any selector which agrees with γ_1 on S, then for each $f \in S$ and every stop rule t, $u(\gamma^\infty(f), t) = u(\gamma_1^\infty(f), t)$, as can be proved using induction on the structure of f_t and the assumption that all available gambles are regular. Hence, $u(\gamma^\infty(f)) = u(\gamma_1^\infty(f)) \geqslant (1 - \varepsilon_1)U(f) > (1 - \varepsilon)U(f)$ for $f \in S$. Set $\gamma = \gamma_2$ on S^c where γ_2 is the analytic selector given by Lemma 8.1. If $f \in S^c$ and $\gamma(f) = \delta(f)$, then $u(\gamma^\infty(f)) = u(f) = U(f)$. If $f \in S^c$ and $\gamma(f) \neq \delta(f)$, then $\gamma(f)(S) = 1$ and, therefore,

$$\begin{aligned} u(\gamma^\infty(f)) &= \int u(\gamma^\infty(f_1))\, d\gamma(f_1|f) \\ &= \int u(\gamma_1^\infty(f_1))\, d\gamma(f_1|f) \\ &\geqslant (1 - \varepsilon_1)\int U(f_1)\, d\gamma(f_1|f) \\ &= (1 - \varepsilon_1)\gamma(f)U \\ &= (1 - \varepsilon_1)\gamma_2(f)U \\ &\geqslant (1 - \varepsilon_1)^2 U(f) \\ &\geqslant (1 - \varepsilon)U(f). \end{aligned}$$

The proof is complete.

If Γ is Borel countably parametrized and u is Borel measurable, then, as is not difficult to verify, the selector γ_2 of Lemma 8.1 and, hence, the selector γ of Theorem 8.1, can be chosen so as to be Borel.

It would not be difficult to use Theorem 8.1 to obtain certain generalizations of itself, but not very interesting ones. What would be interesting would be to ascertain whether the hypothesis that Γ is Borel absolutely continuous can be simply deleted from the statement of the theorem. But this question we leave open.

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